Accelerated Pre-Calculus

1.08 Co-terminal Angles and Reference Angles

Co-terminal angles are angles that have the same ~~terminal~~ (ending) side. Not only are co-terminal angles created by measuring an angle both in the negative and in the positive directions, but they can be created by doing more than one revolution (360°). Yes, angles can measure more than 360°!

Co-terminal angles can be found at the same location just another revolution more or less. So, to find co-terminal angles, we must add or subtract 360° and we will end in the same location for the terminal side.

Example 1: Find one positive and one negative co-terminal angle for the given angle.

a. 30°

$30 + 360 = \boxed{390°}$

$30 - 360 = \boxed{-330°}$

b. -34°

$-34 + 360 = \boxed{326°}$

$-34 - 360 = \boxed{-394°}$

c. $\frac{7\pi}{6}$

$\frac{7\pi}{6} + \frac{12\pi}{6} = \boxed{\frac{19\pi}{6}}$

$\frac{7\pi}{6} - \frac{12\pi}{6} = \boxed{\frac{-5\pi}{6}}$

Can we ever identify *all* co-terminal angles? **No, there are infinitely many!**

We can use this process for angles larger than 360° by subtracting 360 from the larger angle measure until we find a positive and a negative co-terminal angle. [in the range of (-360°, 360°)]

Example 2: Find one positive and one negative co-terminal angle for the given angle. State the quadrant in which the terminal side lies.

a. 800°

-720

$\boxed{80°}$

Q1

80
-360

$\boxed{-280°}$

b. -3732°   10(360)

$+3660$

$\boxed{-132°}$

Q3

-132
$+360$

$\boxed{228°}$

c. 3945°   10(360)

-3600

$\boxed{345°}$
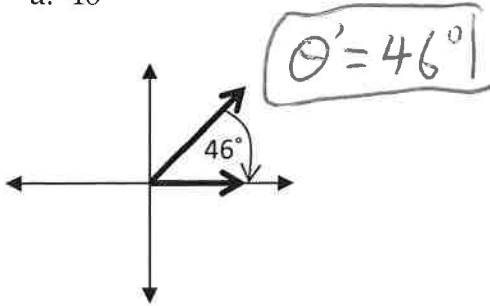
Q4

345
-360

$\boxed{-15°}$

20

Reference Angles

A **Reference angle** (θ') is the angle formed by the *terminal side* of the angle and the closest part of the _x-axi_
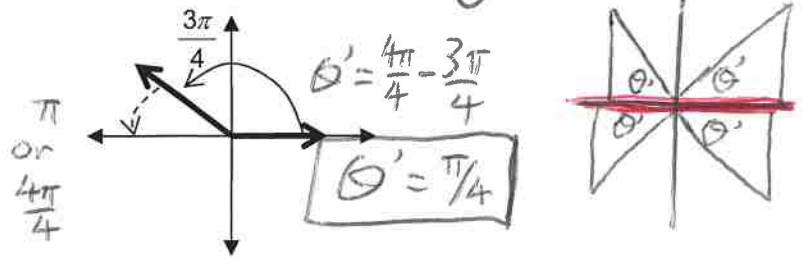
Examples: Find the measure of the reference angle (θ') for each angle.

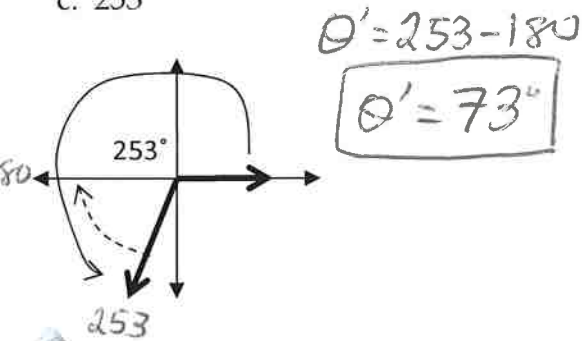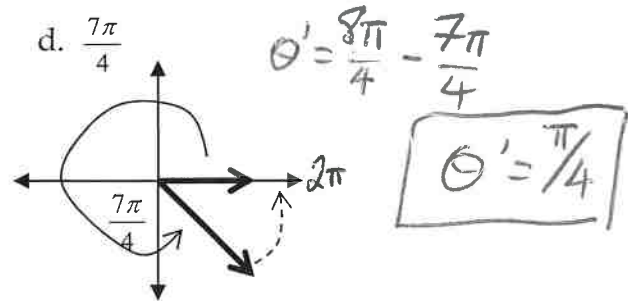*Reference angle is
$0° < x < 90°$
$0 < \theta < \pi/2$*

a. $46°$

$$\boxed{\theta' = 46°}$$

$46°$

b. $\dfrac{3\pi}{4}$

$\dfrac{3\pi}{4}$

π or $\dfrac{4\pi}{4}$

$\theta' = \dfrac{4\pi}{4} - \dfrac{3\pi}{4}$

$$\boxed{\theta' = \pi/4}$$

θ' θ'
θ' θ'

c. $253°$

$\theta' = 253 - 180$

$$\boxed{\theta' = 73°}$$

$253°$ 180

253

d. $\dfrac{7\pi}{4}$

$\theta' = \dfrac{8\pi}{4} - \dfrac{7\pi}{4}$

$$\boxed{\theta' = \pi/4}$$

2π

$\dfrac{7\pi}{4}$

e. $487°$ (first find a positive co-terminal angle between 0° and 360°)

$127°$

$487°$

$90°$

$\theta' = 180 - 127$

$$\boxed{\theta' = 53°}$$

f. $-\dfrac{13\pi}{3}$ (first find a positive co-terminal angle between 0 and 2π)

$-\dfrac{13\pi}{3}$ $\dfrac{6\pi}{3}$

$-\dfrac{13\pi}{3} + \dfrac{6\pi}{3} = \dfrac{-7\pi}{3} + \dfrac{6\pi}{3} = \dfrac{-\pi}{3}$

$-\dfrac{\pi}{3} + \dfrac{6\pi}{3} = \dfrac{5\pi}{3}$

$$\boxed{\theta' = \dfrac{6\pi}{3} - \dfrac{5\pi}{3} = \dfrac{\pi}{3}}$$

21
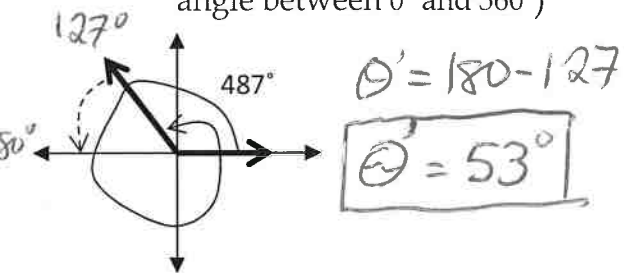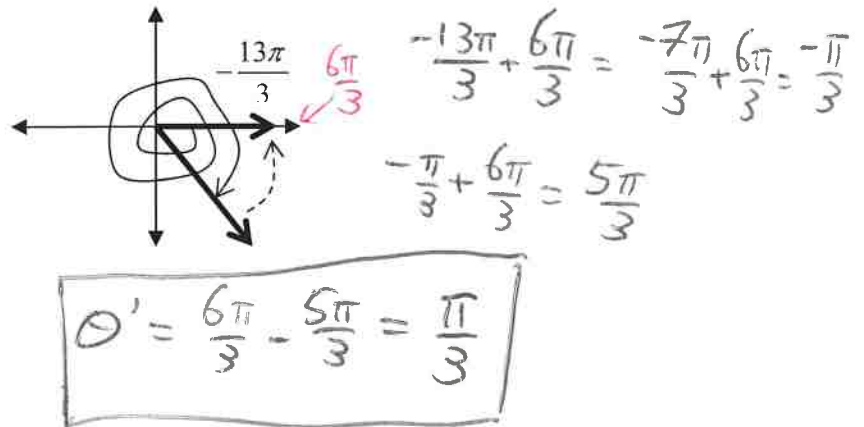
How large can the *reference angle* be? **Up to 90° - reference angles are <u>always</u> <u>acute</u> and <u>positive</u>!**

In summary, to find the reference angle (θ') based on the quadrant in which the terminal side of θ lies

QII $\theta' = \lvert 180 - \theta \rvert$ $\theta' = \theta - 180°$	QI $\theta' = \theta$
QIII $\theta' = \theta - 180°$ $\theta' = 180 - \theta$	QIV $\theta' = 360 - \theta$ $\theta' = -\theta$

Reminder: to use the rules in this table, the angle θ must be between 0° and 360° (or between 0 and 2π).

If this is not the case, then find a positive, co-terminal angle for θ between 0° and 360° to use the table.

∈ is an element of:

$n \in \mathbb{Z}$ ← set of all integers

$360n$ ← multiples of 360
$2\pi n$ ← multiples of 2π

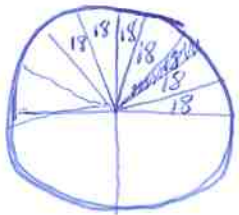**1.08 Coterminal & Reference Angles**

Directions: Complete #18 – 26 all

Identify all angles that are coterminal with the given angle. Then find and draw one positive and one negative angle coterminal with the given angle. (Example 3)

18. 120° $120° + 360n \quad n \in \mathbb{Z}$
19. −75° $-75 + n360 \quad n \in \mathbb{Z}$
20. 225° $225 + 360n \quad n \in \mathbb{Z}$
21. −150° $-150° + n360° \quad n \in \mathbb{Z}$
22. $\frac{\pi}{3}$ $\frac{\pi}{3} + 2\pi n \quad n \in \mathbb{Z}$
23. $-\frac{3\pi}{4}$ $-\frac{3\pi}{4} + 2\pi n \quad n \in \mathbb{Z}$
24. $-\frac{\pi}{12}$ $-\frac{\pi}{12} + 2\pi n \quad n \in \mathbb{Z}$
25. $\frac{3\pi}{2}$ $\frac{3\pi}{2} + 2\pi n \quad n \in \mathbb{Z}$

26. **GAME SHOW** Sofia is spinning a wheel on a game show. There are 20 values in equal-sized spaces around the circumference of the wheel. The value that Sofia needs to win is two spaces above the space where she starts her spin, and the wheel must make at least one full rotation for the spin to count. Describe a spin rotation in degrees that will give Sofia a winning result. (Example 3)

$\frac{360}{20} = 18$

$360 + 2(18) = 396°$

18. 480°   −240°

19. 285°   −435°

20. −135°   585°

21. 210°   −510°

22. $\frac{7\pi}{3}$ $-\frac{5\pi}{3}$

23. $-\frac{11\pi}{4}$ $\frac{5\pi}{4}$

24. $-\frac{25\pi}{12}$ $\frac{23\pi}{12}$

25. $\frac{7\pi}{2}$ $-\frac{\pi}{2}$

Directions: Complete #17 – 24 all

Sketch each angle. Then find its reference angle. (Example 3)

17. 135°

18. 210°

19. $\frac{7\pi}{12}$

20. $\frac{11\pi}{3}$

21. −405°

22. −75°

23. $\frac{5\pi}{6}$

24. $\frac{13\pi}{6}$

1.09 Coterminal and Reference Angles