Summary

Two Basic Limits

- $\lim_{x\to c} A = A$, where A is a constant
- $\lim_{x\to c} x = c$, c a real number

Properties of Limits

If f and g are functions for which $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist, and k is a constant, then

- **Limit of a Sum or a Difference:** $\lim_{x\to c}[f(x) \pm g(x)] = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$
- **Limit of a Product:** $\lim_{x\to c}[f(x) \cdot g(x)] = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$
- **Limit of a Constant Times a Function:** $\lim_{x\to c}[kg(x)] = k \lim_{x\to c} g(x)$
- **Limit of a Power:** $\lim_{x\to c} [f(x)]^n = \left[\lim_{x\to c} f(x)\right]^n$ where $n \geq 2$ is an integer
- **Limit of a Root:** $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)}$ provided $f(x) > 0$ if $n \geq 2$ is even
- **Limit of $[f(x)]^{m/n}$:** $\lim_{x\to c}[f(x)]^{m/n} = \left[\lim_{x\to c} f(x)\right]^{m/n}$ provided $[f(x)]^{m/n}$ is defined for positive integers m and n
- **Limit of a Quotient:** $\lim_{x\to c} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ provided $\lim_{x\to c} g(x) \neq 0$
- **Limit of a Polynomial Function:** $\lim_{x\to c} P(x) = P(c)$
- **Limit of a Rational Function:** $\lim_{x\to c} R(x) = R(c)$ if c is in the domain of R

1.2 Assess Your Understanding

Concepts and Vocabulary

1. (a) $\lim_{x\to 4} (-3) =$ ________; (b) $\lim_{x\to 0} \pi =$ ________
2. If $\lim_{x\to c} f(x) = 3$, then $\lim_{x\to c}[f(x)]^5 =$ ________.
3. If $\lim_{x\to c} f(x) = 64$, then $\lim_{x\to c} \sqrt[3]{f(x)} =$ ________.
4. (a) $\lim_{x\to -1} x =$ ________; (b) $\lim_{x\to e} x =$ ________
5. (a) $\lim_{x\to 0} (x-2) =$ ____; (b) $\lim_{x\to 1/2} (3+x) =$ ________
6. (a) $\lim_{x\to 2} (-3x) =$ ________; (b) $\lim_{x\to 0} (3x) =$ ________
7. *True or False* If p is a polynomial function, then $\lim_{x\to 5} p(x) = p(5)$.
8. If the domain of a rational function R is $\{x \mid x \neq 0\}$, then $\lim_{x\to 2} R(x) = R($____$)$.
9. *True or False* Properties of limits cannot be used for one-sided limits.
10. *True or False* If $f(x) = \frac{(x+1)(x+2)}{x+1}$ and $g(x) = x+2$, then $\lim_{x\to -1} f(x) = \lim_{x\to -1} g(x)$.

Skill Building

In Problems 11–44, find each limit using algebraic properties of limits.

11. $\lim_{x\to 3} [2(x+4)]$
12. $\lim_{x\to -2} [3(x+1)]$
13. PAGE 92 $\lim_{x\to -2} [x(3x-1)(x+2)]$
14. $\lim_{x\to -1} [x(x-1)(x+10)]$
15. PAGE 93 $\lim_{t\to 1} (3t-2)^3$
16. $\lim_{x\to 0} (-3x+1)^2$
17. $\lim_{x\to 4} (3\sqrt{x})$
18. $\lim_{x\to 8} \left(\frac{1}{4}\sqrt[3]{x}\right)$
19. PAGE 94 $\lim_{x\to 3} \sqrt{5x-4}$
20. $\lim_{t\to 2} \sqrt{3t+4}$
21. $\lim_{t\to 2} [t\sqrt{(5t+3)(t+4)}]$
22. $\lim_{t\to -1} [t\sqrt[3]{(t+1)(2t-1)}]$
23. PAGE 94 $\lim_{x\to 3} (\sqrt{x}+x+4)^{1/2}$
24. $\lim_{t\to 2} (t\sqrt{2t}+4)^{1/3}$
25. $\lim_{t\to -1} [4t(t+1)]^{2/3}$
26. $\lim_{x\to 0} (x^2-2x)^{3/5}$
27. $\lim_{t\to 1} (3t^2-2t+4)$
28. $\lim_{x\to 0} (-3x^4+2x+1)$
29. PAGE 95 $\lim_{x\to \frac{1}{2}} (2x^4-8x^3+4x-5)$
30. $\lim_{x\to -\frac{1}{3}} (27x^3+9x+1)$
31. PAGE 96 $\lim_{x\to 4} \frac{x^2+4}{\sqrt{x}}$
32. $\lim_{x\to 3} \frac{x^2+5}{\sqrt{3x}}$
33. PAGE 96 $\lim_{x\to -2} \frac{2x^3+5x}{3x-2}$
34. $\lim_{x\to 1} \frac{2x^4-1}{3x^3+2}$
35. PAGE 97 $\lim_{x\to 2} \frac{x^2-4}{x-2}$
36. $\lim_{x\to -2} \frac{x+2}{x^2-4}$
37. $\lim_{x\to -1} \frac{x^3-x}{x+1}$
38. $\lim_{x\to -1} \frac{x^3+x^2}{x^2-1}$
39. $\lim_{x\to -8} \left(\frac{2x}{x+8}+\frac{16}{x+8}\right)$
40. $\lim_{x\to 2} \left(\frac{3x}{x-2}-\frac{6}{x-2}\right)$
41. PAGE 97 $\lim_{x\to 2} \frac{\sqrt{x}-\sqrt{2}}{x-2}$
42. $\lim_{x\to 3} \frac{\sqrt{x}-\sqrt{3}}{x-3}$
43. $\lim_{x\to 4} \frac{\sqrt{x+5}-3}{(x-4)(x+1)}$
44. $\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x(x-3)}$

In Problems 45–50, find each one-sided limit using properties of limits.

45. $\lim\limits_{x\to 3^-} (x^2-4)$

46. $\lim\limits_{x\to 2^+} (3x^2+x)$

47. $\lim\limits_{x\to 3^-} \dfrac{x^2-9}{x-3}$

48. $\lim\limits_{x\to 3^+} \dfrac{x^2-9}{x-3}$

49. $\lim\limits_{x\to 3^-} (\sqrt{9-x^2}+x)^2$

50. $\lim\limits_{x\to 2^+} (2\sqrt{x^2-4}+3x)$

In Problems 51–58, use the information below to find each limit.

$$\lim_{x\to c} f(x)=5 \qquad \lim_{x\to c} g(x)=2 \qquad \lim_{x\to c} h(x)=0$$

51. $\lim\limits_{x\to c} [f(x)-3g(x)]$

52. $\lim\limits_{x\to c} [5f(x)]$

53. $\lim\limits_{x\to c} [g(x)]^3$

54. $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)-h(x)}$

55. $\lim\limits_{x\to c} \dfrac{h(x)}{g(x)}$

56. $\lim\limits_{x\to c} [4f(x)\cdot g(x)]$

57. $\lim\limits_{x\to c} \left[\dfrac{1}{g(x)}\right]^2$

58. $\lim\limits_{x\to c} \sqrt[3]{5g(x)-3}$

In Problems 59 and 60, use the graphs of the functions and properties of limits to find each limit, if it exists. If the limit does not exist, write, "the limit does not exist," and explain why.

59. (a) $\lim\limits_{x\to 4} [f(x)+g(x)]$

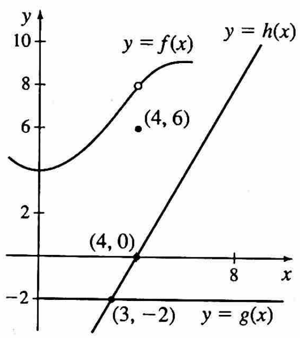
(b) $\lim\limits_{x\to 4} \{f(x)[g(x)-h(x)]\}$

(c) $\lim\limits_{x\to 4} [f(x)\cdot g(x)]$

(d) $\lim\limits_{x\to 4} [2h(x)]$

(e) $\lim\limits_{x\to 4} \dfrac{g(x)}{f(x)}$

(f) $\lim\limits_{x\to 4} \dfrac{h(x)}{f(x)}$



60. (a) $\lim\limits_{x\to 3} \{2[f(x)+h(x)]\}$

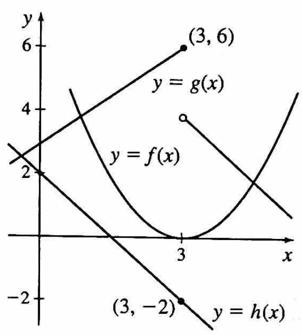
(b) $\lim\limits_{x\to 3^-} [g(x)+h(x)]$

(c) $\lim\limits_{x\to 3} \sqrt[3]{h(x)}$

(d) $\lim\limits_{x\to 3} \dfrac{f(x)}{h(x)}$

(e) $\lim\limits_{x\to 3} [h(x)]^3$

(f) $\lim\limits_{x\to 3} [f(x)-2h(x)]^{3/2}$



In Problems 61–66, for each function f, find the limit as x approaches c of the average rate of change of f from c to x. That is, find

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$$

61. $f(x)=3x^2$, $c=1$

62. $f(x)=8x^3$, $c=2$

PAGE 98 **63.** $f(x)=-2x^2+4$, $c=1$

64. $f(x)=20-0.8x^2$, $c=3$

65. $f(x)=\sqrt{x}$, $c=1$

66. $f(x)=\sqrt{2x}$, $c=5$

In Problems 67–72, find the limit of the difference quotient for each function f. That is, find $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$.

67. $f(x)=4x-3$

68. $f(x)=3x+5$

PAGE 98 **69.** $f(x)=3x^2+4x+1$

70. $f(x)=2x^2+x$

71. $f(x)=\dfrac{2}{x}$

72. $f(x)=\dfrac{3}{x^2}$

In Problems 73–80, find $\lim\limits_{x\to c^-} f(x)$ *and* $\lim\limits_{x\to c^+} f(x)$ *for the given number c. Based on the results, determine whether* $\lim\limits_{x\to c} f(x)$ *exists.*

PAGE 92 **73.** $f(x)=\begin{cases} 2x-3 & \text{if } x\le 1 \\ 3-x & \text{if } x>1 \end{cases}$ at $c=1$

74. $f(x)=\begin{cases} 5x+2 & \text{if } x<2 \\ 1+3x & \text{if } x\ge 2 \end{cases}$ at $c=2$

75. $f(x)=\begin{cases} 3x-1 & \text{if } x<1 \\ 4 & \text{if } x=1 \\ 2x & \text{if } x>1 \end{cases}$ at $c=1$

76. $f(x)=\begin{cases} 3x-1 & \text{if } x<1 \\ 2 & \text{if } x=1 \\ 2x & \text{if } x>1 \end{cases}$ at $c=1$

77. $f(x)=\begin{cases} x-1 & \text{if } x<1 \\ \sqrt{x-1} & \text{if } x>1 \end{cases}$ at $c=1$

78. $f(x)=\begin{cases} \sqrt{9-x^2} & \text{if } 0<x<3 \\ \sqrt{x^2-9} & \text{if } x>3 \end{cases}$ at $c=3$

79. $f(x)=\begin{cases} \dfrac{x^2-9}{x-3} & \text{if } x\ne 3 \\ 6 & \text{if } x=3 \end{cases}$ at $c=3$

80. $f(x)=\begin{cases} \dfrac{x-2}{x^2-4} & \text{if } x\ne 2 \\ 1 & \text{if } x=2 \end{cases}$ at $c=2$

Applications and Extensions

Heaviside Functions *In Problems 81 and 82, find the limit, if it exists, of the given Heaviside function at c.*

PAGE 93 **81.** $u_1(t)=\begin{cases} 0 & \text{if } t<1 \\ 1 & \text{if } t\ge 1 \end{cases}$ at $c=1$

82. $u_3(t)=\begin{cases} 0 & \text{if } t<3 \\ 1 & \text{if } t\ge 3 \end{cases}$ at $c=3$

In Problems 83–92, find each limit.

83. $\lim\limits_{h\to 0} \dfrac{(x+h)^2-x^2}{h}$

84. $\lim\limits_{h\to 0} \dfrac{\sqrt{x+h}-\sqrt{x}}{h}$

85. $\lim\limits_{h\to 0} \dfrac{\dfrac{1}{x+h}-\dfrac{1}{x}}{h}$

86. $\lim\limits_{h\to 0} \dfrac{\dfrac{1}{(x+h)^3}-\dfrac{1}{x^3}}{h}$

87. $\lim_{x\to 0}\left[\frac{1}{x}\left(\frac{1}{4+x}-\frac{1}{4}\right)\right]$

88. $\lim_{x\to -1}\left[\frac{2}{x+1}\left(\frac{1}{3}-\frac{1}{x+4}\right)\right]$

89. $\lim_{x\to 7}\frac{x-7}{\sqrt{x+2}-3}$

90. $\lim_{x\to 2}\frac{x-2}{\sqrt{x+2}-2}$

91. $\lim_{x\to 1}\frac{x^3-3x^2+3x-1}{x^2-2x+1}$

92. $\lim_{x\to -3}\frac{x^3+7x^2+15x+9}{x^2+6x+9}$

93. **Cost of Water** The Jericho Water District determines quarterly water costs, in dollars, using the following rate schedule:

Water used (in thousands of gallons)	Cost
$0 \le x \le 10$	\$9.00
$10 < x \le 30$	\$9.00 + 0.95 for each thousand gallons in excess of 10,000 gallons
$30 < x \le 100$	\$28.00 + 1.65 for each thousand gallons in excess of 30,000 gallons
$x > 100$	\$143.50 + 2.20 for each thousand gallons in excess of 100,000 gallons

Source: Jericho Water District, Syosset, NY.

(a) Find a function C that models the quarterly cost, in dollars, of using x thousand gallons of water.

(b) What is the domain of the function C?

(c) Find each of the following limits. If the limit does not exist, explain why.

$\lim_{x\to 5} C(x)$ $\quad\lim_{x\to 10} C(x)$ $\quad\lim_{x\to 30} C(x)$ $\quad\lim_{x\to 100} C(x)$

(d) What is $\lim_{x\to 0^+} C(x)$?

(e) Graph the function C.

94. **Cost of Electricity** In January 2019, Florida Power and Light charged customers living in single-family residences for their electric usage according to the following table.

Monthly customer charge	for electricity:
\$7.98	per household, plus
\$0.08692	for each kWH used less than or equal to 1000 kWH, plus
\$0.10708	for each kWH used in excess of 1000 kWH

Source: Florida Power and Light, Miami, FL.

(a) Find a function C that models the monthly cost, in dollars, of using x kWH of electricity.

(b) What is the domain of the function C?

(c) Find $\lim_{x\to 1000} C(x)$, if it exists. If the limit does not exist, explain why.

(d) What is $\lim_{x\to 0^+} C(x)$?

(e) Graph the function C.

95. **Low-Temperature Physics** In thermodynamics, the average molecular kinetic energy (energy of motion) of a gas having molecules of mass m is directly proportional to its temperature T on the absolute (or Kelvin) scale. This can be expressed as $\frac{1}{2}mv^2=\frac{3}{2}kT$, where $v=v(T)$ is the speed of a typical molecule at time t, and k is a constant, known as the **Boltzmann constant.**

(a) What limit does the molecular speed v approach as the gas temperature T approaches absolute zero (0 K or −273 °C or −469 °F)?

(b) What does this limit suggest about the behavior of a gas as its temperature approaches absolute zero?

96. For the function $f(x)=\begin{cases}3x+5 & \text{if } x\le 2\\ 13-x & \text{if } x>2\end{cases}$, find

(a) $\lim_{h\to 0^-}\frac{f(2+h)-f(2)}{h}$

(b) $\lim_{h\to 0^+}\frac{f(2+h)-f(2)}{h}$

(c) Does $\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$ exist?

97. Use the fact that $|x|=\begin{cases}x & \text{if } x\ge 0\\ -x & \text{if } x<0\end{cases}$ to show that $\lim_{x\to 0}|x|=0$.

98. Use the fact that $|x|=\sqrt{x^2}$ to show that $\lim_{x\to 0}|x|=0$.

99. Find functions f and g for which $\lim_{x\to c}[f(x)+g(x)]$ may exist even though $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ do not exist.

100. Find functions f and g for which $\lim_{x\to c}[f(x)g(x)]$ may exist even though $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ do not exist.

101. Find functions f and g for which $\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right]$ may exist even though $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ do not exist.

102. Find a function f for which $\lim_{x\to c}|f(x)|$ may exist even though $\lim_{x\to c} f(x)$ does not exist.

103. Prove that if g is a function for which $\lim_{x\to c} g(x)$ exists and if k is any real number, then $\lim_{x\to c}[kg(x)]$ exists and $\lim_{x\to c}[kg(x)]=k\lim_{x\to c} g(x)$.

104. Prove that if the number c is in the domain of a rational function $R(x)=\frac{p(x)}{q(x)}$, then $\lim_{x\to c} R(x)=R(c)$.

Challenge Problems

105. Find $\lim_{x\to a}\frac{x^n-a^n}{x-a}$, n a positive integer.

106. Find $\lim_{x\to -a}\frac{x^n+a^n}{x+a}$, n a positive integer.

107. Find $\lim_{x\to 1}\frac{x^m-1}{x^n-1}$, m, n positive integers.

108. Find $\lim_{x\to 0}\frac{\sqrt[3]{1+x}-1}{x}$.

109. Find $\lim_{x\to 0}\frac{\sqrt{(1+ax)(1+bx)}-1}{x}$.

110. Find $\lim_{x\to 0}\frac{\sqrt{(1+a_1x)(1+a_2x)\cdots(1+a_nx)}-1}{x}$.

111. Find $\lim_{h\to 0}\frac{f(h)-f(0)}{h}$ if $f(x)=x|x|$.

Preparing for the AP® Exam

AP® Practice Problems

PAGE 92 **1.** Consider the piecewise-defined function f given by

$$f(x) = \begin{cases} -x-2 & \text{if } x < -1 \\ x^2 & \text{if } -1 \le x < 2 \\ -4x+12 & \text{if } x \ge 2 \end{cases}$$

Investigate the limits below and decide which limit does NOT exist.

(A) $\lim_{x\to -1^+} f(x)$ (B) $\lim_{x\to 2^-} f(x)$

(C) $\lim_{x\to 2} f(x)$ (D) $\lim_{x\to -1} f(x)$

PAGE 97 **2.** $\lim_{t\to 5} \dfrac{(5-t)^2}{t-5} =$

(A) -5 (B) 0 (C) 1 (D) 5

PAGE 98 **3.** Find $\lim_{x\to 2} \dfrac{f(x)-f(2)}{x-2}$ for the function $f(x) = 3x^3 - 4$.

(A) 0 (B) 12 (C) 24 (D) 36

PAGE 97 **4.** $\lim_{x\to s} \dfrac{x-s}{\sqrt{x}-\sqrt{s}} =$

(A) $2s$ (B) $2\sqrt{s}$ (C) $\sqrt{2s}$ (D) s

PAGE 92 **5.** For $g(x) = \begin{cases} ax^2 - 5 \text{ if } x < 2 \\ ax + b \text{ if } x > 2 \end{cases}$

find values for a and b so that $\lim_{x\to 2} g(x) = 7$.

(A) $a=1, b=5$ (B) $a=2, b=3$

(C) $a=3, b=1$ (D) $a=6, b=-5$

PAGE 94 **6.** $\lim_{x\to 4^+} (5\sqrt{x^2-16} + 3x) =$

(A) -12 (B) 0 (C) 12 (D) The limit does not exist.

PAGE 94 **7.** If $\lim_{x\to 2} \sqrt{\dfrac{[f(x)]^2 - 8x + 3}{x+1}} = 9$ and $f(x) \ge 0$ for all x, find $\lim_{x\to 2} f(x)$.

(A) $\sqrt{22}$ (B) $2\sqrt{10}$ (C) 16 (D) 256

PAGE 94 **8.** $\lim_{x\to 3} [x^{-1/2}(5x-7)^{1/3}] =$

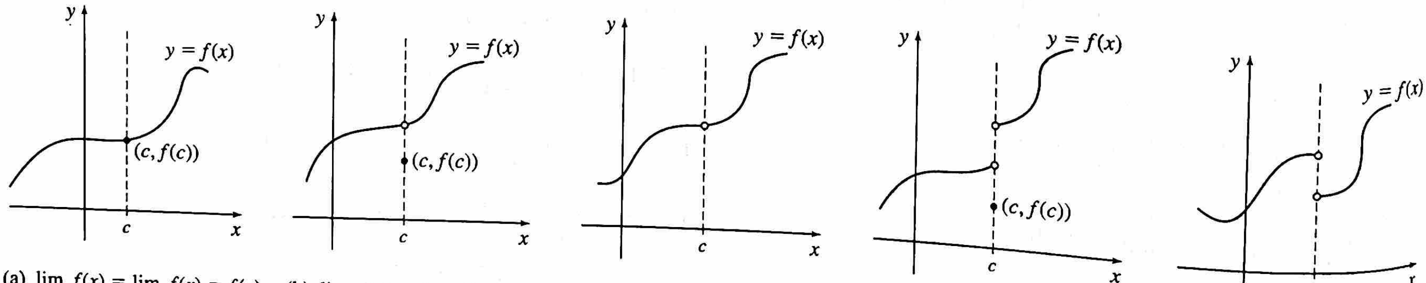
(A) $3^{-1/2}$ (B) $\dfrac{2}{3^{1/2}}$ (C) $\dfrac{8}{3^{1/2}}$ (D) $6^{-1/2}$

1.3 Continuity

OBJECTIVES ***When you finish this section, you should be able to:***

1 **Determine whether a function is continuous at a number (p. 103)**
2 **Determine intervals on which a function is continuous (p. 106)**
3 **Use properties of continuity (p. 108)**
4 **Use the Intermediate Value Theorem (p. 110)**

Sometimes $\lim_{x\to c} f(x)$ equals $f(c)$ and sometimes it does not. In fact, $f(c)$ may not even be defined and yet $\lim_{x\to c} f(x)$ may exist. In this section, we investigate the relationship between $\lim_{x\to c} f(x)$ and $f(c)$. Figure 21 shows some possibilities.



(a) $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = f(c)$

(b) $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) \neq f(c)$

(c) $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)$; $f(c)$ is not defined.

(d) $\lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x)$; $f(c)$ is defined.

(e) $\lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x)$; $f(c)$ is not defined.

Figure 21

Of these five graphs, the "nicest" one is Figure 21(a). There, $\lim_{x\to c} f(x)$ exists and is equal to $f(c)$. Functions that have this property are said to be *continuous at the number c*. This agrees with the intuitive notion that a function is continuous if its graph can be drawn without lifting the pencil. The functions in Figures 21(b)–(e) are