CHAPTER 1
Limits and Their Properties

Section 1.1 A Preview of Calculus

1. Precalculus: (20 ft/sec)(15 sec) = 300 ft

2. Calculus required: Velocity is not constant.

Distance $\approx$ (20 ft/sec)(15 sec) = 300 ft

3. Calculus required: Slope of the tangent line at $x = 2$ is the rate of change, and equals about 0.16.

4. Precalculus: rate of change = slope = 0.08

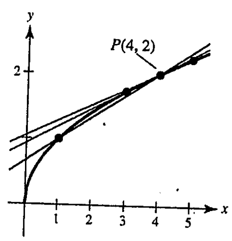
5. (a) Precalculus: Area $= \frac{1}{2}bh = \frac{1}{2}(5)(4) = 10$ sq. units

(b) Calculus required: Area $= bh$

$$\approx 2(2.5)$$
$$= 5 \text{ sq. units}$$

6. $f(x) = \sqrt{x}$

(a)



(b) $\text{slope} = m = \dfrac{\sqrt{x} - 2}{x - 4}$

$$= \frac{\sqrt{x} - 2}{(\sqrt{x} + 2)(\sqrt{x} - 2)}$$

$$= \frac{1}{\sqrt{x} + 2}, x \neq 4$$

$x = 1: m = \dfrac{1}{\sqrt{1} + 2} = \dfrac{1}{3}$

$x = 3: m = \dfrac{1}{\sqrt{3} + 2} \approx 0.2679$

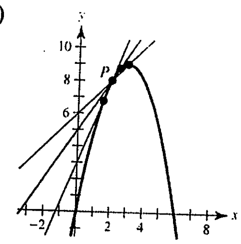
$x = 5: m = \dfrac{1}{\sqrt{5} + 2} \approx 0.2361$

(c) At $P(4, 2)$ the slope is $\dfrac{1}{\sqrt{4} + 2} = \dfrac{1}{4} = 0.25.$

You can improve your approximation of the slope at $x = 4$ by considering x-values very close to 4.

7. $f(x) = 6x - x^2$

(a)



(b) $\text{slope} = m = \dfrac{(6x - x^2) - 8}{x - 2} = \dfrac{(x - 2)(4 - x)}{x - 2}$

$$= (4 - x), x \neq 2$$

For $x = 3$, $m = 4 - 3 = 1$

For $x = 2.5$, $m = 4 - 2.5 = 1.5 = \dfrac{3}{2}$

For $x = 1.5$, $m = 4 - 1.5 = 2.5 = \dfrac{5}{2}$

(c) At $P(2, 8)$, the slope is 2. You can improve your approximation by considering values of x close to 2.

8. Answers will vary. *Sample answer*:

The instantaneous rate of change of an automobile's position is the velocity of the automobile, and can be determined by the speedometer.

9. (a) Area $\approx 5 + \frac{5}{2} + \frac{5}{3} + \frac{5}{4} \approx 10.417$

Area $\approx \frac{1}{2}\left(5 + \frac{5}{1.5} + \frac{5}{2} + \frac{5}{2.5} + \frac{5}{3} + \frac{5}{3.5} + \frac{5}{4} + \frac{5}{4.5}\right) \approx 9.145$

(b) You could improve the approximation by using more rectangles.

10. (a) $D_1 = \sqrt{(5-1)^2 + (1-5)^2} = \sqrt{16+16} \approx 5.66$

(b) $D_2 = \sqrt{1 + \left(\frac{5}{2}\right)^2} + \sqrt{1 + \left(\frac{5}{2} - \frac{5}{3}\right)^2} + \sqrt{1 + \left(\frac{5}{3} - \frac{5}{4}\right)^2} + \sqrt{1 + \left(\frac{5}{4} - 1\right)^2}$

$\approx 2.693 + 1.302 + 1.083 + 1.031 \approx 6.11$

(c) Increase the number of line segments.

Section 1.2 Finding Limits Graphically and Numerically

1.

x	3.9	3.99	3.999	4.001	4.01	4.1
$f(x)$	0.2041	0.2004	0.2000	0.2000	0.1996	0.1961

$\lim_{x \to 4} \frac{x-4}{x^2 - 3x - 4} \approx 0.2000$ (Actual limit is $\frac{1}{5}$.)

2.

x	2.9	2.99	2.999	3	3.001	3.01	3.1
$f(x)$	0.1695	0.1669	0.1667	?	0.1666	0.1664	0.1639

$\lim_{x \to 3} \frac{x-3}{x^2 - 9} \approx 0.1667$ (Actual limit is $\frac{1}{6}$.)

3.

x	−0.1	−0.01	−0.001	0	0.001	0.01	0.1
$f(x)$	0.5132	0.5013	0.5001	?	0.4999	0.4988	0.4881

$\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x} \approx 0.5000$ (Actual limit is $\frac{1}{2}$.)

4.

x	2.9	2.99	2.999	3.001	3.01	3.1
$f(x)$	−0.0641	−0.0627	−0.0625	−0.0625	−0.0623	−0.0610

$\lim_{x \to 3} \frac{[1/(x+1)] - (1/4)}{x-3} \approx -0.0625$ (Actual limit is $-\frac{1}{16}$.)

5.

x	−0.1	−0.01	−0.001	0.001	0.01	0.1
$f(x)$	0.9983	0.99998	1.0000	1.0000	0.99998	0.9983

$\lim_{x \to 0} \frac{\sin x}{x} \approx 1.0000$ (Actual limit is 1.) (Make sure you use radian mode.)

6.

x	−0.1	−0.01	−0.001	0.001	0.01	0.1
$f(x)$	0.0500	0.0050	0.0005	−0.0005	−0.0050	−0.0500

$\lim_{x \to 0} \frac{\cos x - 1}{x} \approx 0.0000$ (Actual limit is 0.) (Make sure you use radian mode.)

7.

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	0.2564	0.2506	0.2501	0.2499	0.2494	0.2439

$\lim_{x\to 1} \frac{x-2}{x^2+x-6} \approx 0.2500$ $\left(\text{Actual limit is } \frac{1}{4}.\right)$

8.

x	−4.1	−4.01	−4.001	−4	−3.999	−3.99	−3.9
$f(x)$	1.1111	1.0101	1.0010	?	0.9990	0.9901	0.9091

$\lim_{x\to -4} \frac{x+4}{x^2+9x+20} \approx 1.0000$ (Actual limit is 1.)

9.

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	0.7340	0.6733	0.6673	0.6660	0.6600	0.6015

$\lim_{x\to 1} \frac{x^4-1}{x^6-1} \approx 0.6666$ $\left(\text{Actual limit is } \frac{2}{3}.\right)$

10.

x	−3.1	−3.01	−3.001	−3	−2.999	−2.99	−2.9
$f(x)$	27.91	27.0901	27.0090	?	26.9910	26.9101	26.11

$\lim_{x\to -3} \frac{x^3+27}{x+3} \approx 27.0000$ (Actual limit is 27.)

11.

x	−6.1	−6.01	−6.001	−6	−5.999	−5.99	−5.9
$f(x)$	−0.1248	−0.1250	−0.1250	?	−0.1250	−0.1250	−0.1252

$\lim_{x\to -6} \frac{\sqrt{10-x}-4}{x+6} \approx -0.1250$ $\left(\text{Actual limit is } -\frac{1}{8}.\right)$

12.

x	1.9	1.99	1.999	2	2.001	2.01	2.1
$f(x)$	0.1149	0.115	0.1111	?	0.1111	0.1107	0.1075

$\lim_{x\to 2} \frac{x/(x+1)-2/3}{x-2} \approx 0.1111$ $\left(\text{Actual limit is } \frac{1}{9}.\right)$

13.

x	−0.1	−0.01	−0.001	0.001	0.01	0.1
$f(x)$	1.9867	1.9999	2.0000	2.0000	1.9999	1.9867

$\lim_{x\to 0} \frac{\sin 2x}{x} \approx 2.0000$ (Actual limit is 2.) (Make sure you use radian mode.)

14.

x	−0.1	−0.01	−0.001	0.001	0.01	0.1
$f(x)$	0.4950	0.5000	0.5000	0.5000	0.5000	0.4950

$\lim_{x\to 0} \frac{\tan x}{\tan 2x} \approx 0.5000$ $\left(\text{Actual limit is } \frac{1}{2}.\right)$

15. $\lim_{x \to 3} (4 - x) = 1$

16. $\lim_{x \to 0} \sec x = 1$

17. $\lim_{x \to 2} f(x) = \lim_{x \to 2} (4 - x) = 2$

18. $\lim_{x \to 1} f(x) = \lim_{x \to 1} (x^2 + 3) = 4$

19. $\lim_{x \to 2} \dfrac{|x - 2|}{x - 2}$ does not exist.

For values of x to the left of 2, $\dfrac{|x - 2|}{(x - 2)} = -1$, whereas for values of x to the right of 2, $\dfrac{|x - 2|}{(x - 2)} = 1$.

20. $\lim_{x \to 5} \dfrac{2}{x - 5}$ does not exist because the function increases and decreases without bound as x approaches 5.

21. $\lim_{x \to 0} \cos(1/x)$ does not exist because the function oscillates between -1 and 1 as x approaches 0.

22. $\lim_{x \to \pi/2} \tan x$ does not exist because the function increases without bound as x approaches $\dfrac{\pi}{2}$ from the left and decreases without bound as x approaches $\dfrac{\pi}{2}$ from the right.

23. (a) $f(1)$ exists. The black dot at $(1, 2)$ indicates that $f(1) = 2$.

(b) $\lim_{x \to 1} f(x)$ does not exist. As x approaches 1 from the left, $f(x)$ approaches 3.5, whereas as x approaches 1 from the right, $f(x)$ approaches 1.

(c) $f(4)$ does not exist. The hollow circle at $(4, 2)$ indicates that f is not defined at 4.

(d) $\lim_{x \to 4} f(x)$ exists. As x approaches 4, $f(x)$ approaches 2: $\lim_{x \to 4} f(x) = 2$.

24. (a) $f(-2)$ does not exist. The vertical dotted line indicates that f is not defined at -2.

(b) $\lim_{x \to -2} f(x)$ does not exist. As x approaches -2, the values of $f(x)$ do not approach a specific number.

(c) $f(0)$ exists. The black dot at $(0, 4)$ indicates that $f(0) = 4$.

(d) $\lim_{x \to 0} f(x)$ does not exist. As x approaches 0 from the left, $f(x)$ approaches $\frac{1}{2}$, whereas as x approaches 0 from the right, $f(x)$ approaches 4.

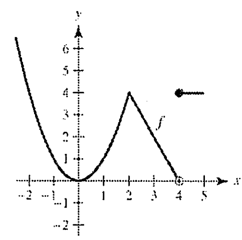
(e) $f(2)$ does not exist. The hollow circle at $\left(2, \frac{1}{2}\right)$ indicates that $f(2)$ is not defined.

(f) $\lim_{x \to 2} f(x)$ exists. As x approaches 2, $f(x)$ approaches $\frac{1}{2}$: $\lim_{x \to 2} f(x) = \frac{1}{2}$.

(g) $f(4)$ exists. The black dot at $(4, 2)$ indicates that $f(4) = 2$.

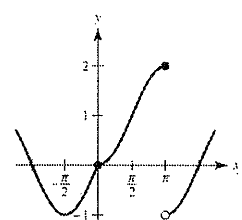
(h) $\lim_{x \to 4} f(x)$ does not exist. As x approaches 4, the values of $f(x)$ do not approach a specific number.

25.



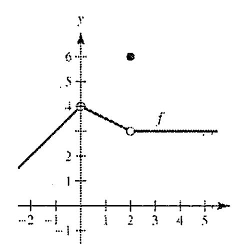
$\lim_{x \to c} f(x)$ exists for all values of $c \neq 4$.

26.

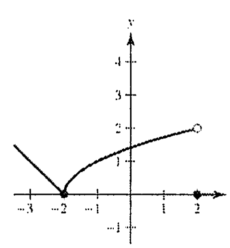


$\lim_{x \to c} f(x)$ exists for all values of $c \neq \pi$.

27. One possible answer is



28. One possible answer is



29. You need $|f(x) - 3| = |(x + 1) - 3| = |x - 2| < 0.4$. So, take $\delta = 0.4$. If $0 < |x - 2| < 0.4$, then $|x - 2| = |(x + 1) - 3| = |f(x) - 3| < 0.4$, as desired.

30. You need $|f(x) - 1| = \left|\frac{1}{x - 1} - 1\right| = \left|\frac{2 - x}{x - 1}\right| < 0.01$. Let $\delta = \frac{1}{101}$. If $0 < |x - 2| < \frac{1}{101}$, then

$$-\frac{1}{101} < x - 2 < \frac{1}{101} \Rightarrow 1 - \frac{1}{101} < x - 1 < 1 + \frac{1}{101}$$
$$\Rightarrow \frac{100}{101} < x - 1 < \frac{102}{101}$$
$$\Rightarrow |x - 1| > \frac{100}{101}$$

and you have

$$|f(x) - 1| = \left|\frac{1}{x - 1} - 1\right| = \left|\frac{2 - x}{x - 1}\right| < \frac{1/101}{100/101} = \frac{1}{100} = 0.01.$$

31. You need to find δ such that $0 < |x - 1| < \delta$ implies $|f(x) - 1| = \left|\frac{1}{x} - 1\right| < 0.1$. That is,

$$-0.1 < \frac{1}{x} - 1 < 0.1$$
$$1 - 0.1 < \frac{1}{x} < 1 + 0.1$$
$$\frac{9}{10} < \frac{1}{x} < \frac{11}{10}$$
$$\frac{10}{9} > x > \frac{10}{11}$$
$$\frac{10}{9} - 1 > x - 1 > \frac{10}{11} - 1$$
$$\frac{1}{9} > x - 1 > -\frac{1}{11}.$$

So take $\delta = \frac{1}{11}$. Then $0 < |x - 1| < \delta$ implies

$$-\frac{1}{11} < x - 1 < \frac{1}{11}$$
$$-\frac{1}{11} < x - 1 < \frac{1}{9}.$$

Using the first series of equivalent inequalities, you obtain

$$|f(x) - 1| = \left|\frac{1}{x} - 1\right| < 0.1.$$

32. You need to find δ such that $0 < |x - 2| < \delta$ implies $|f(x) - 3| = |x^2 - 1 - 3| = |x^2 - 4| < 0.2$. That is,

$$-0.2 < x^2 - 4 < 0.2$$
$$4 - 0.2 < x^2 < 4 + 0.2$$
$$3.8 < x^2 < 4.2$$
$$\sqrt{3.8} < x < \sqrt{4.2}$$
$$\sqrt{3.8} - 2 < x - 2 < \sqrt{4.2} - 2$$

So take $\delta = \sqrt{4.2} - 2 \approx 0.0494$.

Then $0 < |x - 2| < \delta$ implies

$$-\left(\sqrt{4.2} - 2\right) < x - 2 < \sqrt{4.2} - 2$$
$$\sqrt{3.8} - 2 < x - 2 < \sqrt{4.2} - 2.$$

Using the first series of equivalent inequalities, you obtain

$$|f(x) - 3| = |x^2 - 4| < 0.2.$$

33. $\lim_{x\to 2}(3x + 2) = 3(2) + 2 = 8 = L$

$$|(3x + 2) - 8| < 0.01$$
$$|3x - 6| < 0.01$$
$$3|x - 2| < 0.01$$
$$0 < |x - 2| < \tfrac{0.01}{3} \approx 0.0033 = \delta$$

So, if $0 < |x - 2| < \delta = \frac{0.01}{3}$, you have

$$3|x - 2| < 0.01$$
$$|3x - 6| < 0.01$$
$$|(3x + 2) - 8| < 0.01$$
$$|f(x) - L| < 0.01.$$

34. $\lim_{x\to 6}\left(6 - \frac{x}{3}\right) = 6 - \frac{6}{3} = 4 = L$

$$\left|\left(6 - \frac{x}{3}\right) - 4\right| < 0.01$$
$$\left|2 - \frac{x}{3}\right| < 0.01$$
$$\left|-\tfrac{1}{3}(x - 6)\right| < 0.01$$
$$|x - 6| < 0.03$$
$$0 < |x - 6| < 0.03 = \delta$$

So, if $0 < |x - 6| < \delta = 0.03$, you have

$$\left|-\tfrac{1}{3}(x - 6)\right| < 0.01$$
$$\left|2 - \frac{x}{3}\right| < 0.01$$
$$\left|\left(6 - \frac{x}{3}\right) - 4\right| < 0.01$$
$$|f(x) - L| < 0.01.$$

35. $\lim_{x\to 2}\left(x^2 - 3\right) = 2^2 - 3 = 1 = L$

$$\left|\left(x^2 - 3\right) - 1\right| < 0.01$$
$$\left|x^2 - 4\right| < 0.01$$
$$|(x + 2)(x - 2)| < 0.01$$
$$|x + 2||x - 2| < 0.01$$
$$|x - 2| < \frac{0.01}{|x + 2|}$$

If you assume $1 < x < 3$, then $\delta \approx 0.01/5 = 0.002$.

So, if $0 < |x - 2| < \delta \approx 0.002$, you have

$$|x - 2| < 0.002 = \frac{1}{5}(0.01) < \frac{1}{|x + 2|}(0.01)$$
$$|x + 2||x - 2| < 0.01$$
$$\left|x^2 - 4\right| < 0.01$$
$$\left|\left(x^2 - 3\right) - 1\right| < 0.01$$
$$|f(x) - L| < 0.01.$$

36. $\lim_{x\to 4}\left(x^2 + 6\right) = 4^2 + 6 = 22 = L$

$$\left|\left(x^2 + 6\right) - 22\right| < 0.01$$
$$\left|x^2 - 16\right| < 0.01$$
$$|(x + 4)(x - 4)| < 0.01$$
$$|x - 4| < \frac{0.01}{|x + 4|}$$

If you assume $3 < x < 5$, then $\delta = \frac{0.01}{9} \approx 0.00111$.

So, if $0 < |x - 4| < \delta \approx \frac{0.01}{9}$, you have

$$|x - 4| < \frac{0.01}{9} < \frac{0.01}{|x + 4|}$$
$$|(x + 4)(x - 4)| < 0.01$$
$$\left|x^2 - 16\right| < 0.01$$
$$\left|\left(x^2 + 6\right) - 22\right| < 0.01$$
$$|f(x) - L| < 0.01.$$

37. $\lim_{x\to 4}(x + 2) = 4 + 2 = 6$

Given $\varepsilon > 0$:

$$|(x + 2) - 6| < \varepsilon$$
$$|x - 4| < \varepsilon = \delta$$

So, let $\delta = \varepsilon$. So, if $0 < |x - 4| < \delta = \varepsilon$, you have

$$|x - 4| < \varepsilon$$
$$|(x + 2) - 6| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

38. $\lim_{x\to -2}(4x + 5) = 4(-2) + 5 = -3$

Given $\varepsilon > 0$:

$$|(4x + 5) - (-3)| < \varepsilon$$
$$|4x + 8| < \varepsilon$$
$$4|x + 2| < \varepsilon$$
$$|x + 2| < \frac{\varepsilon}{4} = \delta$$

So, let $\delta = \frac{\varepsilon}{4}$.

So, if $0 < |x + 2| < \delta = \frac{\varepsilon}{4}$, you have

$$|x + 2| < \frac{\varepsilon}{4}$$
$$|4x + 8| < \varepsilon$$
$$|(4x + 5) - (-3)| < \varepsilon$$
$$|f(x) - L| < \varepsilon.$$

39. $\lim_{x\to -4}\left(\frac{1}{2}x - 1\right) = \frac{1}{2}(-4) - 1 = -3$

Given $\varepsilon > 0$:

$$\left|\left(\tfrac{1}{2}x - 1\right) - (-3)\right| < \varepsilon$$
$$\left|\tfrac{1}{2}x + 2\right| < \varepsilon$$
$$\tfrac{1}{2}\left|x - (-4)\right| < \varepsilon$$
$$\left|x - (-4)\right| < 2\varepsilon$$

So, let $\delta = 2\varepsilon$.

So, if $0 < \left|x - (-4)\right| < \delta = 2\varepsilon$, you have

$$\left|x - (-4)\right| < 2\varepsilon$$
$$\left|\tfrac{1}{2}x + 2\right| < \varepsilon$$
$$\left|\left(\tfrac{1}{2}x - 1\right) + 3\right| < \varepsilon$$
$$\left|f(x) - L\right| < \varepsilon.$$

40. $\lim_{x\to 3}\left(\frac{3}{4}x + 1\right) = \frac{3}{4}(3) + 1 = \frac{13}{4}$

Given $\varepsilon > 0$:

$$\left|\left(\tfrac{3}{4}x + 1\right) - \tfrac{13}{4}\right| < \varepsilon$$
$$\left|\tfrac{3}{4}x - \tfrac{9}{4}\right| < \varepsilon$$
$$\tfrac{3}{4}\left|x - 3\right| < \varepsilon$$
$$\left|x - 3\right| < \tfrac{4}{3}\varepsilon$$

So, let $\delta = \frac{4}{3}\varepsilon$.

So, if $0 < \left|x - 3\right| < \delta = \frac{4}{3}\varepsilon$, you have

$$\left|x - 3\right| < \tfrac{4}{3}\varepsilon$$
$$\tfrac{3}{4}\left|x - 3\right| < \varepsilon$$
$$\left|\tfrac{3}{4}x - \tfrac{9}{4}\right| < \varepsilon$$
$$\left|\left(\tfrac{3}{4}x + 1\right) - \tfrac{13}{4}\right| < \varepsilon$$
$$\left|f(x) - L\right| < \varepsilon.$$

41. $\lim_{x\to 6} 3 = 3$

Given $\varepsilon > 0$:

$$\left|3 - 3\right| < \varepsilon$$
$$0 < \varepsilon$$

So, any $\delta > 0$ will work.

So, for any $\delta > 0$, you have

$$\left|3 - 3\right| < \varepsilon$$
$$\left|f(x) - L\right| < \varepsilon.$$

42. $\lim_{x\to 2}(-1) = -1$

Given $\varepsilon > 0$: $\left|-1 - (-1)\right| < \varepsilon$

$$0 < \varepsilon$$

So, any $\delta > 0$ will work.

So, for any $\delta > 0$, you have

$$\left|(-1) - (-1)\right| < \varepsilon$$
$$\left|f(x) - L\right| < \varepsilon.$$

43. $\lim_{x\to 0}\sqrt[3]{x} = 0$

Given $\varepsilon > 0$: $\left|\sqrt[3]{x} - 0\right| < \varepsilon$

$$\left|\sqrt[3]{x}\right| < \varepsilon$$
$$\left|x\right| < \varepsilon^3 = \delta$$

So, let $\delta = \varepsilon^3$.

So, for $0\left|x - 0\right|\delta = \varepsilon^3$, you have

$$\left|x\right| < \varepsilon^3$$
$$\left|\sqrt[3]{x}\right| < \varepsilon$$
$$\left|\sqrt[3]{x} - 0\right| < \varepsilon$$
$$\left|f(x) - L\right| < \varepsilon.$$

44. $\lim_{x\to 4}\sqrt{x} = \sqrt{4} = 2$

Given $\varepsilon > 0$: $\left|\sqrt{x} - 2\right| < \varepsilon$

$$\left|\sqrt{x} - 2\right|\left|\sqrt{x} + 2\right| < \varepsilon\left|\sqrt{x} + 2\right|$$
$$\left|x - 4\right| < \varepsilon\left|\sqrt{x} + 2\right|$$

Assuming $1 < x < 9$, you can choose $\delta = 3\varepsilon$. Then,

$$0 < \left|x - 4\right| < \delta = 3\varepsilon \Rightarrow \left|x - 4\right| < \varepsilon\left|\sqrt{x} + 2\right|$$
$$\Rightarrow \left|\sqrt{x} - 2\right| < \varepsilon.$$

45. $\lim_{x \to -5} |x - 5| = |(-5) - 5| = |-10| = 10$

Given $\varepsilon > 0$: $\big||x - 5| - 10\big| < \varepsilon$

$$\big|-(x - 5) - 10\big| < \varepsilon \qquad (x - 5 < 0)$$

$$|-x - 5| < \varepsilon$$

$$|x - (-5)| < \varepsilon$$

So, let $\delta = \varepsilon$.

So for $|x - (-5)| < \delta = \varepsilon$, you have

$$|-(x + 5)| < \varepsilon$$

$$|-(x - 5) - 10| < \varepsilon$$

$$\big||x - 5| - 10\big| < \varepsilon \qquad \text{(because } x - 5 < 0\text{)}$$

$$|f(x) - L| < \varepsilon.$$

46. $\lim_{x \to 3} |x - 3| = |3 - 3| = 0$

Given $\varepsilon > 0$: $\big||x - 3| - 0\big| < \varepsilon$

$$|x - 3| < \varepsilon$$

So, let $\delta = \varepsilon$.

So, for $0 < |x - 3| < \delta = \varepsilon$, you have

$$|x - 3| < \varepsilon$$

$$\big||x - 3| - 0\big| < \varepsilon$$

$$|f(x) - L| < \varepsilon.$$

47. $\lim_{x \to 1} (x^2 + 1) = 1^2 + 1 = 2$

Given $\varepsilon > 0$:

$$\big|(x^2 + 1) - 2\big| < \varepsilon$$

$$|x^2 - 1| < \varepsilon$$

$$|(x + 1)(x - 1)| < \varepsilon$$

$$|x - 1| < \frac{\varepsilon}{|x + 1|}$$

If you assume $0 < x < 2$, then $\delta = \varepsilon/3$.

So for $0 < |x - 1| < \delta = \dfrac{\varepsilon}{3}$, you have

$$|x - 1| < \frac{1}{3}\varepsilon < \frac{1}{|x + 1|}\varepsilon$$

$$|x^2 - 1| < \varepsilon$$

$$\big|(x^2 + 1) - 2\big| < \varepsilon$$

$$|f(x) - 2| < \varepsilon.$$

48. $\lim_{x \to -4} (x^2 + 4x) = (-4)^2 + 4(-4) = 0$

Given $\varepsilon > 0$:

$$\big|(x^2 + 4x) - 0\big| < \varepsilon$$

$$|x(x + 4)| < \varepsilon$$

$$|x + 4| < \frac{\varepsilon}{|x|}$$

If you assume $-5 < x < -3$, then $\delta = \dfrac{\varepsilon}{5}$.

So for $0 < |x - (-4)| < \delta = \dfrac{\varepsilon}{5}$, you have

$$|x + 4| < \frac{\varepsilon}{5} < \frac{1}{|x|}\varepsilon$$

$$|x(x + 4)| < \varepsilon$$

$$\big|(x^2 + 4x) - 0\big| < \varepsilon$$

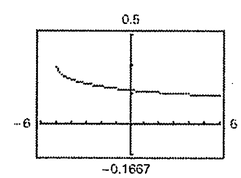
$$|f(x) - L| < \varepsilon.$$

49. $\lim_{x \to \pi} f(x) = \lim_{x \to \pi} 4 = 4$

50. $\lim_{x \to \pi} f(x) = \lim_{x \to \pi} x = \pi$

51. $f(x) = \dfrac{\sqrt{x + 5} - 3}{x - 4}$

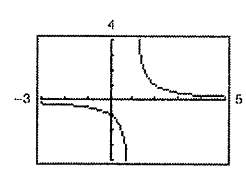
$$\lim_{x \to 4} f(x) = \frac{1}{6}$$



The domain is $[-5, 4) \cup (4, \infty)$. The graphing utility does not show the hole at $\left(4, \dfrac{1}{6}\right)$.

52. $f(x) = \dfrac{x - 3}{x^2 - 4x + 3}$

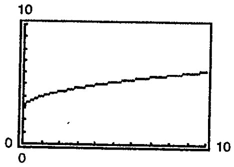
$$\lim_{x \to 3} f(x) = \frac{1}{2}$$



The domain is all $x \neq 1, 3$. The graphing utility does not show the hole at $\left(3, \dfrac{1}{2}\right)$.

53. $f(x) = \dfrac{x - 9}{\sqrt{x} - 3}$

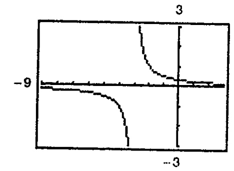
$\lim_{x \to 9} f(x) = 6$



The domain is all $x \geq 0$ except $x = 9$. The graphing utility does not show the hole at $(9, 6)$.

54. $f(x) = \dfrac{x - 3}{x^2 - 9}$

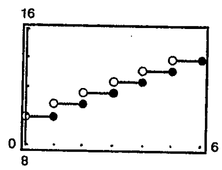
$\lim_{x \to 3} f(x) = \dfrac{1}{6}$



The domain is all $x \neq \pm 3$. The graphing utility does not show the hole at $\left(3, \dfrac{1}{6}\right)$.

55. $C(t) = 9.99 - 0.79[[-(t - 1)]]$

(a)



(b)

t	3	3.3	3.4	3.5	3.6	3.7	4
C	11.57	12.36	12.36	12.36	12.36	12.36	12.36

$\lim_{t \to 3.5} C(t) = 12.36$

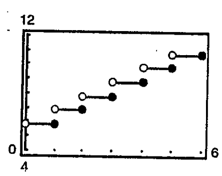
(c)

t	2	2.5	2.9	3	3.1	3.5	4
C	10.78	11.57	11.57	11.57	12.36	12.36	12.36

The $\lim_{t \to 3} C(t)$ does not exist because the values of C approach different values as t approaches 3 from both sides.

56. $C(t) = 5.79 - 0.99[[-(t - 1)]]$

(a)



(b)

t	3	3.3	3.4	3.5	3.6	3.7	4
C	7.77	8.76	8.76	8.76	8.76	8.76	8.76

$\lim_{t \to 3.5} C(t) = 8.76$

(c)

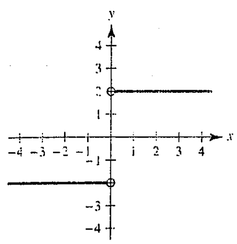
t	2	2.5	2.9	3	3.1	3.5	4
C	6.78	7.77	7.77	7.77	8.76	8.76	8.76

The limit $\lim_{t \to 3} C(t)$ does not exist because the values of C approach different values as t approaches 3 from both sides.

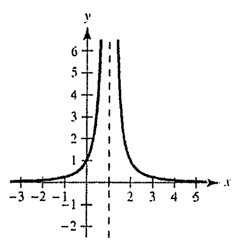
57. $\lim_{x\to 8} f(x) = 25$ means that the values of f approach 25 as x gets closer and closer to 8.

58. In the definition of $\lim_{x\to c} f(x)$, f must be defined on both sides of c, but does not have to be defined at c itself. The value of f at c has no bearing on the limit as x approaches c.

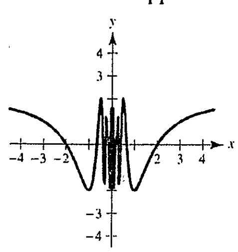
59. (i) The values of f approach different numbers as x approaches c from different sides of c:



(ii) The values of f increase without bound as x approaches c:



(iii) The values of f oscillate between two fixed numbers as x approaches c:



60. (a) No. The fact that $f(2) = 4$ has no bearing on the existence of the limit of $f(x)$ as x approaches 2.

(b) No. The fact that $\lim_{x\to 2} f(x) = 4$ has no bearing on the value of f at 2.

61. (a) $C = 2\pi r$

$$r = \frac{C}{2\pi} = \frac{6}{2\pi} = \frac{3}{\pi} \approx 0.9549 \text{ cm}$$

(b) When $C = 5.5$: $r = \dfrac{5.5}{2\pi} \approx 0.87535$ cm

When $C = 6.5$: $r = \dfrac{6.5}{2\pi} \approx 1.03451$ cm

So $0.87535 < r < 1.03451$.

(c) $\lim_{x\to 3/\pi} (2\pi r) = 6$; $\varepsilon = 0.5$; $\delta \approx 0.0796$

62. $V = \frac{4}{3}\pi r^3$, $V = 2.48$

(a) $2.48 = \frac{4}{3}\pi r^3$

$$r^3 = \frac{1.86}{\pi}$$

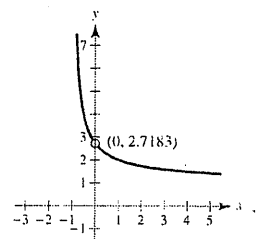
$$r \approx 0.8397 \text{ in.}$$

(b)
$$2.45 \le V \le 2.51$$
$$2.45 \le \frac{4}{3}\pi r^3 \le 2.51$$
$$0.5849 \le r^3 \le 0.5992$$
$$0.8363 \le r \le 0.8431$$

(c) For $\varepsilon = 2.51 - 2.48 = 0.03$, $\delta \approx 0.003$

63. $f(x) = (1 + x)^{1/x}$

$$\lim_{x\to 0}(1 + x)^{1/x} = e \approx 2.71828$$



x	$f(x)$
−0.1	2.867972
−0.01	2.731999
−0.001	2.719642
−0.0001	2.718418
−0.00001	2.718295
−0.000001	2.718283

x	$f(x)$
0.1	2.593742
0.01	2.704814
0.001	2.716942
0.0001	2.718146
0.00001	2.718268
0.000001	2.718280

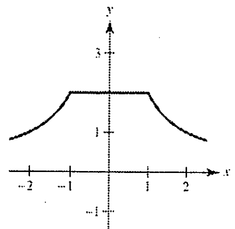
64. $f(x) = \dfrac{|x+1| - |x-1|}{x}$

x	-1	-0.5	-0.1	0	0.1	0.5	1.0
$f(x)$	2	2	2	Undef.	2	2	2

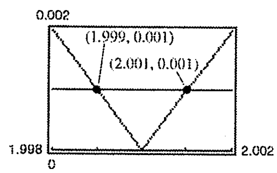
$\lim_{x\to 0} f(x) = 2$

Note that for

$-1 < x < 1, x \neq 0, f(x) = \dfrac{(x+1)+(x-1)}{x} = 2.$



65.



Using the zoom and trace feature, $\delta = 0.001$. So $(2-\delta, 2+\delta) = (1.999, 2.001)$.

Note: $\dfrac{x^2-4}{x-2} = x + 2$ for $x \neq 2$.

66. (a) $\lim_{x\to c} f(x)$ exists for all $c \neq -3$.

(b) $\lim_{x\to c} f(x)$ exists for all $c \neq -2, 0$.

67. False. The existence or nonexistence of $f(x)$ at $x = c$ has no bearing on the existence of the limit of $f(x)$ as $x \to c$.

68. True

69. False. Let

$$f(x) = \begin{cases} x-4, & x \neq 2 \\ 0, & x = 2 \end{cases}$$

$f(2) = 0$

$\lim_{x\to 2} f(x) = \lim_{x\to 2}(x-4) = 2 \neq 0$

70. False. Let

$$f(x) = \begin{cases} x-4, & x \neq 2 \\ 0, & x = 2 \end{cases}$$

$\lim_{x\to 2} f(x) = \lim_{x\to 2}(x-4) = 2$ and $f(2) = 0 \neq 2$

71. $f(x) = \sqrt{x}$

$\lim_{x\to 0.25} \sqrt{x} = 0.5$ is true.

As x approaches $0.25 = \frac{1}{4}$ from either side, $f(x) = \sqrt{x}$ approaches $\frac{1}{2} = 0.5$.

72. $f(x) = \sqrt{x}$

$\lim_{x\to 0} \sqrt{x} = 0$ is false.

$f(x) = \sqrt{x}$ is not defined on an open interval containing 0 because the domain of f is $x \geq 0$.

73. Using a graphing utility, you see that

$\lim_{x\to 0} \dfrac{\sin x}{x} = 1$

$\lim_{x\to 0} \dfrac{\sin 2x}{x} = 2$, etc.

So, $\lim_{x\to 0} \dfrac{\sin nx}{x} = n$.

74. Using a graphing utility, you see that

$\lim_{x\to 0} \dfrac{\tan x}{x} = 1$

$\lim_{x\to 0} \dfrac{\tan 2x}{x} = 2$, etc.

So, $\lim_{x\to 0} \dfrac{\tan(nx)}{x} = n$.

75. If $\lim_{x\to c} f(x) = L_1$ and $\lim_{x\to c} f(x) = L_2$, then for every $\varepsilon > 0$, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that $|x - c| < \delta_1 \Rightarrow |f(x) - L_1| < \varepsilon$ and $|x - c| < \delta_2 \Rightarrow |f(x) - L_2| < \varepsilon$. Let δ equal the smaller of δ_1 and δ_2. Then for $|x - c| < \delta$, you have $|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \leq |L_1 - f(x)| + |f(x) - L_2| < \varepsilon + \varepsilon$. Therefore, $|L_1 - L_2| < 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that $L_1 = L_2$.

76. $f(x) = mx + b,\ m \neq 0$. Let $\varepsilon > 0$ be given. Take

$\delta = \dfrac{\varepsilon}{|m|}$.

If $0 < |x - c| < \delta = \dfrac{\varepsilon}{|m|}$, then

$$|m||x - c| < \varepsilon$$
$$|mx - mc| < \varepsilon$$
$$|(mx + b) - (mc + b)| < \varepsilon$$

which shows that $\lim_{x \to c} (mx + b) = mc + b$.

77. $\lim_{x \to c}[f(x) - L] = 0$ means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if

$0 < |x - c| < \delta$,

then

$|(f(x) - L) - 0| < \varepsilon$.

This means the same as $|f(x) - L| < \varepsilon$ when

$0 < |x - c| < \delta$.

So, $\lim_{x \to c} f(x) = L$.

78. (a) $(3x + 1)(3x - 1)x^2 + 0.01 = (9x^2 - 1)x^2 + \dfrac{1}{100}$

$$= 9x^4 - x^2 + \frac{1}{100}$$
$$= \frac{1}{100}(10x^2 - 1)(90x^2 - 1)$$

So, $(3x + 1)(3x - 1)x^2 + 0.01 > 0$ if

$10x^2 - 1 < 0$ and $90x^2 - 1 < 0$.

Let $(a, b) = \left(-\dfrac{1}{\sqrt{90}}, \dfrac{1}{\sqrt{90}}\right)$.

For all $x \neq 0$ in (a, b), the graph is positive.

You can verify this with a graphing utility.

(b) You are given $\lim_{x \to c} g(x) = L > 0$. Let

$\varepsilon = \dfrac{1}{2}L$. There exists $\delta > 0$ such that

$0 < |x - c| < \delta$ implies that

$|g(x) - L| < \varepsilon = \dfrac{L}{2}$. That is,

$$-\frac{L}{2} < g(x) - L < \frac{L}{2}$$
$$\frac{L}{2} < g(x) < \frac{3L}{2}$$

For x in the interval $(c - \delta, c + \delta)$, $x \neq c$, you have $g(x) > \dfrac{L}{2} > 0$, as desired.

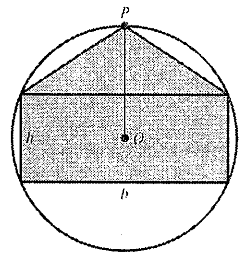
79. The radius OP has a length equal to the altitude z of the triangle plus $\dfrac{h}{2}$. So, $z = 1 - \dfrac{h}{2}$.

Area triangle $= \dfrac{1}{2}b\left(1 - \dfrac{h}{2}\right)$

Area rectangle $= bh$

Because these are equal, $\dfrac{1}{2}b\left(1 - \dfrac{h}{2}\right) = bh$

$$1 - \frac{h}{2} = 2h$$
$$\frac{5}{2}h = 1$$
$$h = \frac{2}{5}.$$



80. Consider a cross section of the cone, where EF is a diagonal of the inscribed cube. $AD = 3$, $BC = 2$.

Let x be the length of a side of the cube.

Then $EF = x\sqrt{2}$.

By similar triangles,

$$\frac{EF}{BC} = \frac{AG}{AD}$$
$$\frac{x\sqrt{2}}{2} = \frac{3 - x}{3}$$

Solving for x,

$$3\sqrt{2}x = 6 - 2x$$
$$(3\sqrt{2} + 2)x = 6$$
$$x = \frac{6}{3\sqrt{2} + 2} = \frac{9\sqrt{2} - 6}{7} \approx 0.96.$$