Section 1.4 Continuity and One-Sided Limits

1. (a) $\lim_{x\to 4^+} f(x) = 3$

(b) $\lim_{x\to 4^-} f(x) = 3$

(c) $\lim_{x\to 4} f(x) = 3$

The function is continuous at $x = 4$ and is continuous on $(-\infty, \infty)$.

2. (a) $\lim_{x\to -2^+} f(x) = -2$

(b) $\lim_{x\to -2^-} f(x) = -2$

(c) $\lim_{x\to -2} f(x) = -2$

The function is continuous at $x = -2$.

3. (a) $\lim_{x\to 3^+} f(x) = 0$

(b) $\lim_{x\to 3^-} f(x) = 0$

(c) $\lim_{x\to 3} f(x) = 0$

The function is NOT continuous at $x = 3$.

4. (a) $\lim_{x\to -3^+} f(x) = 3$

(b) $\lim_{x\to -3^-} f(x) = 3$

(c) $\lim_{x\to -3} f(x) = 3$

The function is NOT continuous at $x = -3$ because $f(-3) = 4 \neq \lim_{x\to -3} f(x)$.

5. (a) $\lim_{x\to 2^+} f(x) = -3$

(b) $\lim_{x\to 2^-} f(x) = 3$

(c) $\lim_{x\to 2} f(x)$ does not exist

The function is NOT continuous at $x = 2$.

6. (a) $\lim_{x\to -1^+} f(x) = 0$

(b) $\lim_{x\to -1^-} f(x) = 2$

(c) $\lim_{x\to -1} f(x)$ does not exist.

The function is NOT continuous at $x = -1$.

7. $\lim_{x\to 8^+} \frac{1}{x+8} = \frac{1}{8+8} = \frac{1}{16}$

8. $\lim_{x\to 2^-} \frac{2}{x+2} = \frac{2}{2+2} = \frac{1}{2}$

9. $$\lim_{x\to 5^+} \frac{x-5}{x^2-25} = \lim_{x\to 5^+} \frac{x-5}{(x+5)(x-5)} = \lim_{x\to 5^+} \frac{1}{x+5} = \frac{1}{10}$$

10. $$\lim_{x\to 4^+} \frac{4-x}{x^2-16} = \lim_{x\to 4^+} \frac{-(x-4)}{(x+4)(x-4)} = \lim_{x\to 4^+} \frac{-1}{x+4} = \frac{-1}{4+4} = -\frac{1}{8}$$

11. $\lim_{x\to -3^-} \frac{x}{\sqrt{x^2-9}}$ does not exist because $\frac{x}{\sqrt{x^2-9}}$ decreases without bound as $x \to -3^-$.

12. $$\lim_{x\to 4^-} \frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4^-} \frac{\sqrt{x}-2}{x-4} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} = \lim_{x\to 4^-} \frac{x-4}{(x-4)(\sqrt{x}+2)} = \lim_{x\to 4^-} \frac{1}{\sqrt{x}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}$$

13. $\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = -1$

14. $\lim_{x\to 10^+} \frac{|x-10|}{x-10} = \lim_{x\to 10^+} \frac{x-10}{x-10} = 1$

15. $$\lim_{\Delta x\to 0^-} \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x} = \lim_{\Delta x\to 0^-} \frac{x-(x+\Delta x)}{x(x+\Delta x)} \cdot \frac{1}{\Delta x} = \lim_{\Delta x\to 0^-} \frac{-\Delta x}{x(x+\Delta x)} \cdot \frac{1}{\Delta x}$$
$$= \lim_{\Delta x\to 0^-} \frac{-1}{x(x+\Delta x)} = \frac{-1}{x(x+0)} = -\frac{1}{x^2}$$

16. $\lim_{\Delta x \to 0^+} \frac{(x + \Delta x)^2 + (x + \Delta x) - (x^2 + x)}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{x^2 + 2x(\Delta x) + (\Delta x)^2 + x + \Delta x - x^2 - x}{\Delta x}$

$$= \lim_{\Delta x \to 0^+} \frac{2x(\Delta x) + (\Delta x)^2 + \Delta x}{\Delta x}$$
$$= \lim_{\Delta x \to 0^+} (2x + \Delta x + 1)$$
$$= 2x + 0 + 1 = 2x + 1$$

17. $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \frac{x + 2}{2} = \frac{5}{2}$

18. $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (x^2 - 4x + 6) = 9 - 12 + 6 = 3$

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (-x^2 + 4x - 2) = -9 + 12 - 2 = 1$

Since these one-sided limits disagree, $\lim_{x \to 3} f(x)$ does not exist.

19. $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x + 1) = 2$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^3 + 1) = 2$

$\lim_{x \to 1} f(x) = 2$

20. $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (1 - x) = 0$

21. $\lim_{x \to \pi} \cot x$ does not exist because

$\lim_{x \to \pi^+} \cot x$ and $\lim_{x \to \pi^-} \cot x$ do not exist.

22. $\lim_{x \to \pi/2} \sec x$ does not exist because

$\lim_{x \to (\pi/2)^+} \sec x$ and $\lim_{x \to (\pi/2)^-} \sec x$ do not exist.

23. $\lim_{x \to 4^-} (5[\![x]\!] - 7) = 5(3) - 7 = 8$

$([\![x]\!] = 3 \text{ for } 3 \le x < 4)$

24. $\lim_{x \to 2^+} (2x - [\![x]\!]) = 2(2) - 2 = 2$

25. $\lim_{x \to 3} (2 - [\![-x]\!])$ does not exist because

$\lim_{x \to 3^-} (2 - [\![-x]\!]) = 2 - (-3) = 5$

and

$\lim_{x \to 3^+} (2 - [\![-x]\!]) = 2 - (-4) = 6.$

26. $\lim_{x \to 1} \left(1 - \left[\!\!\left[-\frac{x}{2}\right]\!\!\right]\right) = 1 - (-1) = 2$

27. $f(x) = \frac{1}{x^2 - 4}$

has discontinuities at $x = -2$ and $x = 2$ because $f(-2)$ and $f(2)$ are not defined.

28. $f(x) = \frac{x^2 - 1}{x + 1}$

has a discontinuity at $x = -1$ because $f(-1)$ is not defined.

29. $f(x) = \frac{[\![x]\!]}{2} + x$

has discontinuities at each integer k because $\lim_{x \to k^-} f(x) \ne \lim_{x \to k^+} f(x)$.

30. $f(x) = \begin{cases} x, & x < 1 \\ 2, & x = 1 \\ 2x - 1, & x > 1 \end{cases}$ has a discontinuity at

$x = 1$ because $f(1) = 2 \ne \lim_{x \to 1} f(x) = 1$.

31. $g(x) = \sqrt{49 - x^2}$ is continuous on $[-7, 7]$.

32. $f(t) = 3 - \sqrt{9 - t^2}$ is continuous on $[-3, 3]$.

33. $\lim_{x \to 0^-} f(x) = 3 = \lim_{x \to 0^+} f(x)$. f is continuous on $[-1, 4]$.

34. $g(2)$ is not defined. g is continuous on $[-1, 2)$.

35. $f(x) = \frac{6}{x}$ has a nonremovable discontinuity at $x = 0$ because $\lim_{x \to 0} f(x)$ does not exist.

36. $f(x) = \frac{4}{x - 6}$ has a nonremovable discontinuity at $x = 6$ because $\lim_{x \to 6} f(x)$ does not exist.

37. $f(x) = x^2 - 9$ is continuous for all real x.

38. $f(x) = x^2 - 4x + 4$ is continuous for all real x.

39. $f(x) = \frac{1}{4 - x^2} = \frac{1}{(2 - x)(2 + x)}$ has nonremovable discontinuities at $x = \pm 2$ because $\lim_{x \to 2} f(x)$ and $\lim_{x \to -2} f(x)$ do not exist.

40. $f(x) = \frac{1}{x^2 + 1}$ is continuous for all real x.

41. $f(x) = 3x - \cos x$ is continuous for all real x.

42. $f(x) = \cos\frac{\pi x}{2}$ is continuous for all real x.

43. $f(x) = \frac{x}{x^2 - x}$ is not continuous at $x = 0, 1$.

Because $\frac{x}{x^2 - x} = \frac{1}{x - 1}$ for $x \neq 0$, $x = 0$ is a removable discontinuity, whereas $x = 1$ is a nonremovable discontinuity.

44. $f(x) = \frac{x}{x^2 - 4}$ has nonremovable discontinuities at $x = 2$ and $x = -2$ because $\lim_{x \to 2} f(x)$ and $\lim_{x \to -2} f(x)$ do not exist.

45. $f(x) = \frac{x}{x^2 + 1}$ is continuous for all real x.

46. $f(x) = \frac{x - 5}{x^2 - 25} = \frac{x - 5}{(x + 5)(x - 5)}$

has a nonremovable discontinuity at $x = -5$ because $\lim_{x \to -5} f(x)$ does not exist, and has a removable discontinuity at $x = 5$ because

$$\lim_{x \to 5} f(x) = \lim_{x \to 5} \frac{1}{x + 5} = \frac{1}{10}.$$

47. $f(x) = \frac{x + 2}{x^2 - 3x - 10} = \frac{x + 2}{(x + 2)(x - 5)}$

has a nonremovable discontinuity at $x = 5$ because $\lim_{x \to 5} f(x)$ does not exist, and has a removable discontinuity at $x = -2$ because

$$\lim_{x \to -2} f(x) = \lim_{x \to -2} \frac{1}{x - 5} = -\frac{1}{7}.$$

48. $f(x) = \frac{x + 2}{x^2 - x - 6} = \frac{x + 2}{(x - 3)(x + 2)}$

has a nonremovable discontinuity at $x = 3$ because $\lim_{x \to 3} f(x)$ does not exist, and has a removable discontinuity at $x = -2$ because

$$\lim_{x \to -2} f(x) = \lim_{x \to -2} \frac{1}{x - 3} = -\frac{1}{5}.$$

49. $f(x) = \frac{|x + 7|}{x + 7}$

has a nonremovable discontinuity at $x = -7$ because $\lim_{x \to -7} f(x)$ does not exist.

50. $f(x) = \frac{|x - 5|}{x - 5}$

has a nonremovable discontinuity at $x = 5$ because $\lim_{x \to 5} f(x)$ does not exist.

51. $f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$

has a **possible** discontinuity at $x = 1$.

1. $f(1) = 1$

2. $$\left.\begin{aligned} \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} x = 1 \\ \lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} x^2 = 1 \end{aligned}\right\} \lim_{x \to 1} f(x) = 1$$

3. $f(-1) = \lim_{x \to 1} f(x)$

f is continuous at $x = 1$, therefore, f is continuous for all real x.

52. $f(x) = \begin{cases} -2x + 3, & x < 1 \\ x^2, & x \ge 1 \end{cases}$

has a **possible** discontinuity at $x = 1$.

1. $f(1) = 1^2 = 1$

2. $$\left.\begin{aligned} \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} (-2x + 3) = 1 \\ \lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} x^2 = 1 \end{aligned}\right\} \lim_{x \to 1} f(x) = 1$$

3. $f(1) = \lim_{x \to 1} f(x)$

f is continuous at $x = 1$, therefore, f is continuous for all real x.

53. $f(x) = \begin{cases} \dfrac{x}{2} + 1, & x \le 2 \\ 3 - x, & x > 2 \end{cases}$

has a **possible** discontinuity at $x = 2$.

1. $f(2) = \dfrac{2}{2} + 1 = 2$

2. $\left.\begin{array}{l} \lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} \left(\dfrac{x}{2} + 1\right) = 2 \\ \lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (3 - x) = 1 \end{array}\right\}$ $\lim\limits_{x \to 2} f(x)$ does not exist.

Therefore, f has a nonremovable discontinuity at $x = 2$.

54. $f(x) = \begin{cases} -2x, & x \le 2 \\ x^2 - 4x + 1, & x > 2 \end{cases}$

has a **possible** discontinuity at $x = 2$.

1. $f(2) = -2(2) = -4$

2. $\left.\begin{array}{l} \lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (-2x) = -4 \\ \lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} \left(x^2 - 4x + 1\right) = -3 \end{array}\right\}$ $\lim\limits_{x \to 2} f(x)$ does not exist.

Therefore, f has a nonremovable discontinuity at $x = 2$.

55. $f(x) = \begin{cases} \tan\dfrac{\pi x}{4}, & |x| < 1 \\ x, & |x| \ge 1 \end{cases}$

$= \begin{cases} \tan\dfrac{\pi x}{4}, & -1 < x < 1 \\ x, & x \le -1 \text{ or } x \ge 1 \end{cases}$

has **possible** discontinuities at $x = -1, x = 1$.

1. $f(-1) = -1$ $\qquad f(1) = 1$

2. $\lim\limits_{x \to -1} f(x) = -1$ $\qquad \lim\limits_{x \to 1} f(x) = 1$

3. $f(-1) = \lim\limits_{x \to -1} f(x)$ $\qquad f(1) = \lim\limits_{x \to 1} f(x)$

f is continuous at $x = \pm 1$, therefore, f is continuous for all real x.

56. $f(x) = \begin{cases} \csc\dfrac{\pi x}{6}, & |x - 3| \le 2 \\ 2, & |x - 3| > 2 \end{cases}$

$= \begin{cases} \csc\dfrac{\pi x}{6}, & 1 \le x \le 5 \\ 2, & x < 1 \text{ or } x > 5 \end{cases}$

has **possible** discontinuities at $x = 1, x = 5$.

1. $f(1) = \csc\dfrac{\pi}{6} = 2$ $\qquad f(5) = \csc\dfrac{5\pi}{6} = 2$

2. $\lim\limits_{x \to 1} f(x) = 2$ $\qquad \lim\limits_{x \to 5} f(x) = 2$

3. $f(1) = \lim\limits_{x \to 1} f(x)$ $\qquad f(5) = \lim\limits_{x \to 5} f(x)$

f is continuous at $x = 1$ and $x = 5$, therefore, f is continuous for all real x.

57. $f(x) = \csc 2x$ has nonremovable discontinuities at integer multiples of $\pi/2$.

58. $f(x) = \tan\dfrac{\pi x}{2}$ has nonremovable discontinuities at each $2k + 1$, k is an integer.

59. $f(x) = [\![x - 8]\!]$ has nonremovable discontinuities at each integer k.

60. $f(x) = 5 - [\![x]\!]$ has nonremovable discontinuities at each integer k.

61. $f(1) = 3$

Find a so that $\lim_{x\to 1^-}(ax - 4) = 3$

$$a(1) - 4 = 3$$
$$a = 7.$$

62. $f(1) = 3$

Find a so that $\lim_{x\to 1^+}(ax + 5) = 3$

$$a(1) + 5 = 3$$
$$a = -2.$$

63. $f(2) = 8$

Find a so that $\lim_{x\to 2^+} ax^2 = 8 \Rightarrow a = \frac{8}{2^2} = 2.$

64. $\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} \frac{4\sin x}{x} = 4$

$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+}(a - 2x) = a$

Let $a = 4$.

65. Find a and b such that $\lim_{x\to -1^+}(ax + b) = -a + b = 2$ and $\lim_{x\to 3^-}(ax + b) = 3a + b = -2$.

$$\begin{aligned} a - b &= -2 \\ (+)3a + b &= -2 \\ \hline 4a &= -4 \\ a &= -1 \\ b &= 2 + (-1) = 1 \end{aligned}$$

$$f(x) = \begin{cases} 2, & x \le -1 \\ -x + 1, & -1 < x < 3 \\ -2, & x \ge 3 \end{cases}$$

66. $\lim_{x\to a} g(x) = \lim_{x\to a} \frac{x^2 - a^2}{x - a}$

$= \lim_{x\to a}(x + a) = 2a$

Find a such $2a = 8 \Rightarrow a = 4$.

67. $f(g(x)) = (x - 1)^2$

Continuous for all real x

68. $f(g(x)) = 5(x^3) + 1 = 5x^3 + 1$

Continuous for all real x

69. $f(g(x)) = \frac{1}{(x^2 + 5) - 6} = \frac{1}{x^2 - 1}$

Nonremovable discontinuities at $x = \pm 1$

70. $f(g(x)) = \frac{1}{\sqrt{x - 1}}$

Nonremovable discontinuity at $x = 1$; continuous for all $x > 1$

71. $f(g(x)) = \tan\frac{x}{2}$

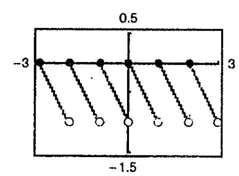
Not continuous at $x = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$ Continuous on the open intervals $\ldots, (-3\pi, -\pi), (-\pi, \pi), (\pi, 3\pi), \ldots$

72. $f(g(x)) = \sin x^2$

Continuous for all real x

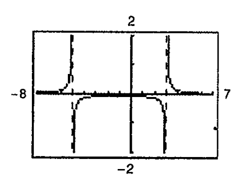
73. $y = [\![x]\!] - x$

Nonremovable discontinuity at each integer



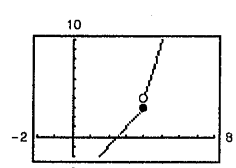
74. $h(x) = \frac{1}{x^2 + 2x - 15} = \frac{1}{(x + 5)(x - 3)}$

Nonremovable discontinuities at $x = -5$ and $x = 3$



75. $g(x) = \begin{cases} x^2 - 3x, & x > 4 \\ 2x - 5, & x \le 4 \end{cases}$

Nonremovable discontinuity at $x = 4$

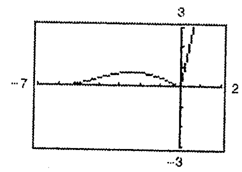


76. $f(x) = \begin{cases} \dfrac{\cos x - 1}{x}, & x < 0 \\ 5x, & x \geq 0 \end{cases}$

$f(0) = 5(0) = 0$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \dfrac{(\cos x - 1)}{x} = 0$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (5x) = 0$



Therefore, $\lim_{x\to 0} f(x) = 0 = f(0)$ and f is continuous on the entire real line.
($x = 0$ was the only possible discontinuity.)

77. $f(x) = \dfrac{x}{x^2 + x + 2}$

Continuous on $(-\infty, \infty)$

78. $f(x) = \dfrac{x + 1}{\sqrt{x}}$

Continuous on $(0, \infty)$

79. $f(x) = 3 - \sqrt{x}$

Continuous on $[0, \infty)$

80. $f(x) = x\sqrt{x + 3}$

Continuous on $[-3, \infty)$

81. $f(x) = \sec\dfrac{\pi x}{4}$

Continuous on:

$\ldots, (-6, -2), (-2, 2), (2, 6), (6, 10), \ldots$

82. $f(x) = \cos\dfrac{1}{x}$

Continuous on $(-\infty, 0)$ and $(0, \infty)$

83. $f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$

Since $\lim_{x\to 1} f(x) = \lim_{x\to 1} \dfrac{x^2 - 1}{x - 1} = \lim_{x\to 1} \dfrac{(x - 1)(x + 1)}{x - 1}$
$= \lim_{x\to 1} (x + 1) = 2,$

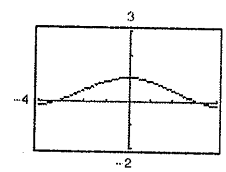
f is continuous on $(-\infty, \infty)$.

84. $f(x) = \begin{cases} 2x - 4, & x \neq 3 \\ 1, & x = 3 \end{cases}$

Since $\lim_{x\to 3} f(x) = \lim_{x\to 3} (2x - 4) = 2 \neq 1$,

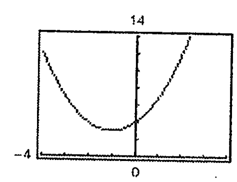
f is continuous on $(-\infty, 3)$ and $(3, \infty)$.

85. $f(x) = \dfrac{\sin x}{x}$



The graph **appears** to be continuous on the interval $[-4, 4]$. Because $f(0)$ is not defined, you know that f has a discontinuity at $x = 0$. This discontinuity is removable so it does not show up on the graph.

86. $f(x) = \dfrac{x^3 - 8}{x - 2}$



The graph **appears** to be continuous on the interval $[-4, 4]$. Because $f(2)$ is not defined, you know that f has a discontinuity at $x = 2$. This discontinuity is removable so it does not show up on the graph.

87. $f(x) = \frac{1}{12}x^4 - x^3 + 4$ is continuous on the interval $[1, 2]$. $f(1) = \frac{37}{12}$ and $f(2) = -\frac{8}{3}$. By the Intermediate Value Theorem, there exists a number c in $[1, 2]$ such that $f(c) = 0$.

88. $f(x) = x^3 + 5x - 3$ is continuous on the interval $[0, 1]$.
$f(0) = -3$ and $f(1) = 3$. By the Intermediate Value Theorem, there exists a number c in $[0, 1]$ such that $f(c) = 0$.

89. $f(x) = x^2 - 2 - \cos x$ is continuous on $[0, \pi]$.
$f(0) = -3$ and $f(\pi) = \pi^2 - 1 \approx 8.87 > 0$. By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of c between 0 and π.

90. $f(x) = -\frac{5}{x} + \tan\left(\frac{\pi x}{10}\right)$ is continuous on the interval $[1, 4]$.
$f(1) = -5 + \tan\left(\frac{\pi}{10}\right) \approx -4.7$ and
$f(4) = -\frac{5}{4} + \tan\left(\frac{2\pi}{5}\right) \approx 1.8$. By the Intermediate Value Theorem, there exists a number c in $[1, 4]$ such that $f(c) = 0$.

91. $f(x) = x^3 + x - 1$
$f(x)$ is continuous on $[0, 1]$.
$f(0) = -1$ and $f(1) = 1$
By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of c between 0 and 1. Using a graphing utility to zoom in on the graph of $f(x)$, you find that $x \approx 0.68$. Using the *root* feature, you find that $x \approx 0.6823$.

92. $f(x) = x^4 - x^2 + 3x - 1$
$f(x)$ is continuous on $[0, 1]$.
$f(0) = -1$ and $f(1) = 2$
By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of c between 0 and 1. Using a graphing utility to zoom in on the graph of $f(x)$, you find that $x \approx 0.37$. Using the *root* feature, you find that $x \approx 0.3733$.

93. $g(t) = 2\cos t - 3t$
g is continuous on $[0, 1]$.
$g(0) = 2 > 0$ and $g(1) \approx -1.9 < 0$.
By the Intermediate Value Theorem, $g(c) = 0$ for at least one value of c between 0 and 1. Using a graphing utility to zoom in on the graph of $g(t)$, you find that $t \approx 0.56$. Using the root feature, you find that $t \approx 0.5636$.

94. $h(\theta) = \tan\theta + 3\theta - 4$ is continuous on $[0, 1]$.
$h(0) = -4$ and $h(1) = \tan(1) - 1 \approx 0.557$.
By the Intermediate Value Theorem, $h(c) = 0$ for at least one value of c between 0 and 1. Using a graphing utility to zoom in on the graph of $h(\theta)$, you find that $\theta \approx 0.91$. Using the *root* feature, you obtain $\theta \approx 0.9071$.

95. $f(x) = x^2 + x - 1$
f is continuous on $[0, 5]$.
$f(0) = -1$ and $f(5) = 29$
$-1 < 11 < 29$
The Intermediate Value Theorem applies.
$$x^2 + x - 1 = 11$$
$$x^2 + x - 12 = 0$$
$$(x + 4)(x - 3) = 0$$
$$x = -4 \text{ or } x = 3$$
$c = 3$ ($x = -4$ is not in the interval.)
So, $f(3) = 11$.

96. $f(x) = x^2 - 6x + 8$
f is continuous on $[0, 3]$.
$f(0) = 8$ and $f(3) = -1$
$-1 < 0 < 8$
The Intermediate Value Theorem applies.
$$x^2 - 6x + 8 = 0$$
$$(x - 2)(x - 4) = 0$$
$$x = 2 \text{ or } x = 4$$
$c = 2$ ($x = 4$ is not in the interval.)
So, $f(2) = 0$.

97. $f(x) = x^3 - x^2 + x - 2$

f is continuous on $[0, 3]$.

$f(0) = -2$ and $f(3) = 19$

$-2 < 4 < 19$

The Intermediate Value Theorem applies.

$$x^3 - x^2 + x - 2 = 4$$
$$x^3 - x^2 + x - 6 = 0$$
$$(x - 2)(x^2 + x + 3) = 0$$
$$x = 2$$

$(x^2 + x + 3$ has no real solution.$)$

$$c = 2$$

So, $f(2) = 4$.

98. $f(x) = \dfrac{x^2 + x}{x - 1}$

f is continuous on $\left[\frac{5}{2}, 4\right]$. The nonremovable discontinuity, $x = 1$, lies outside the interval.

$f\left(\frac{5}{2}\right) = \frac{35}{6}$ and $f(4) = \frac{20}{3}$

$\frac{35}{6} < 6 < \frac{20}{3}$

The Intermediate Value Theorem applies.

$$\frac{x^2 + x}{x - 1} = 6$$
$$x^2 + x = 6x - 6$$
$$x^2 - 5x + 6 = 0$$
$$(x - 2)(x - 3) = 0$$
$$x = 2 \text{ or } x = 3$$

$c = 3$ ($x = 2$ is not in the interval.)

So, $f(3) = 6$.

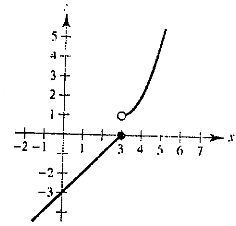
99. (a) The limit does not exist at $x = c$.

(b) The function is not defined at $x = c$.

(c) The limit exists at $x = c$, but it is not equal to the value of the function at $x = c$.

(d) The limit does not exist at $x = c$.

100. Answers will vary. *Sample answer:*



The function is not continuous at $x = 3$ because $\lim_{x\to 3^+} f(x) = 1 \neq 0 = \lim_{x\to 3^-} f(x)$.

101. If f and g are continuous for all real x, then so is $f + g$ (Theorem 1.11, part 2). However, f/g might not be continuous if $g(x) = 0$. For example, let $f(x) = x$ and $g(x) = x^2 - 1$. Then f and g are continuous for all real x, but f/g is not continuous at $x = \pm 1$.

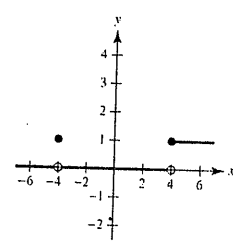
102. A discontinuity at c is removable if the function f can be made continuous at c by appropriately defining (or redefining) $f(c)$. Otherwise, the discontinuity is nonremovable.

(a) $f(x) = \dfrac{|x - 4|}{x - 4}$

(b) $f(x) = \dfrac{\sin(x + 4)}{x + 4}$

(c) $f(x) = \begin{cases} 1, & x \geq 4 \\ 0, & -4 < x < 4 \\ 1, & x = -4 \\ 0, & x < -4 \end{cases}$

$x = 4$ is nonremovable, $x = -4$ is removable



103. True

1. $f(c) = L$ is defined.
2. $\lim_{x\to c} f(x) = L$ exists.
3. $f(c) = \lim_{x\to c} f(x)$

All of the conditions for continuity are met.

104. True. If $f(x) = g(x)$, $x \neq c$, then $\lim_{x\to c} f(x) = \lim_{x\to c} g(x)$ (if they exist) and at least one of these limits then does not equal the corresponding function value at $x = c$.

105. False. A rational function can be written as $P(x)/Q(x)$ where P and Q are polynomials of degree m and n, respectively. It can have, at most, n discontinuities.

106. False. $f(1)$ is not defined and $\lim_{x\to 1} f(x)$ does not exist.

107. The functions agree for integer values of x:

$$\left.\begin{aligned} g(x) &= 3 - [\![-x]\!] = 3 - (-x) = 3 + x \\ f(x) &= 3 + [\![x]\!] = 3 + x \end{aligned}\right\} \text{for } x \text{ an integer}$$

However, for non-integer values of x, the functions differ by 1.

$$f(x) = 3 + [\![x]\!] = g(x) - 1 = 2 - [\![-x]\!].$$

For example,

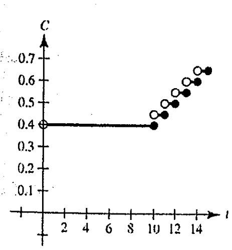
$$f\left(\tfrac{1}{2}\right) = 3 + 0 = 3,\ g\left(\tfrac{1}{2}\right) = 3 - (-1) = 4.$$

108. $\lim_{t\to 4^-} f(t) \approx 28$

$\lim_{t\to 4^+} f(t) \approx 56$

At the end of day 3, the amount of chlorine in the pool has decreased to about 28 oz. At the beginning of day 4, more chlorine was added, and the amount is now about 56 oz.

109. $$C(t) = \begin{cases} 0.40, & 0 < t \le 10 \\ 0.40 + 0.05[\![t - 9]\!], & t > 10,\ t \text{ not an integer} \\ 0.40 + 0.05(t - 10), & t > 10,\ t \text{ an integer} \end{cases}$$



There is a nonremovable discontinuity at each integer greater than or equal to 10.

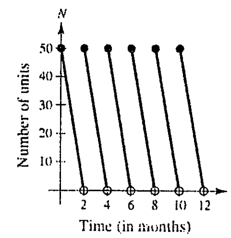
Note: You could also express C as

$$C(t) = \begin{cases} 0.40, & 0 < t \le 10 \\ 0.40 - 0.05[\![10 - t]\!], & t > 10 \end{cases}$$

110. $N(t) = 25\left(2\left[\!\!\left[\dfrac{t + 2}{2}\right]\!\!\right] - t\right)$

t	0	1	1.8	2	3	3.8
$N(t)$	50	25	5	50	25	5

Discontinuous at every positive even integer. The company replenishes its inventory every two months.



111. Let $s(t)$ be the position function for the run up to the campsite. $s(0) = 0$ ($t = 0$ corresponds to 8:00 A.M., $s(20) = k$ (distance to campsite)). Let $r(t)$ be the position function for the run back down the mountain: $r(0) = k$, $r(10) = 0$. Let $f(t) = s(t) - r(t)$.

When $t = 0$ (8:00 A.M.),
$f(0) = s(0) - r(0) = 0 - k < 0$.

When $t = 10$ (8:00 A.M.), $f(10) = s(10) - r(10) > 0$.

Because $f(0) < 0$ and $f(10) > 0$, then there must be a value t in the interval $[0, 10]$ such that $f(t) = 0$. If $f(t) = 0$, then $s(t) - r(t) = 0$, which gives us $s(t) = r(t)$. Therefore, at some time t, where $0 \le t \le 10$, the position functions for the run up and the run down are equal.

112. Let $V = \dfrac{4}{3}\pi r^3$ be the volume of a sphere with radius r.

V is continuous on $[5, 8]$. $V(5) = \dfrac{500\pi}{3} \approx 523.6$ and

$V(8) = \dfrac{2048\pi}{3} \approx 2144.7$. Because

$523.6 < 1500 < 2144.7$, the Intermediate Value Theorem guarantees that there is at least one value r between 5 and 8 such that $V(r) = 1500$. (In fact, $r \approx 7.1012$.)

113. Suppose there exists x_1 in $[a, b]$ such that $f(x_1) > 0$ and there exists x_2 in $[a, b]$ such that $f(x_2) < 0$. Then by the Intermediate Value Theorem, $f(x)$ must equal zero for some value of x in $[x_1, x_2]$ (or $[x_2, x_1]$ if $x_2 < x_1$). So, f would have a zero in $[a, b]$, which is a contradiction. Therefore, $f(x) > 0$ for all x in $[a, b]$ or $f(x) < 0$ for all x in $[a, b]$.

114. Let c be any real number. Then $\lim_{x \to c} f(x)$ does not exist because there are both rational and irrational numbers arbitrarily close to c. Therefore, f is not continuous at c.

115. If $x = 0$, then $f(0) = 0$ and $\lim_{x \to 0} f(x) = 0$. So, f is continuous at $x = 0$.

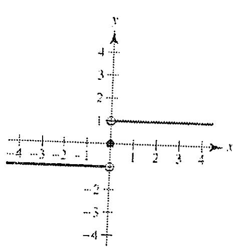
If $x \neq 0$, then $\lim_{t \to x} f(t) = 0$ for x rational, whereas $\lim_{t \to x} f(t) = \lim_{t \to x} kt = kx \neq 0$ for x irrational. So, f is not continuous for all $x \neq 0$.

116. $\operatorname{sgn}(x) = \begin{cases} -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases}$

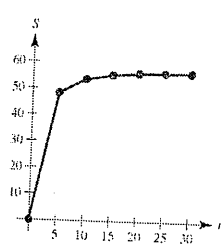
(a) $\lim_{x \to 0^-} \operatorname{sgn}(x) = -1$

(b) $\lim_{x \to 0^+} \operatorname{sgn}(x) = 1$

(c) $\lim_{x \to 0} \operatorname{sgn}(x)$ does not exist.

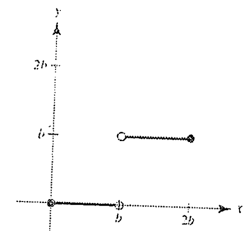


117. (a)



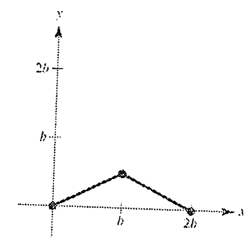
(b) There appears to be a limiting speed and a possible cause is air resistance.

118. (a) $f(x) = \begin{cases} 0, & 0 \le x < b \\ b, & b < x \le 2b \end{cases}$



NOT continuous at $x = b$.

(b) $g(x) = \begin{cases} \dfrac{x}{2}, & 0 \le x \le b \\ b - \dfrac{x}{2}, & b < x \le 2b \end{cases}$



Continuous on $[0, 2b]$.

119. $f(x) = \begin{cases} 1 - x^2, & x \le c \\ x, & x > c \end{cases}$

f is continuous for $x < c$ and for $x > c$. At $x = c$, you need $1 - c^2 = c$. Solving $c^2 + c - 1$, you obtain

$$c = \frac{-1 \pm \sqrt{1 + 4}}{2} = \frac{-1 \pm \sqrt{5}}{2}.$$

120. Let y be a real number. If $y = 0$, then $x = 0$. If $y > 0$, then let $0 < x_0 < \pi/2$ such that $M = \tan x_0 > y$ (this is possible since the tangent function increases without bound on $[0, \pi/2)$). By the Intermediate Value Theorem, $f(x) = \tan x$ is continuous on $[0, x_0]$ and $0 < y < M$, which implies that there exists x between 0 and x_0 such that $\tan x = y$. The argument is similar if $y < 0$.

121. $f(x) = \dfrac{\sqrt{x + c^2} - c}{x}, c > 0$

Domain: $x + c^2 \geq 0 \Rightarrow x \geq -c^2$ and $x \neq 0, [-c^2, 0) \cup (0, \infty)$

$$\lim_{x\to 0} \frac{\sqrt{x + c^2} - c}{x} = \lim_{x\to 0} \frac{\sqrt{x + c^2} - c}{x} \cdot \frac{\sqrt{x + c^2} + c}{\sqrt{x + c^2} + c} = \lim_{x\to 0} \frac{(x + c^2) - c^2}{x\left[\sqrt{x + c^2} + c\right]} = \lim_{x\to 0} \frac{1}{\sqrt{x + c^2} + c} = \frac{1}{2c}$$

Define $f(0) = 1/(2c)$ to make f continuous at $x = 0$.

122. **1.** $f(c)$ is defined.

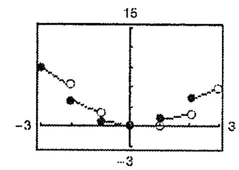
2. $\lim_{x\to c} f(x) = \lim_{\Delta x\to 0} f(c + \Delta x) = f(c)$ exists.

[Let $x = c + \Delta x$. As $x \to c, \Delta x \to 0$]

3. $\lim_{x\to c} f(x) = f(c)$.

Therefore, f is continuous at $x = c$.

123. $h(x) = x[\![x]\!]$



h has nonremovable discontinuities at $x = \pm 1, \pm 2, \pm 3, \ldots$.

124. (a) Define $f(x) = f_2(x) - f_1(x)$. Because f_1 and f_2 are continuous on $[a, b]$, so is f.

$f(a) = f_2(a) - f_1(a) > 0$ and $f(b) = f_2(b) - f_1(b) < 0$

By the Intermediate Value Theorem, there exists c in $[a, b]$ such that $f(c) = 0$.

$f(c) = f_2(c) - f_1(c) = 0 \Rightarrow f_1(c) = f_2(c)$

(b) Let $f_1(x) = x$ and $f_2(x) = \cos x$, continuous on $[0, \pi/2]$, $f_1(0) < f_2(0)$ and $f_1(\pi/2) > f_2(\pi/2)$.

So by part (a), there exists c in $[0, \pi/2]$ such that $c = \cos(c)$.

Using a graphing utility, $c \approx 0.739$.

125. The statement is true.

If $y \geq 0$ and $y \leq 1$, then $y(y - 1) \leq 0 \leq x^2$, as desired. So assume $y > 1$. There are now two cases.

Case 1: If $x \leq y - \frac{1}{2}$, then $2x + 1 \leq 2y$ and

$$\begin{aligned} y(y - 1) &= y(y + 1) - 2y \\ &\leq (x + 1)^2 - 2y \\ &= x^2 + 2x + 1 - 2y \\ &\leq x^2 + 2y - 2y \\ &= x^2 \end{aligned}$$

Case 2: If $x \geq y - \frac{1}{2}$

$$\begin{aligned} x^2 &\geq \left(y - \tfrac{1}{2}\right)^2 \\ &= y^2 - y + \tfrac{1}{4} \\ &> y^2 - y \\ &= y(y - 1) \end{aligned}$$

In both cases, $y(y - 1) \leq x^2$.

126. $P(1) = P(0^2 + 1) = P(0)^2 + 1 = 1$

$P(2) = P(1^2 + 1) = P(1)^2 + 1 = 2$

$P(5) = P(2^2 + 1) = P(2)^2 + 1 = 5$

Continuing this pattern, you see that $P(x) = x$ for infinitely many values of x. So, the finite degree polynomial must be constant: $P(x) = x$ for all x.