1.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Estimating Limits **In Exercises 1–4, use a graphing utility to graph the function and visually estimate the limits.**

1. $h(x) = -x^2 + 4x$
 (a) $\lim_{x\to 4} h(x)$
 (b) $\lim_{x\to -1} h(x)$

2. $g(x) = \dfrac{12(\sqrt{x} - 3)}{x - 9}$
 (a) $\lim_{x\to 4} g(x)$
 (b) $\lim_{x\to 9} g(x)$

3. $f(x) = x\cos x$
 (a) $\lim_{x\to 0} f(x)$
 (b) $\lim_{x\to \pi/3} f(x)$

4. $f(t) = t|t - 4|$
 (a) $\lim_{t\to 4} f(t)$
 (b) $\lim_{t\to -1} f(t)$

Finding a Limit **In Exercises 5–22, find the limit.**

5. $\lim_{x\to 2} x^3$
6. $\lim_{x\to -3} x^4$
7. $\lim_{x\to 0} (2x - 1)$
8. $\lim_{x\to -4} (2x + 3)$
9. $\lim_{x\to -3} (x^2 + 3x)$
10. $\lim_{x\to 2} (-x^3 + 1)$
11. $\lim_{x\to -3} (2x^2 + 4x + 1)$
12. $\lim_{x\to 1} (2x^3 - 6x + 5)$
13. $\lim_{x\to 3} \sqrt{x + 1}$
14. $\lim_{x\to 2} \sqrt[3]{12x + 3}$
15. $\lim_{x\to -4} (x + 3)^2$
16. $\lim_{x\to 0} (3x - 2)^4$
17. $\lim_{x\to 2} \dfrac{1}{x}$
18. $\lim_{x\to -5} \dfrac{5}{x + 3}$
19. $\lim_{x\to 1} \dfrac{x}{x^2 + 4}$
20. $\lim_{x\to 1} \dfrac{3x + 5}{x + 1}$
21. $\lim_{x\to 7} \dfrac{3x}{\sqrt{x + 2}}$
22. $\lim_{x\to 3} \dfrac{\sqrt{x + 6}}{x + 2}$

Finding Limits **In Exercises 23–26, find the limits.**

23. $f(x) = 5 - x,\ g(x) = x^3$
 (a) $\lim_{x\to 1} f(x)$ (b) $\lim_{x\to 4} g(x)$ (c) $\lim_{x\to 1} g(f(x))$

24. $f(x) = x + 7,\ g(x) = x^2$
 (a) $\lim_{x\to -3} f(x)$ (b) $\lim_{x\to 4} g(x)$ (c) $\lim_{x\to -3} g(f(x))$

25. $f(x) = 4 - x^2,\ g(x) = \sqrt{x + 1}$
 (a) $\lim_{x\to 1} f(x)$ (b) $\lim_{x\to 3} g(x)$ (c) $\lim_{x\to 1} g(f(x))$

26. $f(x) = 2x^2 - 3x + 1,\ g(x) = \sqrt[3]{x + 6}$
 (a) $\lim_{x\to 4} f(x)$ (b) $\lim_{x\to 21} g(x)$ (c) $\lim_{x\to 4} g(f(x))$

Finding a Limit of a Trigonometric Function **In Exercises 27–36, find the limit of the trigonometric function.**

27. $\lim_{x\to \pi/2} \sin x$
28. $\lim_{x\to \pi} \tan x$
29. $\lim_{x\to 1} \cos \dfrac{\pi x}{3}$
30. $\lim_{x\to 2} \sin \dfrac{\pi x}{2}$
31. $\lim_{x\to 0} \sec 2x$
32. $\lim_{x\to \pi} \cos 3x$
33. $\lim_{x\to 5\pi/6} \sin x$
34. $\lim_{x\to 5\pi/3} \cos x$
35. $\lim_{x\to 3} \tan\left(\dfrac{\pi x}{4}\right)$
36. $\lim_{x\to 7} \sec\left(\dfrac{\pi x}{6}\right)$

Evaluating Limits **In Exercises 37–40, use the information to evaluate the limits.**

37. $\lim_{x\to c} f(x) = 3$
 $\lim_{x\to c} g(x) = 2$
 (a) $\lim_{x\to c} [5g(x)]$
 (b) $\lim_{x\to c} [f(x) + g(x)]$
 (c) $\lim_{x\to c} [f(x)g(x)]$
 (d) $\lim_{x\to c} \dfrac{f(x)}{g(x)}$

38. $\lim_{x\to c} f(x) = 2$
 $\lim_{x\to c} g(x) = \frac{3}{4}$
 (a) $\lim_{x\to c} [4f(x)]$
 (b) $\lim_{x\to c} [f(x) + g(x)]$
 (c) $\lim_{x\to c} [f(x)g(x)]$
 (d) $\lim_{x\to c} \dfrac{f(x)}{g(x)}$

39. $\lim_{x\to c} f(x) = 4$
 (a) $\lim_{x\to c} [f(x)]^3$
 (b) $\lim_{x\to c} \sqrt{f(x)}$
 (c) $\lim_{x\to c} [3f(x)]$
 (d) $\lim_{x\to c} [f(x)]^{3/2}$

40. $\lim_{x\to c} f(x) = 27$
 (a) $\lim_{x\to c} \sqrt[3]{f(x)}$
 (b) $\lim_{x\to c} \dfrac{f(x)}{18}$
 (c) $\lim_{x\to c} [f(x)]^2$
 (d) $\lim_{x\to c} [f(x)]^{2/3}$

Finding a Limit **In Exercises 41–46, write a simpler function that agrees with the given function at all but one point. Then find the limit of the function. Use a graphing utility to confirm your result.**

41. $\lim_{x\to 0} \dfrac{x^2 + 3x}{x}$
42. $\lim_{x\to 0} \dfrac{x^4 - 5x^2}{x^2}$
43. $\lim_{x\to -1} \dfrac{x^2 - 1}{x + 1}$
44. $\lim_{x\to -2} \dfrac{3x^2 + 5x - 2}{x + 2}$
45. $\lim_{x\to 2} \dfrac{x^3 - 8}{x - 2}$
46. $\lim_{x\to -1} \dfrac{x^3 + 1}{x + 1}$

Finding a Limit **In Exercises 47–62, find the limit.**

47. $\lim_{x\to 0} \dfrac{x}{x^2 - x}$
48. $\lim_{x\to 0} \dfrac{2x}{x^2 + 4x}$
49. $\lim_{x\to 4} \dfrac{x - 4}{x^2 - 16}$
50. $\lim_{x\to 5} \dfrac{5 - x}{x^2 - 25}$
51. $\lim_{x\to -3} \dfrac{x^2 + x - 6}{x^2 - 9}$
52. $\lim_{x\to 2} \dfrac{x^2 + 2x - 8}{x^2 - x - 2}$
53. $\lim_{x\to 4} \dfrac{\sqrt{x + 5} - 3}{x - 4}$
54. $\lim_{x\to 3} \dfrac{\sqrt{x + 1} - 2}{x - 3}$
55. $\lim_{x\to 0} \dfrac{\sqrt{x + 5} - \sqrt{5}}{x}$
56. $\lim_{x\to 0} \dfrac{\sqrt{2 + x} - \sqrt{2}}{x}$
57. $\lim_{x\to 0} \dfrac{[1/(3 + x)] - (1/3)}{x}$
58. $\lim_{x\to 0} \dfrac{[1/(x + 4)] - (1/4)}{x}$

59. $\lim_{\Delta x\to 0} \dfrac{2(x+\Delta x)-2x}{\Delta x}$ **60.** $\lim_{\Delta x\to 0} \dfrac{(x+\Delta x)^2-x^2}{\Delta x}$

61. $\lim_{\Delta x\to 0} \dfrac{(x+\Delta x)^2-2(x+\Delta x)+1-(x^2-2x+1)}{\Delta x}$

62. $\lim_{\Delta x\to 0} \dfrac{(x+\Delta x)^3-x^3}{\Delta x}$

Finding a Limit of a Trigonometric Function **In Exercises 63–74, find the limit of the trigonometric function.**

63. $\lim_{x\to 0} \dfrac{\sin x}{5x}$ **64.** $\lim_{x\to 0} \dfrac{3(1-\cos x)}{x}$

65. $\lim_{x\to 0} \dfrac{\sin x(1-\cos x)}{x^2}$ **66.** $\lim_{\theta\to 0} \dfrac{\cos\theta\tan\theta}{\theta}$

67. $\lim_{x\to 0} \dfrac{\sin^2 x}{x}$ **68.** $\lim_{x\to 0} \dfrac{\tan^2 x}{x}$

69. $\lim_{h\to 0} \dfrac{(1-\cos h)^2}{h}$ **70.** $\lim_{\phi\to\pi} \phi\sec\phi$

71. $\lim_{x\to\pi/2} \dfrac{\cos x}{\cot x}$ **72.** $\lim_{x\to\pi/4} \dfrac{1-\tan x}{\sin x-\cos x}$

73. $\lim_{t\to 0} \dfrac{\sin 3t}{2t}$

74. $\lim_{x\to 0} \dfrac{\sin 2x}{\sin 3x}$ $\left[\textit{Hint: } \text{Find } \lim_{x\to 0}\left(\dfrac{2\sin 2x}{2x}\right)\left(\dfrac{3x}{3\sin 3x}\right).\right]$

Graphical, Numerical, and Analytic Analysis **In Exercises 75–82, use a graphing utility to graph the function and estimate the limit. Use a table to reinforce your conclusion. Then find the limit by analytic methods.**

75. $\lim_{x\to 0} \dfrac{\sqrt{x+2}-\sqrt{2}}{x}$ **76.** $\lim_{x\to 16} \dfrac{4-\sqrt{x}}{x-16}$

77. $\lim_{x\to 0} \dfrac{[1/(2+x)]-(1/2)}{x}$ **78.** $\lim_{x\to 2} \dfrac{x^5-32}{x-2}$

79. $\lim_{t\to 0} \dfrac{\sin 3t}{t}$ **80.** $\lim_{x\to 0} \dfrac{\cos x-1}{2x^2}$

81. $\lim_{x\to 0} \dfrac{\sin x^2}{x}$ **82.** $\lim_{x\to 0} \dfrac{\sin x}{\sqrt[3]{x}}$

Finding a Limit **In Exercises 83–88, find**

$$\lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}.$$

83. $f(x)=3x-2$ **84.** $f(x)=-6x+3$

85. $f(x)=x^2-4x$ **86.** $f(x)=\sqrt{x}$

87. $f(x)=\dfrac{1}{x+3}$ **88.** $f(x)=\dfrac{1}{x^2}$

Using the Squeeze Theorem **In Exercises 89 and 90, use the Squeeze Theorem to find $\lim_{x\to c} f(x)$.**

89. $c=0$

$4-x^2 \le f(x) \le 4+x^2$

90. $c=a$

$b-|x-a| \le f(x) \le b+|x-a|$

Using the Squeeze Theorem **In Exercises 91–94, use a graphing utility to graph the given function and the equations $y=|x|$ and $y=-|x|$ in the same viewing window. Using the graphs to observe the Squeeze Theorem visually, find $\lim_{x\to 0} f(x)$.**

91. $f(x)=|x|\sin x$ **92.** $f(x)=|x|\cos x$

93. $f(x)=x\sin\dfrac{1}{x}$ **94.** $h(x)=x\cos\dfrac{1}{x}$

WRITING ABOUT CONCEPTS

95. Functions That Agree at All but One Point

(a) In the context of finding limits, discuss what is meant by two functions that agree at all but one point.

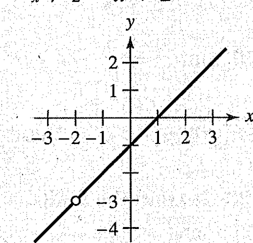
(b) Give an example of two functions that agree at all but one point.

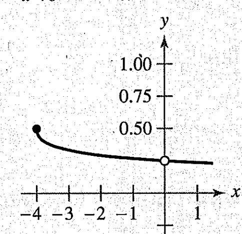
96. Indeterminate Form What is meant by an indeterminate form?

97. Squeeze Theorem In your own words, explain the Squeeze Theorem.

98. HOW DO YOU SEE IT? Would you use the dividing out technique or the rationalizing technique to find the limit of the function? Explain your reasoning.

(a) $\lim_{x\to -2} \dfrac{x^2+x-2}{x+2}$ (b) $\lim_{x\to 0} \dfrac{\sqrt{x+4}-2}{x}$





99. Writing Use a graphing utility to graph

$$f(x)=x,\quad g(x)=\sin x,\quad \text{and}\quad h(x)=\frac{\sin x}{x}$$

in the same viewing window. Compare the magnitudes of $f(x)$ and $g(x)$ when x is close to 0. Use the comparison to write a short paragraph explaining why

$\lim_{x\to 0} h(x)=1.$

100. Writing Use a graphing utility to graph

$$f(x)=x,\quad g(x)=\sin^2 x,\quad \text{and}\quad h(x)=\frac{\sin^2 x}{x}$$

in the same viewing window. Compare the magnitudes of $f(x)$ and $g(x)$ when x is close to 0. Use the comparison to write a short paragraph explaining why

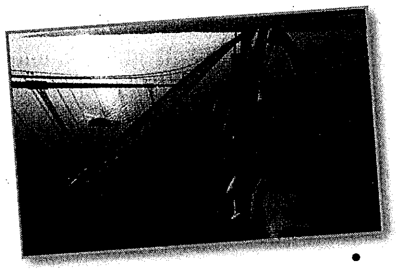
$\lim_{x\to 0} h(x)=0.$

Free-Falling Object **In Exercises 101 and 102, use the position function $s(t) = -16t^2 + 500$, which gives the height (in feet) of an object that has fallen for t seconds from a height of 500 feet. The velocity at time $t = a$ seconds is given by**

$$\lim_{t \to a} \frac{s(a) - s(t)}{a - t}.$$

101. A construction worker drops a full paint can from a height of 500 feet. How fast will the paint can be falling after 2 seconds?

102. A construction worker drops a full paint can from a height of 500 feet. When will the paint can hit the ground? At what velocity will the paint can impact the ground?



Free-Falling Object **In Exercises 103 and 104, use the position function $s(t) = -4.9t^2 + 200$, which gives the height (in meters) of an object that has fallen for t seconds from a height of 200 meters. The velocity at time $t = a$ seconds is given by**

$$\lim_{t \to a} \frac{s(a) - s(t)}{a - t}.$$

103. Find the velocity of the object when $t = 3$.

104. At what velocity will the object impact the ground?

105. Finding Functions Find two functions f and g such that $\lim_{x \to 0} f(x)$ and $\lim_{x \to 0} g(x)$ do not exist, but

$$\lim_{x \to 0} [f(x) + g(x)]$$

does exist.

106. Proof Prove that if $\lim_{x \to c} f(x)$ exists and $\lim_{x \to c} [f(x) + g(x)]$ does not exist, then $\lim_{x \to c} g(x)$ does not exist.

107. Proof Prove Property 1 of Theorem 1.1.

108. Proof Prove Property 3 of Theorem 1.1. (You may use Property 3 of Theorem 1.2.)

109. Proof Prove Property 1 of Theorem 1.2.

110. Proof Prove that if $\lim_{x \to c} f(x) = 0$, then $\lim_{x \to c} |f(x)| = 0$.

111. Proof Prove that if $\lim_{x \to c} f(x) = 0$ and $|g(x)| \le M$ for a fixed number M and all $x \neq c$, then $\lim_{x \to c} f(x)g(x) = 0$.

112. Proof

(a) Prove that if $\lim_{x \to c} |f(x)| = 0$, then $\lim_{x \to c} f(x) = 0$.

(*Note:* This is the converse of Exercise 110.)

(b) Prove that if $\lim_{x \to c} f(x) = L$, then $\lim_{x \to c} |f(x)| = |L|$.

[*Hint:* Use the inequality $\big||f(x)| - |L|\big| \le |f(x) - L|$.]

113. Think About It Find a function f to show that the converse of Exercise 112(b) is not true. [*Hint:* Find a function f such that $\lim_{x \to c} |f(x)| = |L|$ but $\lim_{x \to c} f(x)$ does not exist.]

114. Think About It When using a graphing utility to generate a table to approximate

$$\lim_{x \to 0} \frac{\sin x}{x}$$

a student concluded that the limit was 0.01745 rather than 1. Determine the probable cause of the error.

True or False? **In Exercises 115–120, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

115. $\lim_{x \to 0} \frac{|x|}{x} = 1$

116. $\lim_{x \to \pi} \frac{\sin x}{x} = 1$

117. If $f(x) = g(x)$ for all real numbers other than $x = 0$, and $\lim_{x \to 0} f(x) = L$, then $\lim_{x \to 0} g(x) = L$.

118. If $\lim_{x \to c} f(x) = L$, then $f(c) = L$.

119. $\lim_{x \to 2} f(x) = 3$, where $f(x) = \begin{cases} 3, & x \le 2 \\ 0, & x > 2 \end{cases}$

120. If $f(x) < g(x)$ for all $x \neq a$, then $\lim_{x \to a} f(x) < \lim_{x \to a} g(x)$.

121. Proof Prove the second part of Theorem 1.9.

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

122. Piecewise Functions Let

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$$

and

$$g(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ x, & \text{if } x \text{ is irrational} \end{cases}.$$

Find (if possible) $\lim_{x \to 0} f(x)$ and $\lim_{x \to 0} g(x)$.

123. Graphical Reasoning Consider $f(x) = \dfrac{\sec x - 1}{x^2}$.

(a) Find the domain of f.

(b) Use a graphing utility to graph f. Is the domain of f obvious from the graph? If not, explain.

(c) Use the graph of f to approximate $\lim_{x \to 0} f(x)$.

(d) Confirm your answer to part (c) analytically.

124. Approximation

(a) Find $\lim_{x \to 0} \dfrac{1 - \cos x}{x^2}$.

(b) Use your answer to part (a) to derive the approximation $\cos x \approx 1 - \frac{1}{2}x^2$ for x near 0.

(c) Use your answer to part (b) to approximate $\cos(0.1)$.

(d) Use a calculator to approximate $\cos(0.1)$ to four decimal places. Compare the result with part (c).

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