Key

Ch. 1.3b (More) Evaluating Limits Algebraically

Recap Steps: **Simplify/Reduction Method** (Factor/Simplify/Substitute) Steps:

1) Evaluate argument First! (plug in value into expression)
2) If direct substitution produces $\frac{0}{0}$ (indeterminate form), we need to evaluate further

*Note: $\frac{0}{0}$ does not mean the Limit is Undefined. $\frac{0}{0}$ just means our problem is incomplete and unfinished. (It's true that the function value is undefined because there's a hole in the graph, but the limit most times does still exist)

3) **Factor/Reduce/Simplify: Try finding common factors in order to reduce expression**
4) Using the Reduced expression, re-evaluate the limit
5) Confirm resulting value is now a Real Number, therefore the limit (answer)

I. Simplify using conjugate method

- If there is a sum or difference of 2 terms in the numerator, then multiply the numerator and denominator by the **conjugate** term.

*leave the denominator in factored form, unexpanded.

Example 1: $\lim_{x\to 4} \frac{6-\sqrt{x+32}}{x-4} \to \frac{6-\sqrt{4+32}}{4-4} \to \frac{0}{0}$

$$\lim_{x\to 4} \frac{(6-\sqrt{x+32})}{(x-4)} \cdot \frac{(6+\sqrt{x+32})}{(6+\sqrt{x+32})} \to \lim_{x\to 4} \frac{36-(x+32)}{(x-4)(6+\sqrt{x+32})}$$

$$\lim_{x\to 4} \frac{36-x-32}{(x-4)(6+\sqrt{x+32})} \qquad 4-x$$

$$\lim_{x\to 4} \frac{-1\cancel{(x-4)}}{\cancel{(x-4)}(6+\sqrt{x+32})}$$

$$\frac{-1}{6+\sqrt{4+32}} = \frac{-1}{6+\sqrt{36}} = \boxed{\frac{-1}{12}}$$

II. Simplify by finding Common Denominator

Example 2: $\lim_{x\to 0} \frac{\frac{1}{x+4}-\frac{1}{4}}{x}$

$$\lim_{x\to 0} \frac{\frac{1}{0+4}-\frac{1}{4}}{0} \to \frac{0}{0}$$

$$\lim_{x\to 0} \frac{\left(\frac{1}{x+4}-\frac{1}{4}\right)}{x} \cdot \frac{4(x+4)}{4(x+4)} \to \frac{\frac{1}{\cancel{x+4}}\cdot\frac{4\cancel{(x+4)}}{1} - \frac{1}{\cancel{4}}\cdot\frac{\cancel{4}(x+4)}{1}}{}$$

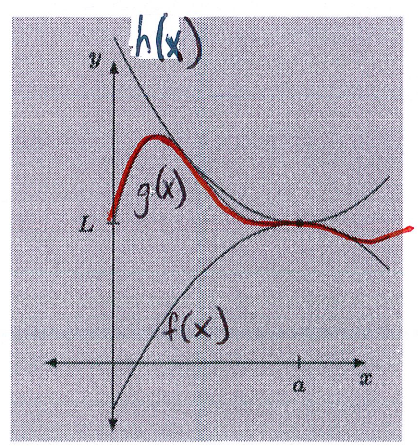
$$\lim_{x\to 0} \frac{4-(x+4)}{4(x+4)\cdot x} \to \frac{4-x-4}{4x(x+4)}$$

$$\lim_{x\to 0} \frac{-\cancel{x}}{4\cancel{x}(x+4)} = \frac{-1}{4(0+4)} = \frac{-1}{4(4)} = \boxed{\frac{-1}{16}}$$

III. Squeeze Theorem

In the graph below, the lower and upper functions have the same limit value at x=a. The middle function has the same limit value because it is trapped between the two outer functions.

The middle function is "squeezed" to Limit **L** as x approaches **a**



Definition: *Suppose* $f(x) \le g(x) \le h(x)$ *for all x in an open interval*

Suppose that $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$ *Then,* $\lim_{x\to a} g(x) = L$

Example 3: Let $h(x) = 1$, $f(x) = x^2 + 1$. *If* $f(x) \le g(x) \le h(x)$ *find* $\lim_{x\to 0} g(x)$

$$\lim_{x\to 0} x^2+1 \le \lim_{x\to 0} g(x) \le \lim_{x\to 0} 1$$

$$1 \le \lim_{x\to 0} g(x) \le 1$$

By squeeze theorem

$$\boxed{\lim_{x\to 0} g(x) = 1}$$

1.3b Practice Problems:

Simplify/Reduction Method (Factor/Simplify/Substitute) Steps:

a) Evaluate argument First! (plug in value into expression)
b) If direct substitution produces $\frac{0}{0}$ (indeterminate form), we need to evaluate further
c) Factor/Reduce/Simplify: Try finding common factors in order to reduce expression
d) Using the Reduced expression, re-evaluate the limit
e) Confirm resulting value is now a Real Number, therefore the limit (answer)

1)
$$\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4} \qquad \frac{2-2}{4-4}\to\frac{0}{0}$$

$$\lim_{x\to 4}\frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x-4)(\sqrt{x}+2)}$$

$$\lim_{x\to 4}\frac{\cancel{x-4}}{(\cancel{x-4})(\sqrt{x}+2)}$$

$$\lim_{x\to 4}\frac{1}{\sqrt{x}+2}=\frac{1}{\sqrt{4}+2}=\boxed{\frac{1}{4}}$$

2)
$$\lim_{x\to 0}\frac{\frac{1}{3+x}-\frac{1}{3}}{x}\to\frac{0}{0}\to\frac{\frac{3}{3(3+x)}-\frac{3+x}{3(3+x)}}{x}$$

$$\lim_{x\to 0}\frac{\frac{3-3-x}{3(3+x)}}{x}\cdot\frac{\frac{1}{x}}{\frac{1}{x}}\to\lim_{x\to 0}\frac{-\cancel{x}}{3(3+x)}\cdot\frac{1}{\cancel{x}}\to\lim_{x\to 0}\frac{-1}{3(3+x)}$$

$$\frac{-1}{3(3+0)}=\boxed{-\frac{1}{9}}$$

3)
$$\lim_{x\to 0}\frac{x^4-5x^2}{x^2}\to\frac{0}{0}$$

$$\lim_{x\to 0}\frac{\cancel{x^2}(x^2-5)}{\cancel{x^2}}\to 0^2-5=\boxed{-5}$$

4)
$$\lim_{x\to 16}\frac{4-\sqrt{x}}{x-16}\to\frac{0}{0}$$

$$\lim_{x\to 16}\frac{4-\sqrt{x}}{x-16}\cdot\frac{(4+\sqrt{x})}{(4+\sqrt{x})}$$

$$\lim_{x\to 16}\frac{16-x}{(x-16)(4+\sqrt{x})}$$

$$\lim_{x\to 16}\frac{-1(\cancel{x-16})}{(\cancel{x-16})(4+\sqrt{x})}$$

$$\frac{-1}{4+\sqrt{16}}=\frac{-1}{4+4}=\boxed{\frac{-1}{8}}$$

5)
$$\lim_{x\to 4}\frac{x^2+7x-44}{x^2-6x+8}\to\frac{16+28-44}{16-24+8}\to\frac{0}{0}$$

$$\lim_{x\to 4}\frac{(x+11)(\cancel{x-4})}{(\cancel{x-4})(x-2)}\to\frac{4+11}{4-2}=\boxed{\frac{15}{2}}$$

6)
$$\lim_{x\to 0}\frac{\sqrt{1+2x}-1}{3x}\to\frac{0}{0}\cdot\frac{(\sqrt{1+2x}+1)}{(\sqrt{1+2x}+1)}$$

$$\lim_{x\to 0}\frac{1+2x-1}{3x(\sqrt{1+2x}+1)}$$

$$\lim_{x\to 0}\frac{2\cancel{x}}{3\cancel{x}(\sqrt{1+2x}+1)}$$

$$\frac{2}{3(\sqrt{1+0}+1)}=\frac{2}{3(2)}=\boxed{\frac{1}{3}}$$

7)
$$\lim_{x\to 0}\frac{\frac{1}{4}+\frac{1}{x-4}}{x}\to\frac{0}{0}\qquad\frac{\frac{x-4}{4(x-4)}+\frac{4}{4(x-4)}}{x}$$

$$\lim_{x\to 0}\frac{\frac{x-4+4}{4(x-4)}}{x}\cdot\frac{\frac{1}{x}}{\frac{1}{x}}\to\lim_{x\to 0}\frac{\cancel{x}}{4(x-4)}\cdot\frac{1}{\cancel{x}}$$

$$\lim_{x\to 0}\frac{1}{4(x-4)}=\frac{1}{4(0-4)}=\boxed{\frac{-1}{16}}$$

8)
$$\lim_{x\to 5}\frac{\sqrt{x-1}-2}{x^2-25}\to\frac{2-2}{25-25}=\frac{0}{0}$$

$$\lim_{x\to 5}\frac{\sqrt{x-1}-2}{x^2-25}\cdot\frac{(\sqrt{x-1}+2)}{\sqrt{x-1}+2)}$$

$$\lim_{x\to 5}\frac{x-1-4}{(x^2-25)(\sqrt{x-1}+2)}$$

$$\lim_{x\to 5}\frac{\cancel{x-5}}{(\cancel{x-5})(x+5)(\sqrt{x-1}+2)}$$

$$\lim_{x\to 5}\frac{1}{(x+5)(\sqrt{x-1}+2)}$$

$$\frac{1}{(5+5)(2+2)}$$

$$\frac{1}{10(4)}=\boxed{\frac{1}{40}}$$