1.2 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Estimating a Limit Numerically In Exercises 1–6, complete the table and use the result to estimate the limit. Use a graphing utility to graph the function to confirm your result.

1. $\lim_{x\to 4} \dfrac{x-4}{x^2-3x-4}$

x	3.9	3.99	3.999	4	4.001	4.01	4.1
$f(x)$				?			

2. $\lim_{x\to 3} \dfrac{x-3}{x^2-9}$

x	2.9	2.99	2.999	3	3.001	3.01	3.1
$f(x)$				?			

3. $\lim_{x\to 0} \dfrac{\sqrt{x+1}-1}{x}$

x	−0.1	−0.01	−0.001	0	0.001	0.01	0.1
$f(x)$				?			

4. $\lim_{x\to 3} \dfrac{[1/(x+1)]-(1/4)}{x-3}$

x	2.9	2.99	2.999	3	3.001	3.01	3.1
$f(x)$				?			

5. $\lim_{x\to 0} \dfrac{\sin x}{x}$

x	−0.1	−0.01	−0.001	0	0.001	0.01	0.1
$f(x)$				?			

6. $\lim_{x\to 0} \dfrac{\cos x - 1}{x}$

x	−0.1	−0.01	−0.001	0	0.001	0.01	0.1
$f(x)$				?			

Estimating a Limit Numerically In Exercises 7–14, create a table of values for the function and use the result to estimate the limit. Use a graphing utility to graph the function to confirm your result.

7. $\lim_{x\to 1} \dfrac{x-2}{x^2+x-6}$

8. $\lim_{x\to -4} \dfrac{x+4}{x^2+9x+20}$

9. $\lim_{x\to 1} \dfrac{x^4-1}{x^6-1}$

10. $\lim_{x\to -3} \dfrac{x^3+27}{x+3}$

11. $\lim_{x\to -6} \dfrac{\sqrt{10-x}-4}{x+6}$

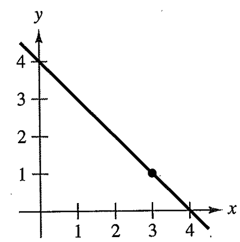
12. $\lim_{x\to 2} \dfrac{[x/(x+1)]-(2/3)}{x-2}$

13. $\lim_{x\to 0} \dfrac{\sin 2x}{x}$

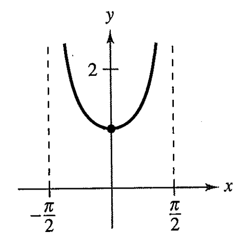
14. $\lim_{x\to 0} \dfrac{\tan x}{\tan 2x}$

Finding a Limit Graphically In Exercises 15–22, use the graph to find the limit (if it exists). If the limit does not exist, explain why.

15. $\lim_{x\to 3} (4-x)$

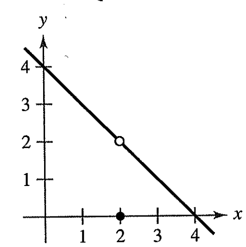


16. $\lim_{x\to 0} \sec x$



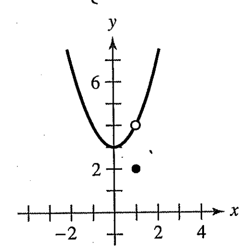
17. $\lim_{x\to 2} f(x)$

$$f(x) = \begin{cases} 4-x, & x \neq 2 \\ 0, & x = 2 \end{cases}$$

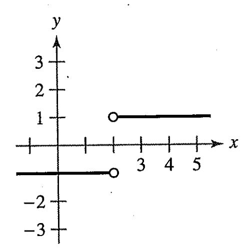


18. $\lim_{x\to 1} f(x)$

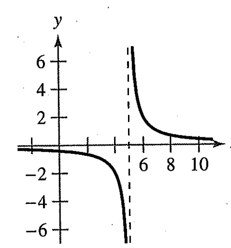
$$f(x) = \begin{cases} x^2+3, & x \neq 1 \\ 2, & x = 1 \end{cases}$$



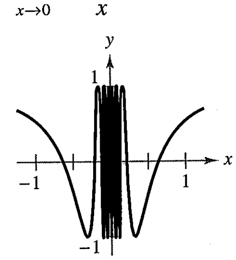
19. $\lim_{x\to 2} \dfrac{|x-2|}{x-2}$



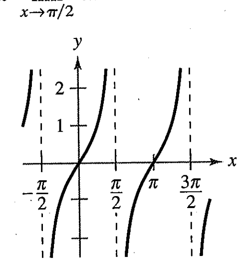
20. $\lim_{x\to 5} \dfrac{2}{x-5}$



21. $\lim_{x\to 0} \cos \dfrac{1}{x}$



22. $\lim_{x\to \pi/2} \tan x$



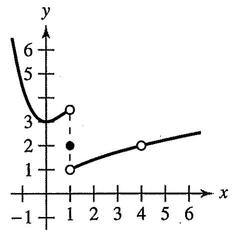
Graphical Reasoning **In Exercises 23 and 24, use the graph of the function f to decide whether the value of the given quantity exists. If it does, find it. If not, explain why.**

23. (a) $f(1)$

(b) $\lim_{x\to1} f(x)$

(c) $f(4)$

(d) $\lim_{x\to4} f(x)$



24. (a) $f(-2)$

(b) $\lim_{x\to-2} f(x)$

(c) $f(0)$

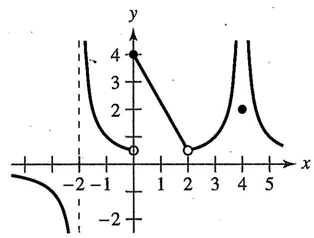
(d) $\lim_{x\to0} f(x)$

(e) $f(2)$

(f) $\lim_{x\to2} f(x)$

(g) $f(4)$

(h) $\lim_{x\to4} f(x)$



Limits of a Piecewise Function **In Exercises 25 and 26, sketch the graph of f. Then identify the values of c for which $\lim_{x\to c} f(x)$ exists.**

25. $f(x) = \begin{cases} x^2, & x \le 2 \\ 8 - 2x, & 2 < x < 4 \\ 4, & x \ge 4 \end{cases}$

26. $f(x) = \begin{cases} \sin x, & x < 0 \\ 1 - \cos x, & 0 \le x \le \pi \\ \cos x, & x > \pi \end{cases}$

Sketching a Graph **In Exercises 27 and 28, sketch a graph of a function f that satisfies the given values. (There are many correct answers.)**

27. $f(0)$ is undefined.

$\lim_{x\to0} f(x) = 4$

$f(2) = 6$

$\lim_{x\to2} f(x) = 3$

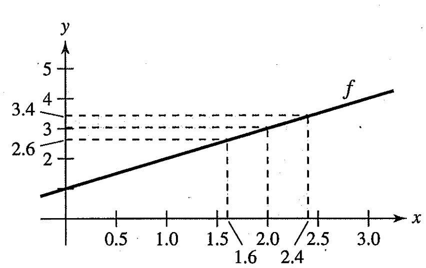
28. $f(-2) = 0$

$f(2) = 0$

$\lim_{x\to-2} f(x) = 0$

$\lim_{x\to2} f(x)$ does not exist.

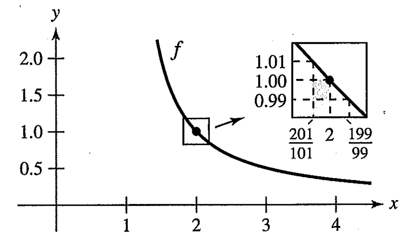
29. Finding a δ for a Given ε The graph of $f(x) = x + 1$ is shown in the figure. Find δ such that if $0 < |x - 2| < \delta$, then $|f(x) - 3| < 0.4$.



30. Finding a δ for a Given ε The graph of

$$f(x) = \frac{1}{x - 1}$$

is shown in the figure. Find δ such that if $0 < |x - 2| < \delta$, then $|f(x) - 1| < 0.01$.



31. Finding a δ for a Given ε The graph of

$$f(x) = 2 - \frac{1}{x}$$

is shown in the figure. Find δ such that if $0 < |x - 1| < \delta$, then $|f(x) - 1| < 0.1$.

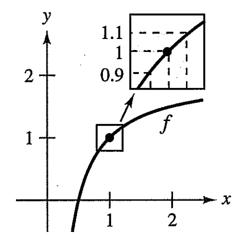


Figure for 31

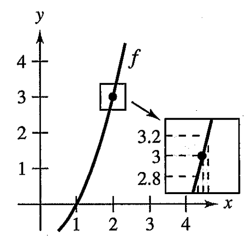


Figure for 32

32. Finding a δ for a Given ε The graph of

$$f(x) = x^2 - 1$$

is shown in the figure. Find δ such that if $0 < |x - 2| < \delta$, then $|f(x) - 3| < 0.2$.

Finding a δ for a Given ε **In Exercises 33–36, find the limit L. Then find $\delta > 0$ such that $|f(x) - L| < 0.01$ whenever $0 < |x - c| < \delta$.**

33. $\lim_{x\to2} (3x + 2)$

34. $\lim_{x\to6} \left(6 - \frac{x}{3}\right)$

35. $\lim_{x\to2} (x^2 - 3)$

36. $\lim_{x\to4} (x^2 + 6)$

Using the ε-δ Definition of Limit **In Exercises 37–48, find the limit L. Then use the ε-δ definition to prove that the limit is L.**

37. $\lim_{x\to4} (x + 2)$

38. $\lim_{x\to-2} (4x + 5)$

39. $\lim_{x\to-4} \left(\frac{1}{2}x - 1\right)$

40. $\lim_{x\to3} \left(\frac{3}{4}x + 1\right)$

41. $\lim_{x\to6} 3$

42. $\lim_{x\to2} (-1)$

43. $\lim_{x\to0} \sqrt[3]{x}$

44. $\lim_{x\to4} \sqrt{x}$

45. $\lim_{x\to-5} |x - 5|$

46. $\lim_{x\to3} |x - 3|$

47. $\lim_{x\to1} (x^2 + 1)$

48. $\lim_{x\to-4} (x^2 + 4x)$

49. Finding a Limit What is the limit of $f(x) = 4$ as x approaches π?

50. Finding a Limit What is the limit of $g(x) = x$ as x approaches π?

Writing In Exercises 51–54, use a graphing utility to graph the function and estimate the limit (if it exists). What is the domain of the function? Can you detect a possible error in determining the domain of a function solely by analyzing the graph generated by a graphing utility? Write a short paragraph about the importance of examining a function analytically as well as graphically.

51. $f(x) = \dfrac{\sqrt{x+5} - 3}{x - 4}$

$\lim_{x \to 4} f(x)$

52. $f(x) = \dfrac{x - 3}{x^2 - 4x + 3}$

$\lim_{x \to 3} f(x)$

53. $f(x) = \dfrac{x - 9}{\sqrt{x} - 3}$

$\lim_{x \to 9} f(x)$

54. $f(x) = \dfrac{x - 3}{x^2 - 9}$

$\lim_{x \to 3} f(x)$

55. Modeling Data For a long distance phone call, a hotel charges \$9.99 for the first minute and \$0.79 for each additional minute or fraction thereof. A formula for the cost is given by

$C(t) = 9.99 - 0.79[\![-(t - 1)]\!]$

where t is the time in minutes.

(*Note:* $[\![x]\!]$ = greatest integer n such that $n \le x$. For example, $[\![3.2]\!] = 3$ and $[\![-1.6]\!] = -2$.)

(a) Use a graphing utility to graph the cost function for $0 < t \le 6$.

(b) Use the graph to complete the table and observe the behavior of the function as t approaches 3.5. Use the graph and the table to find $\lim_{t \to 3.5} C(t)$.

t	3	3.3	3.4	3.5	3.6	3.7	4
C				?			

(c) Use the graph to complete the table and observe the behavior of the function as t approaches 3.

t	2	2.5	2.9	3	3.1	3.5	4
C				?			

Does the limit of $C(t)$ as t approaches 3 exist? Explain.

56. Repeat Exercise 55 for

$C(t) = 5.79 - 0.99[\![-(t - 1)]\!]$.

WRITING ABOUT CONCEPTS

57. Describing Notation Write a brief description of the meaning of the notation

$\lim_{x \to 8} f(x) = 25.$

58. Using the Definition of Limit The definition of limit on page 52 requires that f is a function defined on an open interval containing c, except possibly at c. Why is this requirement necessary?

59. Limits That Fail to Exist Identify three types of behavior associated with the nonexistence of a limit. Illustrate each type with a graph of a function.

60. Comparing Functions and Limits

(a) If $f(2) = 4$, can you conclude anything about the limit of $f(x)$ as x approaches 2? Explain your reasoning.

(b) If the limit of $f(x)$ as x approaches 2 is 4, can you conclude anything about $f(2)$? Explain your reasoning.

61. Jewelry A jeweler resizes a ring so that its inner circumference is 6 centimeters.

(a) What is the radius of the ring?

(b) The inner circumference of the ring varies between 5.5 centimeters and 6.5 centimeters. How does the radius vary?

(c) Use the ε-δ definition of limit to describe this situation. Identify ε and δ.

62. Sports

A sporting goods manufacturer designs a golf ball having a volume of 2.48 cubic inches.

(a) What is the radius of the golf ball?

(b) The volume of the golf ball varies between 2.45 cubic inches and 2.51 cubic inches. How does the radius vary?

(c) Use the ε-δ definition of limit to describe this situation. Identify ε and δ.

63. Estimating a Limit Consider the function

$f(x) = (1 + x)^{1/x}$.

Estimate

$\lim_{x \to 0} (1 + x)^{1/x}$

by evaluating f at x-values near 0. Sketch the graph of f.

The symbol [graphing icon] indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system. The solutions of other exercises may also be facilitated by the use of appropriate technology.

64. Estimating a Limit Consider the function

$$f(x) = \frac{|x+1| - |x-1|}{x}.$$

Estimate

$$\lim_{x \to 0} \frac{|x+1| - |x-1|}{x}$$

by evaluating f at x-values near 0. Sketch the graph of f.

65. Graphical Analysis The statement

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4$$

means that for each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that if $0 < |x - 2| < \delta$, then

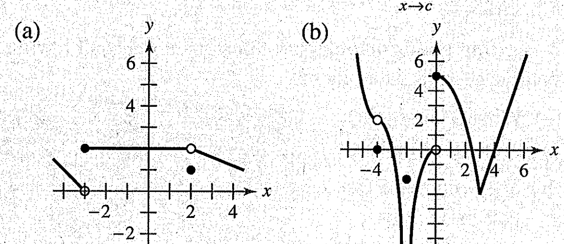
$$\left| \frac{x^2 - 4}{x - 2} - 4 \right| < \varepsilon.$$

If $\varepsilon = 0.001$, then

$$\left| \frac{x^2 - 4}{x - 2} - 4 \right| < 0.001.$$

Use a graphing utility to graph each side of this inequality. Use the *zoom* feature to find an interval $(2 - \delta, 2 + \delta)$ such that the graph of the left side is below the graph of the right side of the inequality.

66. HOW DO YOU SEE IT? Use the graph of f to identify the values of c for which $\lim_{x \to c} f(x)$ exists.



True or False? **In Exercises 67–70, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

67. If f is undefined at $x = c$, then the limit of $f(x)$ as x approaches c does not exist.

68. If the limit of $f(x)$ as x approaches c is 0, then there must exist a number k such that $f(k) < 0.001$.

69. If $f(c) = L$, then $\lim_{x \to c} f(x) = L$.

70. If $\lim_{x \to c} f(x) = L$, then $f(c) = L$.

Determining a Limit **In Exercises 71 and 72, consider the function $f(x) = \sqrt{x}$.**

71. Is $\lim_{x \to 0.25} \sqrt{x} = 0.5$ a true statement? Explain.

72. Is $\lim_{x \to 0} \sqrt{x} = 0$ a true statement? Explain.

73. Evaluating Limits Use a graphing utility to evaluate

$$\lim_{x \to 0} \frac{\sin nx}{x}$$

for several values of n. What do you notice?

74. Evaluating Limits Use a graphing utility to evaluate

$$\lim_{x \to 0} \frac{\tan nx}{x}$$

for several values of n. What do you notice?

75. Proof Prove that if the limit of $f(x)$ as x approaches c exists, then the limit must be unique. [*Hint:* Let $\lim_{x \to c} f(x) = L_1$ and $\lim_{x \to c} f(x) = L_2$ and prove that $L_1 = L_2$.]

76. Proof Consider the line $f(x) = mx + b$, where $m \neq 0$. Use the ε-δ definition of limit to prove that $\lim_{x \to c} f(x) = mc + b$.

77. Proof Prove that

$$\lim_{x \to c} f(x) = L$$

is equivalent to

$$\lim_{x \to c} [f(x) - L] = 0.$$

78. Proof

(a) Given that

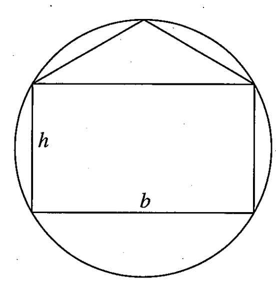
$$\lim_{x \to 0} (3x + 1)(3x - 1)x^2 + 0.01 = 0.01$$

prove that there exists an open interval (a, b) containing 0 such that $(3x + 1)(3x - 1)x^2 + 0.01 > 0$ for all $x \neq 0$ in (a, b).

(b) Given that $\lim_{x \to c} g(x) = L$, where $L > 0$, prove that there exists an open interval (a, b) containing c such that $g(x) > 0$ for all $x \neq c$ in (a, b).

PUTNAM EXAM CHALLENGE

79. Inscribe a rectangle of base b and height h in a circle of radius one, and inscribe an isosceles triangle in a region of the circle cut off by one base of the rectangle (with that side as the base of the triangle). For what value of h do the rectangle and triangle have the same area?



80. A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

These problems were composed by the Committee on the Putnam Prize Competition.