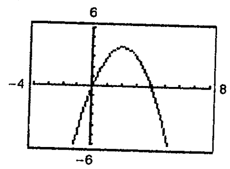
## Section 1.3 Evaluating Limits Analytically

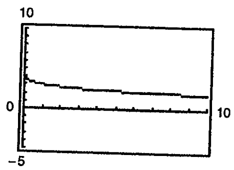
**1.**




(a) $\lim_{x\to4} h(x) = 0$

(b) $\lim_{x\to-1} h(x) = -5$

**2.**

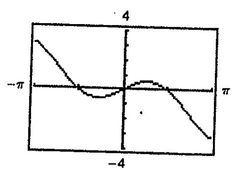



$g(x) = \dfrac{12(\sqrt{x} - 3)}{x - 9}$

(a) $\lim_{x\to4} g(x) = 2.4$

(b) $\lim_{x\to0} g(x) = 4$

**3.**



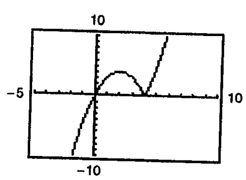

$f(x) = x\cos x$

(a) $\lim_{x\to0} f(x) = 0$

(b) $\lim_{x\to\pi/3} f(x) \approx 0.524$

$\left(= \dfrac{\pi}{6}\right)$

**4.**




$f(t) = t|t - 4|$

(a) $\lim_{t\to4} f(t) = 0$

(b) $\lim_{t\to-1} f(t) = -5$

**5.** $\lim_{x\to2} x^3 = 2^3 = 8$

**6.** $\lim_{x\to-3} x^4 = (-3)^4 = 81$

**7.** $\lim_{x\to0} (2x - 1) = 2(0) - 1 = -1$

**8.** $\lim_{x\to-4} (2x + 3) = 2(-4) + 3 = -8 + 3 = -5$

**9.** $\lim_{x\to-3} (x^2 + 3x) = (-3)^2 + 3(-3) = 9 - 9 = 0$

**10.** $\lim_{x\to2} (-x^3 + 1) = (-2)^3 + 1 = -8 + 1 = -7$

**11.** $\lim_{x\to-3} (2x^2 + 4x + 1) = 2(-3)^2 + 4(-3) + 1$

$= 18 - 12 + 1 = 7$

**12.** $\lim_{x\to1} (2x^3 - 6x + 5) = 2(1)^3 - 6(1) + 5$

$= 2 - 6 + 5 = 1$

**13.** $\lim_{x\to3} \sqrt{x + 1} = \sqrt{3 + 1} = 2$

**14.** $\lim_{x\to2} \sqrt[3]{12x + 3} = \sqrt[3]{12(2) + 3}$

$= \sqrt[3]{24 + 3} = \sqrt[3]{27} = 3$

**15.** $\lim_{x\to-4} (x + 3)^2 = (-4 + 3)^2 = 1$

**16.** $\lim_{x\to0} (3x - 2)^4 = (3(0) - 2)^4 = (-2)^4 = 16$

**17.** $\lim_{x\to2} \dfrac{1}{x} = \dfrac{1}{2}$

**18.** $\lim_{x\to-5} \dfrac{5}{x + 3} = \dfrac{5}{-5 + 3} = -\dfrac{5}{2}$

**19.** $\lim_{x\to1} \dfrac{x}{x^2 + 4} = \dfrac{1}{1^2 + 4} = \dfrac{1}{5}$

**20.** $\lim_{x\to1} \dfrac{3x + 5}{x + 1} = \dfrac{3(1) + 5}{1 + 1} = \dfrac{3 + 5}{2} = \dfrac{8}{2} = 4$

**21.** $\lim_{x\to7} \dfrac{3x}{\sqrt{x + 2}} = \dfrac{3(7)}{\sqrt{7 + 2}} = \dfrac{21}{3} = 7$

**22.** $\lim_{x\to3} \dfrac{\sqrt{x + 6}}{x + 2} = \dfrac{\sqrt{3 + 6}}{3 + 2} = \dfrac{\sqrt{9}}{5} = \dfrac{3}{5}$

**23.** (a) $\lim_{x\to1} f(x) = 5 - 1 = 4$

(b) $\lim_{x\to4} g(x) = 4^3 = 64$

(c) $\lim_{x\to1} g(f(x)) = g(f(1)) = g(4) = 64$

**24.** (a) $\lim_{x\to-3} f(x) = (-3) + 7 = 4$

(b) $\lim_{x\to4} g(x) = 4^2 = 16$

(c) $\lim_{x\to-3} g(f(x)) = g(4) = 16$

**25.** (a) $\lim_{x\to 1} f(x) = 4 - 1 = 3$

(b) $\lim_{x\to 3} g(x) = \sqrt{3+1} = 2$

(c) $\lim_{x\to 1} g(f(x)) = g(3) = 2$

**26.** (a) $\lim_{x\to 4} f(x) = 2(4^2) - 3(4) + 1 = 21$

(b) $\lim_{x\to 21} g(x) = \sqrt[3]{21+6} = 3$

(c) $\lim_{x\to 4} g(f(x)) = g(21) = 3$

**27.** $\lim_{x\to \pi/2} \sin x = \sin \frac{\pi}{2} = 1$

**28.** $\lim_{x\to \pi} \tan x = \tan \pi = 0$

**29.** $\lim_{x\to 1} \cos \frac{\pi x}{3} = \cos \frac{\pi}{3} = \frac{1}{2}$

**30.** $\lim_{x\to 2} \sin \frac{\pi x}{2} = \sin \frac{\pi(2)}{2} = 0$

**31.** $\lim_{x\to 0} \sec 2x = \sec 0 = 1$

**32.** $\lim_{x\to \pi} \cos 3x = \cos 3\pi = -1$

**33.** $\lim_{x\to 5\pi/6} \sin x = \sin \frac{5\pi}{6} = \frac{1}{2}$

**34.** $\lim_{x\to 5\pi/3} \cos x = \cos \frac{5\pi}{3} = \frac{1}{2}$

**35.** $\lim_{x\to 3} \tan\left(\frac{\pi x}{4}\right) = \tan \frac{3\pi}{4} = -1$

**36.** $\lim_{x\to 7} \sec\left(\frac{\pi x}{6}\right) = \sec \frac{7\pi}{6} = \frac{-2\sqrt{3}}{3}$

**37.** (a) $\lim_{x\to c}[5g(x)] = 5 \lim_{x\to c} g(x) = 5(2) = 10$

(b) $\lim_{x\to c}[f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = 3 + 2 = 5$

(c) $\lim_{x\to c}[f(x)g(x)] = \left[\lim_{x\to c} f(x)\right]\left[\lim_{x\to c} g(x)\right] = (3)(2) = 6$

(d) $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{3}{2}$

**38.** (a) $\lim_{x\to c}[4f(x)] = 4 \lim_{x\to c} f(x) = 4(2) = 8$

(b) $\lim_{x\to c}[f(x) + g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = 2 + \frac{3}{4} = \frac{11}{4}$

(c) $\lim_{x\to c}[f(x)g(x)] = \left[\lim_{x\to c} f(x)\right]\left[\lim_{x\to c} g(x)\right] = 2\left(\frac{3}{4}\right) = \frac{3}{2}$

(d) $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{2}{(3/4)} = \frac{8}{3}$

**39.** (a) $\lim_{x\to c}[f(x)]^3 = \left[\lim_{x\to c} f(x)\right]^3 = (4)^3 = 64$

(b) $\lim_{x\to c}\sqrt{f(x)} = \sqrt{\lim_{x\to c} f(x)} = \sqrt{4} = 2$

(c) $\lim_{x\to c}[3f(x)] = 3 \lim_{x\to c} f(x) = 3(4) = 12$

(d) $\lim_{x\to c}[f(x)]^{3/2} = \left[\lim_{x\to c} f(x)\right]^{3/2} = (4)^{3/2} = 8$

**40.** (a) $\lim_{x\to c}\sqrt[3]{f(x)} = \sqrt[3]{\lim_{x\to c} f(x)} = \sqrt[3]{27} = 3$

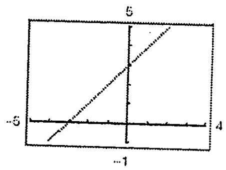
(b) $\lim_{x\to c}\frac{f(x)}{18} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} 18} = \frac{27}{18} = \frac{3}{2}$

(c) $\lim_{x\to c}[f(x)]^2 = \left[\lim_{x\to c} f(x)\right]^2 = (27)^2 = 729$

(d) $\lim_{x\to c}[f(x)]^{2/3} = \left[\lim_{x\to c} f(x)\right]^{2/3} = (27)^{2/3} = 9$

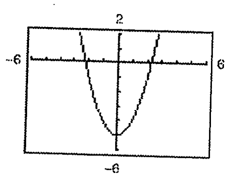
**41.** $f(x) = \dfrac{x^2 + 3x}{x} = \dfrac{x(x+3)}{x}$ and $g(x) = x + 3$ agree except at $x = 0$.

$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = \lim_{x\to 0} (x + 3) = 0 + 3 = 3$



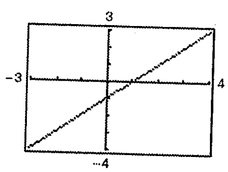

**42.** $f(x) = \dfrac{x^4 - 5x^2}{x^2} = \dfrac{x^2(x^2 - 5)}{x^2}$ and $g(x) = x^2 - 5$ agree except at $x = 0$.

$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = \lim_{x\to 0} (x^2 - 5) = 0^2 - 5 = -5$




**43.** $f(x) = \dfrac{x^2 - 1}{x + 1} = \dfrac{(x+1)(x-1)}{x+1}$ and $g(x) = x - 1$ agree except at $x = -1$.

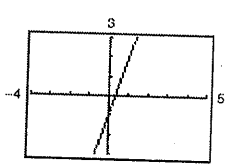
$\lim_{x\to -1} f(x) = \lim_{x\to -1} g(x) = \lim_{x\to -1} (x - 1) = -1 - 1 = -2$




**44.** $f(x) = \dfrac{3x^2 + 5x - 2}{x + 2} = \dfrac{(x+2)(3x-1)}{x+2}$ and $g(x) = 3x - 1$ agree except at $x = -2$.

$\lim_{x\to -2} f(x) = \lim_{x\to -2} g(x) = \lim_{x\to -2} (3x - 1)$

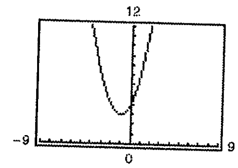
$= 3(-2) - 1 = -7$




**45.** $f(x) = \dfrac{x^3 - 8}{x - 2}$ and $g(x) = x^2 + 2x + 4$ agree except at $x = 2$.

$\lim_{x\to 2} f(x) = \lim_{x\to 2} g(x) = \lim_{x\to 2}(x^2 + 2x + 4)$

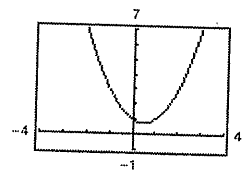
$= 2^2 + 2(2) + 4 = 12$




**46.** $f(x) = \dfrac{x^3 + 1}{x + 1}$ and $g(x) = x^2 - x + 1$ agree except at $x = -1$.

$\lim_{x\to -1} f(x) = \lim_{x\to -1} g(x) = \lim_{x\to -1}(x^2 - x + 1)$

$= (-1)^2 - (-1) + 1 = 3$




**47.** $\lim_{x\to 0} \dfrac{x}{x^2 - x} = \lim_{x\to 0} \dfrac{x}{x(x-1)} = \lim_{x\to 0} \dfrac{1}{x - 1} = \dfrac{1}{0 - 1} = -1$

**48.** $\lim_{x\to 0} \dfrac{2x}{x^2 + 4x} = \lim_{x\to 0} \dfrac{2x}{x(x+4)} = \lim_{x\to 0} \dfrac{2}{x + 4}$

$= \dfrac{2}{0 + 4} = \dfrac{2}{4} = \dfrac{1}{2}$

**49.** $\lim_{x\to 4} \dfrac{x - 4}{x^2 - 16} = \lim_{x\to 4} \dfrac{x - 4}{(x+4)(x-4)}$

$= \lim_{x\to 4} \dfrac{1}{x + 4} = \dfrac{1}{4 + 4} = \dfrac{1}{8}$

**50.** $\lim_{x\to 5} \dfrac{5 - x}{x^2 - 25} = \lim_{x\to 5} \dfrac{-(x-5)}{(x-5)(x+5)}$

$= \lim_{x\to 5} \dfrac{-1}{x + 5} = \dfrac{-1}{5 + 5} = -\dfrac{1}{10}$

**51.** $\lim_{x\to -3} \dfrac{x^2 + x - 6}{x^2 - 9} = \lim_{x\to -3} \dfrac{(x+3)(x-2)}{(x+3)(x-3)}$

$= \lim_{x\to -3} \dfrac{x - 2}{x - 3} = \dfrac{-3 - 2}{-3 - 3} = \dfrac{-5}{-6} = \dfrac{5}{6}$

**52.** $\lim_{x\to 2}\frac{x^2+2x-8}{x^2-x-2} = \lim_{x\to 2}\frac{(x-2)(x+4)}{(x-2)(x+1)}$

$= \lim_{x\to 2}\frac{x+4}{x+1} = \frac{2+4}{2+1} = \frac{6}{3} = 2$

**53.** $\lim_{x\to 4}\frac{\sqrt{x+5}-3}{x-4} = \lim_{x\to 4}\frac{\sqrt{x+5}-3}{x-4}\cdot\frac{\sqrt{x+5}+3}{\sqrt{x+5}+3}$

$= \lim_{x\to 4}\frac{(x+5)-9}{(x-4)\left(\sqrt{x+5}+3\right)} = \lim_{x\to 4}\frac{1}{\sqrt{x+5}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}$

**54.** $\lim_{x\to 3}\frac{\sqrt{x+1}-2}{x-3} = \lim_{x\to 3}\frac{\sqrt{x+1}-2}{x-3}\cdot\frac{\sqrt{x+1}+2}{\sqrt{x+1}+2} = \lim_{x\to 3}\frac{x-3}{(x-3)\left[\sqrt{x+1}+2\right]}$

$= \lim_{x\to 3}\frac{1}{\sqrt{x+1}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}$

**55.** $\lim_{x\to 0}\frac{\sqrt{x+5}-\sqrt{5}}{x} = \lim_{x\to 0}\frac{\sqrt{x+5}-\sqrt{5}}{x}\cdot\frac{\sqrt{x+5}+\sqrt{5}}{\sqrt{x+5}+\sqrt{5}}$

$= \lim_{x\to 0}\frac{(x+5)-5}{x\left(\sqrt{x+5}+\sqrt{5}\right)} = \lim_{x\to 0}\frac{1}{\sqrt{x+5}+\sqrt{5}} = \frac{1}{\sqrt{5}+\sqrt{5}} = \frac{1}{2\sqrt{5}} = \frac{\sqrt{5}}{10}$

**56.** $\lim_{x\to 0}\frac{\sqrt{2+x}-\sqrt{2}}{x} = \lim_{x\to 0}\frac{\sqrt{2+x}-\sqrt{2}}{x}\cdot\frac{\sqrt{2+x}+\sqrt{2}}{\sqrt{2+x}+\sqrt{2}}$

$= \lim_{x\to 0}\frac{2+x-2}{\left(\sqrt{2+x}+\sqrt{2}\right)x} = \lim_{x\to 0}\frac{1}{\sqrt{2+x}+\sqrt{2}} = \frac{1}{\sqrt{2}+\sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$

**57.** $\lim_{x\to 0}\frac{\frac{1}{3+x}-\frac{1}{3}}{x} = \lim_{x\to 0}\frac{3-(3+x)}{(3+x)3(x)} = \lim_{x\to 0}\frac{-x}{(3+x)(3)(x)} = \lim_{x\to 0}\frac{-1}{(3+x)3} = \frac{-1}{(3)3} = -\frac{1}{9}$

**58.** $\lim_{x\to 0}\frac{\frac{1}{x+4}-\frac{1}{4}}{x} = \lim_{x\to 0}\frac{\frac{4-(x+4)}{4(x+4)}}{x}$

$= \lim_{x\to 0}\frac{-1}{4(x+4)} = \frac{-1}{4(4)} = -\frac{1}{16}$

**59.** $\lim_{\Delta x\to 0}\frac{2(x+\Delta x)-2x}{\Delta x} = \lim_{\Delta x\to 0}\frac{2x+2\Delta x-2x}{\Delta x} = \lim_{\Delta x\to 0}\frac{2\Delta x}{\Delta x} = \lim_{\Delta x\to 0} 2 = 2$

**60.** $\lim_{\Delta x\to 0}\frac{(x+\Delta x)^2-x^2}{\Delta x} = \lim_{\Delta x\to 0}\frac{x^2+2x\Delta x+(\Delta x)^2-x^2}{\Delta x} = \lim_{\Delta x\to 0}\frac{\Delta x(2x+\Delta x)}{\Delta x} = \lim_{\Delta x\to 0}(2x+\Delta x) = 2x$

**61.** $\lim_{\Delta x\to 0}\frac{(x+\Delta x)^2-2(x+\Delta x)+1-(x^2-2x+1)}{\Delta x} = \lim_{\Delta x\to 0}\frac{x^2+2x\Delta x+(\Delta x)^2-2x-2\Delta x+1-x^2+2x-1}{\Delta x}$

$= \lim_{\Delta x\to 0}(2x+\Delta x-2) = 2x-2$

**62.** $\lim_{\Delta x\to 0}\frac{(x+\Delta x)^3-x^3}{\Delta x} = \lim_{\Delta x\to 0}\frac{x^3+3x^2\Delta x+3x(\Delta x)^2+(\Delta x)^3-x^3}{\Delta x}$

$= \lim_{\Delta x\to 0}\frac{\Delta x\left(3x^2+3x\Delta x+(\Delta x)^2\right)}{\Delta x} = \lim_{\Delta x\to 0}\left(3x^2+3x\Delta x+(\Delta x)^2\right) = 3x^2$

**63.** $\lim_{x\to 0}\frac{\sin x}{5x} = \lim_{x\to 0}\left[\left(\frac{\sin x}{x}\right)\left(\frac{1}{5}\right)\right] = (1)\left(\frac{1}{5}\right) = \frac{1}{5}$

**64.** $\lim_{x\to 0}\frac{3(1-\cos x)}{x} = \lim_{x\to 0}\left[3\left(\frac{(1-\cos x)}{x}\right)\right] = (3)(0) = 0$

**65.** $\lim_{x\to 0}\frac{\sin x(1-\cos x)}{x^2} = \lim_{x\to 0}\left[\frac{\sin x}{x}\cdot\frac{1-\cos x}{x}\right]$
$= (1)(0) = 0$

**66.** $\lim_{\theta\to 0}\frac{\cos\theta\tan\theta}{\theta} = \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$

**67.** $\lim_{x\to 0}\frac{\sin^2 x}{x} = \lim_{x\to 0}\left[\frac{\sin x}{x}\sin x\right] = (1)\sin 0 = 0$

**68.** $\lim_{x\to 0}\frac{\tan^2 x}{x} = \lim_{x\to 0}\frac{\sin^2 x}{x\cos^2 x} = \lim_{x\to 0}\left[\frac{\sin x}{x}\cdot\frac{\sin x}{\cos^2 x}\right]$
$= (1)(0) = 0$

**69.** $\lim_{h\to 0}\frac{(1-\cos h)^2}{h} = \lim_{h\to 0}\left[\frac{1-\cos h}{h}(1-\cos h)\right]$
$= (0)(0) = 0$

**70.** $\lim_{\phi\to\pi}\phi\sec\phi = \pi(-1) = -\pi$

**71.** $\lim_{x\to\pi/2}\frac{\cos x}{\cot x} = \lim_{x\to\pi/2}\sin x = 1$

**72.** $\lim_{x\to\pi/4}\frac{1-\tan x}{\sin x-\cos x} = \lim_{x\to\pi/4}\frac{\cos x-\sin x}{\sin x\cos x-\cos^2 x}$
$= \lim_{x\to\pi/4}\frac{-(\sin x-\cos x)}{\cos x(\sin x-\cos x)}$
$= \lim_{x\to\pi/4}\frac{-1}{\cos x}$
$= \lim_{x\to\pi/4}(-\sec x)$
$= -\sqrt{2}$

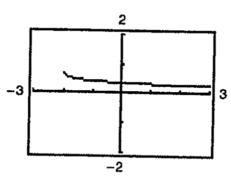
**73.** $\lim_{t\to 0}\frac{\sin 3t}{2t} = \lim_{t\to 0}\left(\frac{\sin 3t}{3t}\right)\left(\frac{3}{2}\right) = (1)\left(\frac{3}{2}\right) = \frac{3}{2}$

**74.** $\lim_{x\to 0}\frac{\sin 2x}{\sin 3x} = \lim_{x\to 0}\left[2\left(\frac{\sin 2x}{2x}\right)\left(\frac{1}{3}\right)\left(\frac{3x}{\sin 3x}\right)\right]$
$= 2(1)\left(\frac{1}{3}\right)(1) = \frac{2}{3}$

**75.** $f(x) = \frac{\sqrt{x+2}-\sqrt{2}}{x}$

| $x$ | −0.1 | −0.01 | −0.001 | 0 | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 0.358 | 0.354 | 0.354 | ? | 0.354 | 0.353 | 0.349 |

It appears that the limit is 0.354.



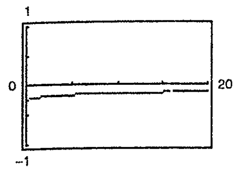

The graph has a hole at $x = 0$.

Analytically, $\lim_{x\to 0}\frac{\sqrt{x+2}-\sqrt{2}}{x} = \lim_{x\to 0}\frac{\sqrt{x+2}-\sqrt{2}}{x}\cdot\frac{\sqrt{x+2}+\sqrt{2}}{\sqrt{x+2}+\sqrt{2}}$
$= \lim_{x\to 0}\frac{x+2-2}{x\left(\sqrt{x+2}+\sqrt{2}\right)} = \lim_{x\to 0}\frac{1}{\sqrt{x+2}+\sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4} \approx 0.354.$

**76.** $f(x) = \dfrac{4 - \sqrt{x}}{x - 16}$

| $x$ | 15.9 | 15.99 | 15.999 | 16 | 16.001 | 16.01 | 16.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | –0.1252 | –0.125 | –0.125 | ? | –0.125 | –0.125 | –0.1248 |

It appears that the limit is –0.125.



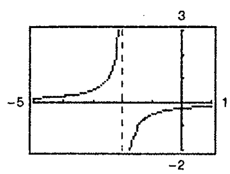

The graph has a hole at $x = 16$.

Analytically, $\lim_{x\to16}\dfrac{4 - \sqrt{x}}{x - 16} = \lim_{x\to16}\dfrac{\left(4 - \sqrt{x}\right)}{\left(\sqrt{x} + 4\right)\left(\sqrt{x} - 4\right)} = \lim_{x\to16}\dfrac{-1}{\sqrt{x} + 4} = -\dfrac{1}{8}.$

**77.** $f(x) = \dfrac{\dfrac{1}{2 + x} - \dfrac{1}{2}}{x}$

| $x$ | –0.1 | –0.01 | –0.001 | 0 | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | –0.263 | –0.251 | –0.250 | ? | –0.250 | –0.249 | –0.238 |

It appears that the limit is –0.250.



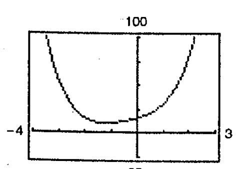

The graph has a hole at $x = 0$.

Analytically, $\lim_{x\to0}\dfrac{\dfrac{1}{2 + x} - \dfrac{1}{2}}{x} = \lim_{x\to0}\dfrac{2 - (2 + x)}{2(2 + x)} \cdot \dfrac{1}{x} = \lim_{x\to0}\dfrac{-x}{2(2 + x)} \cdot \dfrac{1}{x} = \lim_{x\to0}\dfrac{-1}{2(2 + x)} = -\dfrac{1}{4}.$

**78.** $f(x) = \dfrac{x^5 - 32}{x - 2}$

| $x$ | 1.9 | 1.99 | 1.999 | 1.9999 | 2.0 | 2.0001 | 2.001 | 2.01 | 2.1 |
|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | 72.39 | 79.20 | 79.92 | 79.99 | ? | 80.01 | 80.08 | 80.80 | 88.41 |

It appears that the limit is 80.




The graph has a hole at $x = 2$.

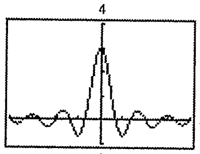
Analytically, $\lim_{x\to2}\dfrac{x^5 - 32}{x - 2} = \lim_{x\to2}\dfrac{(x - 2)\left(x^4 + 2x^3 + 4x^2 + 8x + 16\right)}{x - 2} = \lim_{x\to2}\left(x^4 + 2x^3 + 4x^2 + 8x + 16\right) = 80.$

(*Hint:* Use long division to factor $x^5 - 32$.)

**79.** $f(t) = \frac{\sin 3t}{t}$

| $t$ | −0.1 | −0.01 | −0.001 | 0 | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|---|
| $f(t)$ | 2.96 | 2.9996 | 3 | ? | 3 | 2.9996 | 2.96 |

It appears that the limit is 3.



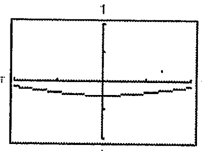

The graph has a hole at $t = 0$.

Analytically, $\lim_{t\to 0}\frac{\sin 3t}{t} = \lim_{t\to 0} 3\left(\frac{\sin 3t}{3t}\right) = 3(1) = 3.$

**80.** $f(x) = \frac{\cos x - 1}{2x^2}$

| $x$ | −1 | −0.1 | −0.01 | 0.01 | 0.1 | 1 |
|---|---|---|---|---|---|---|
| $f(x)$ | −0.2298 | −0.2498 | −0.25 | −0.25 | −0.2498 | −0.2298 |

It appears that the limit is −0.25.




The graph has a hole at $x = 0$.

Analytically, $\frac{\cos x - 1}{2x^2} \cdot \frac{\cos x + 1}{\cos x + 1} = \frac{\cos^2 x - 1}{2x^2(\cos x + 1)}$

$$= \frac{-\sin^2 x}{2x^2(\cos x + 1)}$$

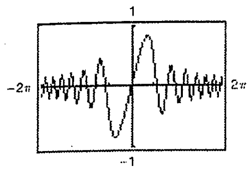
$$= \frac{\sin^2 x}{x^2} \cdot \frac{-1}{2(\cos x + 1)}$$

$$\lim_{x\to 0}\left[\frac{\sin^2 x}{x^2} \cdot \frac{-1}{2(\cos x + 1)}\right] = 1\left(\frac{-1}{4}\right) = -\frac{1}{4} = -0.25$$

**81.** $f(x) = \dfrac{\sin x^2}{x}$

| $x$ | −0.1 | −0.01 | −0.001 | 0 | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | −0.099998 | −0.01 | −0.001 | ? | 0.001 | 0.01 | 0.099998 |

It appears that the limit is 0.



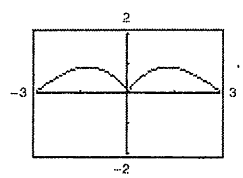

The graph has a hole at $x = 0$.

Analytically, $\lim\limits_{x\to 0}\dfrac{\sin x^2}{x} = \lim\limits_{x\to 0} x\left(\dfrac{\sin x^2}{x}\right) = 0(1) = 0.$

**82.** $f(x) = \dfrac{\sin x}{\sqrt[3]{x}}$

| $x$ | −0.1 | −0.01 | −0.001 | 0 | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 0.215 | 0.0464 | 0.01 | ? | 0.01 | 0.0464 | 0.215 |

It appears that the limit is 0.




The graph has a hole at $x = 0$.

Analytically, $\lim\limits_{x\to 0}\dfrac{\sin x}{\sqrt[3]{x}} = \lim\limits_{x\to 0} \sqrt[3]{x^2}\left(\dfrac{\sin x}{x}\right) = (0)(1) = 0.$

**83.** $\lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x) - f(x)}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{3(x+\Delta x) - 2 - (3x - 2)}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{3x + 3\Delta x - 2 - 3x + 2}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{3\Delta x}{\Delta x} = 3$

**84.** $\lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x) - f(x)}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{[-6(x+\Delta x) + 3] - [-6x + 3]}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{-6x - 6\Delta x + 3 + 6x - 3}{\Delta x}$

$= \lim\limits_{\Delta x\to 0}\dfrac{-6\Delta x}{\Delta x} = \lim\limits_{\Delta x\to 0}(-6) = -6$

**85.** $\lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x) - f(x)}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{(x+\Delta x)^2 - 4(x+\Delta x) - (x^2 - 4x)}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{x^2 + 2x\Delta x + \Delta x^2 - 4x - 4\Delta x - x^2 + 4x}{\Delta x}$

$= \lim\limits_{\Delta x\to 0}\dfrac{\Delta x(2x + \Delta x - 4)}{\Delta x} = \lim\limits_{\Delta x\to 0}(2x + \Delta x - 4) = 2x - 4$

**86.** $\lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x) - f(x)}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x}\cdot\dfrac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}}$

$= \lim\limits_{\Delta x\to 0}\dfrac{x + \Delta x - x}{\Delta x\left(\sqrt{x+\Delta x} + \sqrt{x}\right)} = \lim\limits_{\Delta x\to 0}\dfrac{1}{\sqrt{x+\Delta x} + \sqrt{x}} = \dfrac{1}{2\sqrt{x}}$

87. $\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{1}{x + \Delta x + 3} - \frac{1}{x + 3}}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{x + 3 - (x + \Delta x + 3)}{(x + \Delta x + 3)(x + 3)} \cdot \frac{1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-\Delta x}{(x + \Delta x + 3)(x + 3)\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-1}{(x + \Delta x + 3)(x + 3)} = \frac{-1}{(x + 3)^2}$$

88. $\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{1}{(x + \Delta x)^2} - \frac{1}{x^2}}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{x^2 - (x + \Delta x)^2}{x^2(x + \Delta x)^2 \Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 - \left[x^2 + 2x\Delta x + (\Delta x)^2\right]}{x^2(x + \Delta x)^2 \Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-2x\Delta x - (\Delta x)^2}{x^2(x + \Delta x)^2 \Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-2x - \Delta x}{x^2(x + \Delta x)^2} = \frac{-2x}{x^4} = -\frac{2}{x^3}$$

89. $\lim_{x \to 0}(4 - x^2) \le \lim_{x \to 0} f(x) \le \lim_{x \to 0}(4 + x^2)$

$$4 \le \lim_{x \to 0} f(x) \le 4$$

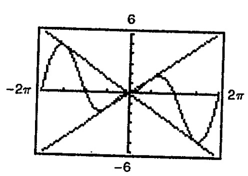
Therefore, $\lim_{x \to 0} f(x) = 4$.

90. $\lim_{x \to a}\left[b - |x - a|\right] \le \lim_{x \to a} f(x) \le \lim_{x \to a}\left[b + |x - a|\right]$

$$b \le \lim_{x \to a} f(x) \le b$$

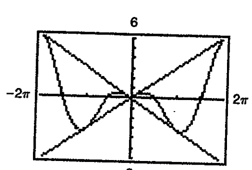
Therefore, $\lim_{x \to a} f(x) = b$.

91. $f(x) = |x|\sin x$



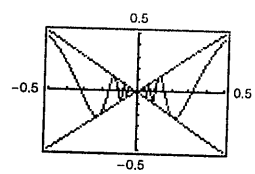

$\lim_{x \to 0} |x|\sin x = 0$

92. $f(x) = |x|\cos x$



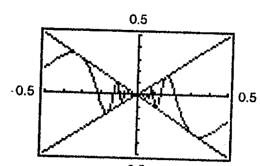

$\lim_{x \to 0} |x|\cos x = 0$

93. $f(x) = x \sin \frac{1}{x}$




$\lim_{x \to 0}\left(x \sin \frac{1}{x}\right) = 0$

94. $h(x) = x \cos \frac{1}{x}$




$\lim_{x \to 0}\left(x \cos \frac{1}{x}\right) = 0$

95. (a) Two functions $f$ and $g$ agree at all but one point (on an open interval) if $f(x) = g(x)$ for all $x$ in the interval except for $x = c$, where $c$ is in the interval.

(b) $f(x) = \frac{x^2 - 1}{x - 1} = \frac{(x + 1)(x - 1)}{x - 1}$ and $g(x) = x + 1$ agree at all points except $x = 1$. (Other answers possible.)

**96.** An indeterminant form is obtained when evaluating a limit using direct substitution produces a meaningless fractional expression such as 0/0. That is,

$$\lim_{x\to c}\frac{f(x)}{g(x)}$$

for which $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$

**97.** If a function $f$ is squeezed between two functions $h$ and $g$, $h(x) \le f(x) \le g(x)$, and $h$ and $g$ have the same limit $L$ as $x \to c$, then $\lim_{x\to c} f(x)$ exists and equals $L$

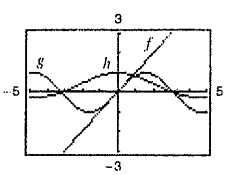
**98.** (a) Use the dividing out technique because the numerator and denominator have a common factor.

$$\lim_{x\to -2}\frac{x^2 + x - 2}{x + 2} = \lim_{x\to -2}\frac{(x + 2)(x - 1)}{x + 2}$$
$$= \lim_{x\to -2}(x - 1) = -2 - 1 = -3$$

(b) Use the rationalizing technique because the numerator involves a radical expression.

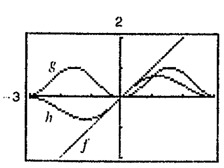
$$\lim_{x\to 0}\frac{\sqrt{x + 4} - 2}{x} = \lim_{x\to 0}\frac{\sqrt{x + 4} - 2}{x}\cdot\frac{\sqrt{x + 4} + 2}{\sqrt{x + 4} + 2}$$
$$= \lim_{x\to 0}\frac{(x + 4) - 4}{x\left(\sqrt{x + 4} + 2\right)}$$
$$= \lim_{x\to 0}\frac{1}{\sqrt{x + 4} + 2} = \frac{1}{\sqrt{4} + 2} = \frac{1}{4}$$

**99.** $f(x) = x,\ g(x) = \sin x,\ h(x) = \dfrac{\sin x}{x}$




When the $x$-values are "close to" 0 the magnitude of $f$ is approximately equal to the magnitude of $g$. So, $|g|/|f| \approx 1$ when $x$ is "close to" 0.

**100.** $f(x) = x,\ g(x) = \sin^2 x,\ h(x) = \dfrac{\sin^2 x}{x}$




When the $x$-values are "close to" 0 the magnitude of $g$ is "smaller" than the magnitude of $f$ and the magnitude of $g$ is approaching zero "faster" than the magnitude of $f$. So, $|g|/|f| \approx 0$ when $x$ is "close to" 0.

**101.** $s(t) = -16t^2 + 500$

$$\lim_{t\to 2}\frac{s(2) - s(t)}{2 - t} = \lim_{t\to 2}\frac{-16(2)^2 + 500 - \left(-16t^2 + 500\right)}{2 - t}$$
$$= \lim_{t\to 2}\frac{436 + 16t^2 - 500}{2 - t}$$
$$= \lim_{t\to 2}\frac{16\left(t^2 - 4\right)}{2 - t}$$
$$= \lim_{t\to 2}\frac{16(t - 2)(t + 2)}{2 - t}$$
$$= \lim_{t\to 2} -16(t + 2) = -64 \text{ ft/sec}$$

The paint can is falling at about 64 feet/second.

**102.** $s(t) = -16t^2 + 500 = 0$ when $t = \sqrt{\frac{500}{16}} = \frac{5\sqrt{5}}{2}$ sec. The velocity at time $a = \frac{5\sqrt{5}}{2}$ is

$$\lim_{t\to\left(\frac{5\sqrt{5}}{2}\right)} \frac{s\left(\frac{5\sqrt{5}}{2}\right) - s(t)}{\frac{5\sqrt{5}}{2} - t} = \lim_{t\to\left(\frac{5\sqrt{5}}{2}\right)} \frac{0 - \left(-16t^2 + 500\right)}{\frac{5\sqrt{5}}{2} - t}$$

$$= \lim_{t\to\left(\frac{5\sqrt{5}}{2}\right)} \frac{16\left(t^2 - \frac{125}{4}\right)}{\frac{5\sqrt{5}}{2} - t}$$

$$= \lim_{t\to\left(\frac{5\sqrt{5}}{2}\right)} \frac{16\left(t + \frac{5\sqrt{5}}{2}\right)\left(t - \frac{5\sqrt{5}}{2}\right)}{\frac{5\sqrt{5}}{2} - t}$$

$$= \lim_{t\to\frac{5\sqrt{5}}{2}} \left[-16\left(t + \frac{5\sqrt{5}}{2}\right)\right] = -80\sqrt{5} \text{ ft/sec}$$

$$\approx -178.9 \text{ ft/sec.}$$

The velocity of the paint can when it hits the ground is about 178.9 ft/sec.

**103.** $s(t) = -4.9t^2 + 200$

$$\lim_{t\to 3}\frac{s(3) - s(t)}{3 - t} = \lim_{t\to 3}\frac{-4.9(3)^2 + 200 - \left(-4.9t^2 + 200\right)}{3 - t}$$

$$= \lim_{t\to 3}\frac{4.9\left(t^2 - 9\right)}{3 - t}$$

$$= \lim_{t\to 3}\frac{4.9(t - 3)(t + 3)}{3 - t}$$

$$= \lim_{t\to 3}\left[-4.9(t + 3)\right]$$

$$= -29.4 \text{ m/sec}$$

The object is falling about 29.4 m/sec.

**104.** $-4.9t^2 + 200 = 0$ when $t = \sqrt{\frac{200}{4.9}} = \frac{20\sqrt{5}}{7}$ sec. The velocity at time $a = \frac{20\sqrt{5}}{7}$ is

$$\lim_{t\to a}\frac{s(a) - s(t)}{a - t} = \lim_{t\to a}\frac{0 - \left[-4.9t^2 + 200\right]}{a - t}$$

$$= \lim_{t\to a}\frac{4.9(t + a)(t - a)}{a - t}$$

$$= \lim_{t\to\frac{20\sqrt{5}}{7}}\left[-4.9\left(t + \frac{20\sqrt{5}}{7}\right)\right] = -28\sqrt{5} \text{ m/sec}$$

$$\approx -62.6 \text{ m/sec.}$$

The velocity of the object when it hits the ground is about 62.6 m/sec.

**105.** Let $f(x) = 1/x$ and $g(x) = -1/x$. $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ do not exist. However,

$$\lim_{x\to 0}\left[f(x) + g(x)\right] = \lim_{x\to 0}\left[\frac{1}{x} + \left(-\frac{1}{x}\right)\right] = \lim_{x\to 0}[0] = 0$$

and therefore does not exist.

**106.** Suppose, on the contrary, that $\lim_{x\to c} g(x)$ exists. Then, because $\lim_{x\to c} f(x)$ exists, so would $\lim_{x\to c}[f(x) + g(x)]$, which is a contradiction. So, $\lim_{x\to c} g(x)$ does not exist.

**107.** Given $f(x) = b$, show that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - b| < \varepsilon$ whenever $|x - c| < \delta$. Because $|f(x) - b| = |b - b| = 0 < \varepsilon$ for every $\varepsilon > 0$, any value of $\delta > 0$ will work.

**108.** Given $f(x) = x^n$, $n$ is a positive integer, then

$$\begin{aligned}\lim_{x\to c} x^n &= \lim_{x\to c}(xx^{n-1})\\ &= \left[\lim_{x\to c} x\right]\left[\lim_{x\to c} x^{n-1}\right] = c\left[\lim_{x\to c}(xx^{n-2})\right]\\ &= c\left[\lim_{x\to c} x\right]\left[\lim_{x\to c} x^{n-2}\right] = c(c)\lim_{x\to c}(xx^{n-3})\\ &= \cdots = c^n.\end{aligned}$$

**109.** If $b = 0$, the property is true because both sides are equal to 0. If $b \neq 0$, let $\varepsilon > 0$ be given. Because $\lim_{x\to c} f(x) = L$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon/|b|$ whenever $0 < |x - c| < \delta$. So, whenever $0 < |x - c| < \delta$, we have

$$|b||f(x) - L| < \varepsilon \quad \text{or} \quad |bf(x) - bL| < \varepsilon$$

which implies that $\lim_{x\to c}[bf(x)] = bL$.

**110.** Given $\lim_{x\to c} f(x) = 0$:

For every $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - 0| < \varepsilon$ whenever $0 < |x - c| < \delta$.

Now $|f(x) - 0| = |f(x)| = ||f(x)| - 0| < \varepsilon$ for $|x - c| < \delta$. Therefore, $\lim_{x\to c}|f(x)| = 0$.

**111.**

$$\begin{aligned}-M|f(x)| &\le f(x)g(x) \le M|f(x)|\\ \lim_{x\to c}(-M|f(x)|) &\le \lim_{x\to c} f(x)g(x) \le \lim_{x\to c}(M|f(x)|)\\ -M(0) &\le \lim_{x\to c} f(x)g(x) \le M(0)\\ 0 &\le \lim_{x\to c} f(x)g(x) \le 0\end{aligned}$$

Therefore, $\lim_{x\to c} f(x)g(x) = 0$.

**112.** (a) If $\lim_{x\to c}|f(x)| = 0$, then $\lim_{x\to c}[-|f(x)|] = 0$.

$$\begin{aligned}-|f(x)| &\le f(x) \le |f(x)|\\ \lim_{x\to c}[-|f(x)|] &\le \lim_{x\to c} f(x) \le \lim_{x\to c}|f(x)|\\ 0 &\le \lim_{x\to c} f(x) \le 0\end{aligned}$$

Therefore, $\lim_{x\to c} f(x) = 0$.

(b) Given $\lim_{x\to c} f(x) = L$:

For every $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$. Since $||f(x)| - |L|| \le |f(x) - L| < \varepsilon$ for $|x - c| < \delta$, then $\lim_{x\to c}|f(x)| = |L|$.

**113.** Let

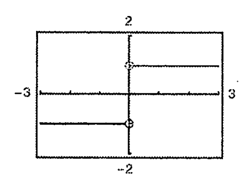
$$f(x) = \begin{cases} 4, & \text{if } x \ge 0\\ -4, & \text{if } x < 0\end{cases}$$

$\lim_{x\to 0}|f(x)| = \lim_{x\to 0} 4 = 4$.

$\lim_{x\to 0} f(x)$ does not exist because for $x < 0$, $f(x) = -4$ and for $x \ge 0$, $f(x) = 4$.

**114.** The graphing utility was set in degree mode, instead of *radian* mode.

**115.** The limit does not exist because the function approaches 1 from the right side of 0 and approaches $-1$ from the left side of 0.




**116.** False. $\lim_{x\to\pi}\dfrac{\sin x}{x} = \dfrac{0}{\pi} = 0$

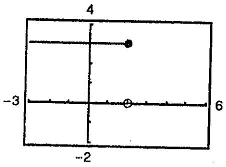
**117.** True.

**118.** False. Let

$$f(x) = \begin{cases} x & x \neq 1\\ 3 & x = 1\end{cases}, \quad c = 1.$$

Then $\lim_{x\to 1} f(x) = 1$ but $f(1) \neq 1$.

**119.** False. The limit does not exist because $f(x)$ approaches 3 from the left side of 2 and approaches 0 from the right side of 2.




**120.** False. Let $f(x) = \frac{1}{2}x^2$ and $g(x) = x^2$.

Then $f(x) < g(x)$ for all $x \neq 0$. But

$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0.$

**121.**
$$\lim_{x\to 0}\frac{1-\cos x}{x} = \lim_{x\to 0}\frac{1-\cos x}{x}\cdot\frac{1+\cos x}{1+\cos x}$$
$$= \lim_{x\to 0}\frac{1-\cos^2 x}{x(1+\cos x)} = \lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)}$$
$$= \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\sin x}{1+\cos x}$$
$$= \left[\lim_{x\to 0}\frac{\sin x}{x}\right]\left[\lim_{x\to 0}\frac{\sin x}{1+\cos x}\right]$$
$$= (1)(0) = 0$$

**122.**
$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$$
$$g(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ x, & \text{if } x \text{ is irrational} \end{cases}$$

$\lim_{x\to 0} f(x)$ does not exist.

No matter how "close to" 0 $x$ is, there are still an infinite number of rational and irrational numbers so that $\lim_{x\to 0} f(x)$ does not exist.

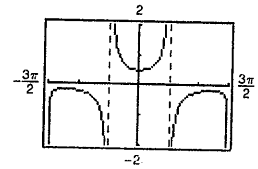
$\lim_{x\to 0} g(x) = 0$

when $x$ is "close to" 0, both parts of the function are "close to" 0.

**123.** $f(x) = \dfrac{\sec x - 1}{x^2}$

(a) The domain of $f$ is all $x \neq 0, \pi/2 + n\pi$.

(b)




The domain is not obvious. The hole at $x = 0$ is not apparent.

(c) $\lim_{x\to 0} f(x) = \dfrac{1}{2}$

(d)
$$\frac{\sec x - 1}{x^2} = \frac{\sec x - 1}{x^2}\cdot\frac{\sec x + 1}{\sec x + 1} = \frac{\sec^2 x - 1}{x^2(\sec x + 1)}$$
$$= \frac{\tan^2 x}{x^2(\sec x + 1)} = \frac{1}{\cos^2 x}\left(\frac{\sin^2 x}{x^2}\right)\frac{1}{\sec x + 1}$$

So, $\lim_{x\to 0}\dfrac{\sec x - 1}{x^2} = \lim_{x\to 0}\dfrac{1}{\cos^2 x}\left(\dfrac{\sin^2 x}{x^2}\right)\dfrac{1}{\sec x + 1}$
$$= 1(1)\left(\frac{1}{2}\right) = \frac{1}{2}.$$

**124.** (a)
$$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{1-\cos x}{x^2}\cdot\frac{1+\cos x}{1+\cos x}$$
$$= \lim_{x\to 0}\frac{1-\cos^2 x}{x^2(1+\cos x)}$$
$$= \lim_{x\to 0}\frac{\sin^2 x}{x^2}\cdot\frac{1}{1+\cos x}$$
$$= (1)\left(\frac{1}{2}\right) = \frac{1}{2}$$

(b) From part (a),
$$\frac{1-\cos x}{x^2} \approx \frac{1}{2} \Rightarrow 1 - \cos x$$
$$\approx \frac{1}{2}x^2 \Rightarrow \cos x$$
$$\approx 1 - \frac{1}{2}x^2 \text{ for } x$$
$$\approx 0.$$

(c) $\cos(0.1) \approx 1 - \frac{1}{2}(0.1)^2 = 0.995$

(d) $\cos(0.1) \approx 0.9950$, which agrees with part (c).