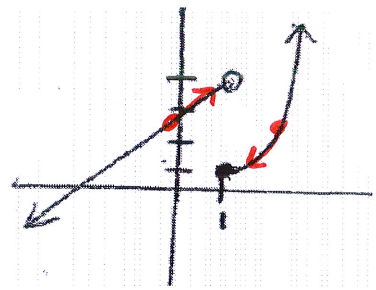
Key

**Calculus** Ch. 1.4a Notes **Continuity and One-Sided Limits**

Definition: **One-Sided Limits** – describes the function's behavior from the left or the right side of an x-value

**Example 1:**

$$f(x)=\begin{cases} x^2 & , \quad x \ge 1 \\ x+3 & , \quad x<1 \end{cases}$$




a) $\lim\limits_{x\to 1} f(x) =$ does not exist

b) left handed limit: $\lim\limits_{x\to 1^-} f(x) =$ 4

In other words: "The Limit (y-value that the graph approaches) **from the left side** of x = 1 is 4
(start left of point, move right)

c) right handed limit: $\lim\limits_{x\to 1^+} f(x) =$ 1

In other words: "The Limit (y-value that the graph approaches) **from the right side** of x = 1 is 1
(start right of point, move left)

- Recall that the limit of $f(x)$ as $x \to c$ exists only if $\lim\limits_{x\to c^-} f(x) = \lim\limits_{x\to c^+} f(x)$.

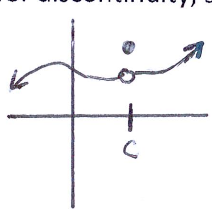
**Continuity**

can I walk along the graph without any interruptions? Can I draw the graph without ever lifting my pen/pencil? If so, the path or graph is continuous at that point.

**Continuity Conditions: (*IMPORTANT: KNOW THIS*)**
For a function, $f$, to be continuous at $c$, the following 3 conditions must be met.

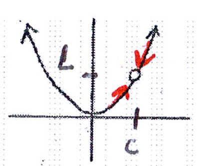
1. $f(c)$ is defined *point exists
2. $\lim\limits_{x\to c} f(x)$ exists $\left(\lim\limits_{x\to c^-} f(x) = \lim\limits_{x\to c^+} f(x)\right)$ *the limit exists
3. $\lim\limits_{x\to c} f(x) = f(c)$ * the limit exists at same location as point

- When checking for discontinuity, step through each of the conditions above in order.




**Types of Continuity:**

1) **Removable Discontinuity** (hole in graph) – a graph with removable discontinuity can be made continuous by filling in a single point.

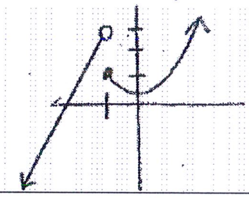



*Removable discontinuity passes the 2nd condition but fails the 3rd condition.

✓ → $\lim\limits_{x\to c} f(x)$ exists

✗ → $\lim\limits_{x\to c} f(x) \neq f(c)$

2) **Nonremovable Discontinuity** (step, jump discontinuity) – this is a discontinuity where the graph jumps from one connected piece of graph to another.



*Non-removable discontinuity fails the 2nd continuity condition:

$\lim\limits_{x\to -1^-} f(x) = 3$  $\lim\limits_{x\to -1^+} f(x) = 1$  then $\lim\limits_{x\to -1} f(x) =$ does not exist

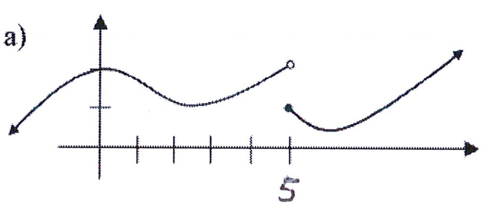
## Continuity Conditions revisited

| | |
|---|---|
| $i.\ f(c) \text{ is defined}$ | *If first condition fails, function not continuous at the point, but continue to test next condition to categorize removable/nonremovable |
| $ii.\ \lim\limits_{x\to c} f(x)$ exists $\left(\lim\limits_{x\to c^-} f(x) = \lim\limits_{x\to c^+} f(x)\right)$ | *If 2nd condition fails, then the limit does not exist, and this function must have **non-removable discontinuity** at that point<br>* Test 3rd condition only if 2nd condition passes. |
| $iii.\ \lim\limits_{x\to c} f(x) = f(c)$ | *If 2nd condition passes, but 3rd condition fails, then this function must have **removable discontinuity** at that point<br>*If all 3 condition passes, then the function is continuous at that point. |

(* since 2nd condition fails, we stop and don't test 3rd condition)

**Class Example 2:** Using continuity conditions, determine the reason why the following graphs are discontinuous. Then categorize as removable or nonremovable discontinuity

a)

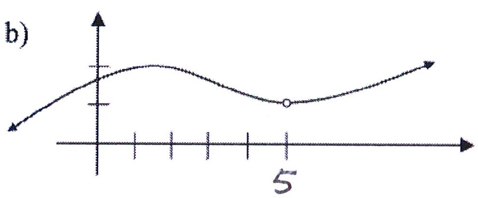


✓ i) $f(5) = 1$

× ii) $\lim\limits_{x\to 5} f(x)$ does not exist $\left[\lim\limits_{x\to 5^-} f(x) \neq \lim\limits_{x\to 5^+} f(x)\right]$

* therefore, nonremovable discontinuity at $x = 5$.

b)



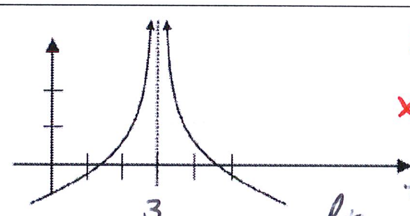
× i) $f(5)$ undefined

✓ ii) $\lim\limits_{x\to 5} f(x) = 1$

× iii) $\lim\limits_{x\to 5} f(x) \neq f(5)$

* therefore removable discontinuity at $x = 5$.

c)



× i) $f(3)$ undefined.

× ii) $\lim\limits_{x\to 3} f(x)$ does not exist

* therefore, nonremovable discontinuity at $x = 3$

* Extension questions:
1) can you sketch example graph condition 1 and 2 passes, 3rd fails?
2) Is there example where #1, 2 fails but #3 passes?

d) Find the point (x-value) of discontinuity for the function $f(x) = \dfrac{x^2 - 9}{x - 3}$. Is it removable? If so, what would we need to set f(x) equal to at that value for the function to be continuous? (Step through continuity conditions to support your answer)

continuity conditions.

* Discontinuity at $x = 3$

i) $f(3) = \dfrac{3^2 - 9}{3 - 3} \to \dfrac{0}{0} \to f(3)$ undefined

ii) $\lim\limits_{x\to 3} \dfrac{x^2 - 9}{x - 3} \to \dfrac{0}{0} \to \lim\limits_{x\to 3} \dfrac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}} \to \lim\limits_{x\to 3} x + 3 = \boxed{6}$

iii) $\lim\limits_{x\to 3} f(x) \neq f(3)$

* therefore, removable discontinuity at $x = 3$.

* To make f(x) continuous, let $f(3) = 6$

* Recall that a graph with removable discontinuity is only 1 point away from becoming continuous.