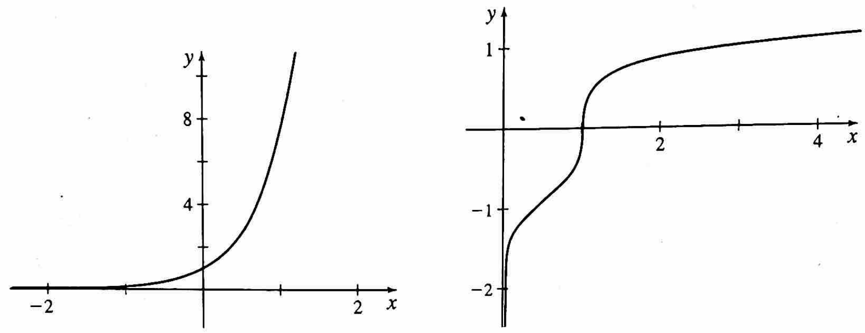
Figures 47(a) and 47(b) illustrate the graphs of f and F.



(a) $f(x) = e^{2x}$ (b) $F(x) = \sqrt[3]{\ln x}$

Figure 47

NOW WORK Problem 45 and AP® Practice Problems 3, 5, and 9.

Summary Basic Limits

- $\lim_{x \to c} A = A$, where A is a constant
- $\lim_{x \to c} x = c$
- $\lim_{x \to 0} \sin x = 0$
- $\lim_{x \to 0} \cos x = 1$
- $\lim_{x \to c} \sin x = \sin c$
- $\lim_{x \to c} \cos x = \cos c$
- $\lim_{\theta \to 0} \dfrac{\sin \theta}{\theta} = 1$
- $\lim_{\theta \to 0} \dfrac{\cos \theta - 1}{\theta} = 0$
- $\lim_{x \to c} a^x = a^c,\ a > 0,\ a \neq 1$
- $\lim_{x \to c} \log_a x = \log_a c,\ \ c > 0,\ a > 0,\ a \neq 1$

1.4 Assess Your Understanding

Concepts and Vocabulary

1. $\lim_{x \to 0} \sin x =$ ________

2. *True or False* $\lim_{x \to 0} \dfrac{\cos x - 1}{x} = 1$

3. The Squeeze Theorem states that if the functions f, g, and h have the property $f(x) \le g(x) \le h(x)$ for all x in an open interval containing c, except possibly at c, and if $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$, then $\lim_{x \to c} g(x) =$ ________.

4. *True or False* $f(x) = \csc x$ is continuous for all real numbers except $x = 0$.

Skill Building

In Problems 5–8, use the Squeeze Theorem to find each limit.

5. [PAGE 118] Suppose $-x^2 + 1 \le g(x) \le x^2 + 1$ for all x in an open interval containing 0. Find $\lim_{x \to 0} g(x)$.

6. Suppose $-(x-2)^2 - 3 \le g(x) \le (x-2)^2 - 3$ for all x in an open interval containing 2. Find $\lim_{x \to 2} g(x)$.

7. Suppose $\cos x \le g(x) \le 1$ for all x in an open interval containing 0. Find $\lim_{x \to 0} g(x)$.

8. Suppose $-x^2 + 1 \le g(x) \le \sec x$ for all x in an open interval containing 0. Find $\lim_{x \to 0} g(x)$.

In Problems 9–22, find each limit.

9. $\lim_{x \to 0} (x^3 + \sin x)$

10. $\lim_{x \to 0} (x^2 - \cos x)$

11. $\lim_{x \to \pi/3} (\cos x + \sin x)$

12. $\lim_{x \to \pi/3} (\sin x - \cos x)$

13. $\lim_{x \to 0} \dfrac{\cos x}{1 + \sin x}$

14. $\lim_{x \to 0} \dfrac{\sin x}{1 + \cos x}$

15. $\lim_{x \to 0} \dfrac{3}{1 + e^x}$

16. $\lim_{x \to 0} \dfrac{e^x - 1}{1 + e^x}$

17. $\lim_{x \to 0} (e^x \sin x)$

18. $\lim_{x \to 0} (e^{-x} \tan x)$

19. $\lim_{x \to 1} \ln \left(\dfrac{e^x}{x} \right)$

20. $\lim_{x \to 1} \ln \left(\dfrac{x}{e^x} \right)$

21. $\lim_{x \to 0} \dfrac{e^{2x}}{1 + e^x}$

22. $\lim_{x \to 0} \dfrac{1 - e^x}{1 - e^{2x}}$

In Problems 23–34, find each limit.

23. [PAGE 121] $\lim_{x \to 0} \dfrac{\sin(7x)}{x}$

24. $\lim_{x \to 0} \dfrac{\sin \frac{x}{3}}{x}$

25. [PAGE 121] $\lim_{\theta \to 0} \dfrac{\theta + 3 \sin \theta}{2\theta}$

26. $\lim_{x \to 0} \dfrac{2x - 5 \sin(3x)}{x}$

27. $\lim_{\theta \to 0} \dfrac{\sin \theta}{\theta + \tan \theta}$

28. $\lim_{\theta \to 0} \dfrac{\tan \theta}{\theta}$

29. $\lim_{\theta \to 0} \dfrac{5}{\theta \cdot \csc \theta}$

30. $\lim_{\theta \to 0} \dfrac{\sin(3\theta)}{\sin(2\theta)}$

31. [PAGE 122] $\lim_{\theta \to 0} \dfrac{1 - \cos^2 \theta}{\theta}$

32. $\lim_{\theta \to 0} \dfrac{\cos(4\theta) - 1}{2\theta}$

33. $\lim_{\theta \to 0} (\theta \cdot \cot \theta)$

34. $\lim_{\theta \to 0} \left[\sin \theta \cdot \dfrac{\cot \theta - \csc \theta}{\theta} \right]$

In Problems 35–38, determine whether f is continuous at the number c.

35. $f(x)=\begin{cases} 3\cos x & \text{if } x<0 \\ 3 & \text{if } x=0 \\ x+3 & \text{if } x>0 \end{cases}$ at $c=0$

36. $f(x)=\begin{cases} \cos x & \text{if } x<0 \\ 0 & \text{if } x=0 \\ e^x & \text{if } x>0 \end{cases}$ at $c=0$

37. $f(\theta)=\begin{cases} \sin\theta & \text{if } \theta\le\frac{\pi}{4} \\ \cos\theta & \text{if } \theta>\frac{\pi}{4} \end{cases}$ at $c=\frac{\pi}{4}$

38. $f(x)=\begin{cases} \tan^{-1}x & \text{if } x<1 \\ \ln x & \text{if } x\ge 1 \end{cases}$ at $c=1$

In Problems 39–46, determine where f is continuous.

39. $f(x)=\sin\left(\dfrac{x^2-4x}{x-4}\right)$

40. $f(x)=\cos\left(\dfrac{x^2-5x+1}{2x}\right)$

41. $f(\theta)=\dfrac{1}{1+\sin\theta}$

42. $f(\theta)=\dfrac{1}{1+\cos^2\theta}$

43. $f(x)=\dfrac{\ln x}{x-3}$

44. $f(x)=\ln(x^2+1)$

PAGE 125 **45.** $f(x)=e^{-x}\sin x$

46. $f(x)=\dfrac{e^x}{1+\sin^2 x}$

Applications and Extensions

In Problems 47–50, use the Squeeze Theorem to find each limit.

PAGE 118 **47.** $\lim\limits_{x\to 0}\left(x^2\sin\dfrac{1}{x}\right)$

48. $\lim\limits_{x\to 0}\left[x\left(1-\cos\dfrac{1}{x}\right)\right]$

49. $\lim\limits_{x\to 0}\left[x^2\left(1-\cos\dfrac{1}{x}\right)\right]$

50. $\lim\limits_{x\to 0}\left[\sqrt{x^3+3x^2}\sin\dfrac{1}{x}\right]$

In Problems 51–54, show that each statement is true.

51. $\lim\limits_{x\to 0}\dfrac{\sin(ax)}{\sin(bx)}=\dfrac{a}{b};\quad b\neq 0$

52. $\lim\limits_{x\to 0}\dfrac{\cos(ax)}{\cos(bx)}=1$

53. $\lim\limits_{x\to 0}\dfrac{\sin(ax)}{bx}=\dfrac{a}{b};\quad b\neq 0$

54. $\lim\limits_{x\to 0}\dfrac{1-\cos(ax)}{bx}=0;\quad a\neq 0, b\neq 0$

55. Show that $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}=\dfrac{1}{2}$.

56. Squeeze Theorem If $0\le f(x)\le 1$ for every number x, show that $\lim\limits_{x\to 0}[x^2 f(x)]=0$.

57. Squeeze Theorem If $0\le f(x)\le M$ for every x, show that $\lim\limits_{x\to 0}[x^2 f(x)]=0$.

58. The function $f(x)=\dfrac{\sin(\pi x)}{x}$ is not defined at 0. Decide how to define $f(0)$ so that f is continuous at 0.

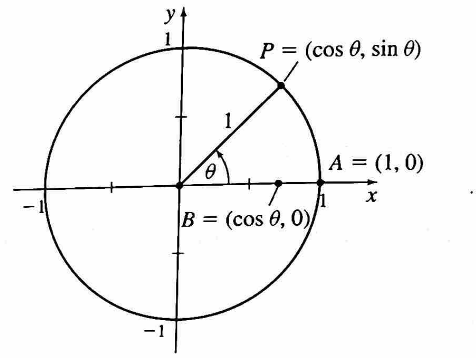
59. Define $f(0)$ and $f(1)$ so that the function $f(x)=\dfrac{\sin(\pi x)}{x(1-x)}$ is continuous on the interval $[0, 1]$.

60. Is $f(x)=\begin{cases} \dfrac{\sin x}{x} & \text{if } x\neq 0 \\ 1 & \text{if } x=0 \end{cases}$ continuous at 0?

61. Is $f(x)=\begin{cases} \dfrac{1-\cos x}{x} & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$ continuous at 0?

62. Squeeze Theorem Show that $\lim\limits_{x\to 0}\left[x^n\sin\left(\dfrac{1}{x}\right)\right]=0$, where n is a positive integer. *Hint:* Look first at Problem 56.

63. Prove $\lim\limits_{\theta\to 0}\sin\theta=0$. *Hint:* Use a unit circle as shown in the figure, first assuming $0<\theta<\dfrac{\pi}{2}$. Then use the fact that $\sin\theta$ is less than the length of the arc AP, and the Squeeze Theorem, to show that $\lim\limits_{\theta\to 0^+}\sin\theta=0$. Then use a similar argument with $-\dfrac{\pi}{2}<\theta<0$ to show $\lim\limits_{\theta\to 0^-}\sin\theta=0$.



64. Prove $\lim\limits_{\theta\to 0}\cos\theta=1$. Use either the proof outlined in Problem 63 or a proof using the result $\lim\limits_{\theta\to 0}\sin\theta=0$ and a Pythagorean identity.

65. Without using limits, explain how you can decide whether $f(x)=\cos(5x^3+2x^2-8x+1)$ is continuous.

66. Explain the Squeeze Theorem. Draw a graph to illustrate your explanation.

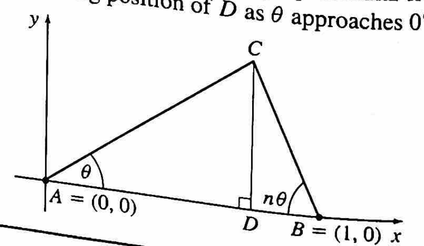
Challenge Problems

67. Use the Sum Formulas $\sin(a+b)=\sin a\cos b+\cos a\sin b$ and $\cos(a+b)=\cos a\cos b-\sin a\sin b$ to show that the sine function and cosine function are continuous on their domains.

68. Find $\lim\limits_{x\to 0}\dfrac{\sin x^2}{x}$.

69. Squeeze Theorem If $f(x)=\begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$ show that $\lim\limits_{x\to 0}[xf(x)]=0$.

70. Suppose points A and B with coordinates $(0, 0)$ and $(1, 0)$, respectively, are given. Let n be a number greater than 0, and let θ be an angle with the property $0<\theta<\dfrac{\pi}{1+n}$. Construct a triangle ABC where $\overline{AC}$ and $\overline{AB}$ form the angle θ, and $\overline{BA}$ and $\overline{BC}$ form the angle $n\theta$ (see the figure below). Let D be the point of intersection of $\overline{AB}$ with the perpendicular from C to $\overline{AB}$. What is the limiting position of D as θ approaches 0?



Preparing for the AP® Exam

AP® Practice Problems

PAGE 121 1. The function $g(x)=\begin{cases}\dfrac{\sin(2x)}{2x} & \text{if } x\neq 0\\ k & \text{if } x=0\end{cases}$

is continuous at $x=0$. What is the value of k?

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) 2

PAGE 121 2. $\lim_{x\to 0}\dfrac{\sin(4x)}{2x}=$

(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) 2

PAGE 125 3. The function $f(x)=\begin{cases}x^3+2x^2 & \text{if } x\leq -2\\ e^{2x+4} & \text{if } x>-2\end{cases}$.

Find $\lim_{x\to -2} f(x)$ if it exists.

(A) 0 (B) 1 (C) 16 (D) The limit does not exist.

PAGE 121 4. $\lim_{x\to 0}\dfrac{1-\cos^2(3x)}{x^2}=$

(A) 0 (B) 1 (C) 3 (D) 9

PAGE 125 5. Which of the following functions are continuous for all real numbers x?

I. $f(x)=x^{1/3}$

II. $g(x)=\sec x$

III. $h(x)=e^{-x}$

(A) I only (B) I and II only

(C) I and III only (D) I, II, and III

PAGE 121 6. Find $\lim_{x\to 0}\dfrac{1}{x\csc x}$ if it exists.

(A) -1 (B) 0 (C) 1 (D) The limit does not exist.

PAGE 121 7. $\lim_{x\to\pi/3}\dfrac{\sin\left(x-\frac{\pi}{3}\right)}{x-\frac{\pi}{3}}=$

(A) $-\frac{\pi}{3}$ (B) 0 (C) 1 (D) $\frac{\pi}{3}$

PAGE 119 8. $\lim_{x\to 0}\dfrac{1-\cos x}{3\sin^2 x}=$

(A) $\frac{1}{6}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) 1

PAGE 125 9. If $f(x)=\begin{cases}\ln x & \text{if } 0<x<3\\ (2x-3)\ln 3 & \text{if } x\geq 3\end{cases}$,

then $\lim_{x\to 3} f(x)=$

(A) $\ln 3$ (B) 3 (C) $\ln 9$ (D) The limit does not exist.

PAGE 121 10. $\lim_{x\to 0}\dfrac{\tan(2x)}{3x}=$

(A) $\frac{1}{3}$ (B) $\frac{1}{2}$ (C) $\frac{2}{3}$ (D) 2

PAGE 118 11. $\lim_{x\to 0}\left(x^3\sin\frac{1}{x}\right)=$

(A) -1 (B) 0 (C) 1 (D) The limit does not exist.

1.5 Infinite Limits; Limits at Infinity; Asymptotes

OBJECTIVES ***When you finish this section, you should be able to:***

1 Investigate infinite limits (p. 128)

2 Find the vertical asymptotes of a graph (p. 131)

3 Investigate limits at infinity (p. 131)

4 Find the horizontal asymptotes of a graph (p. 138)

5 Find the asymptotes of the graph of a rational function (p. 139)

RECALL The symbols ∞ (infinity) and $-\infty$ (negative infinity) are not numbers. The symbol ∞ expresses unboundedness in the positive direction and $-\infty$ expresses unboundedness in the negative direction.

We have described $\lim_{x\to c} f(x)=L$ by saying if a function f is defined everywhere in an open interval containing c, except possibly at c, then the value $f(x)$ can be made as close as we please to L by choosing numbers x sufficiently close to c. Here c and L are real numbers. In this section, we extend the language of limits to allow c to be ∞ or $-\infty$ (*limits at infinity*) and to allow L to be ∞ or $-\infty$ (*infinite limits*). These limits are useful for locating *asymptotes* that aid in graphing some functions.

We begin with infinite limits.