Key

Calculus **Ch. 1.4b Notes** **Intermediate Value Theorem (IVT)**

Review continuity conditions:

i) $f(c)$ is defined **ii)** $\lim_{x\to c} f(x)$ exists ($\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)$) **iii)** $\lim_{x\to c} f(x) = f(c)$

Warm-up problem: Prove that the following is discontinuous at x = 2. Is it removable? If so, redefine f(2) to make the function continuous. (step through continuity conditions)

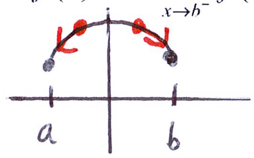
$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x \neq 2 \\ x+5, & x = 2 \end{cases}$$

i) $f(2) = 2+5 = 7$

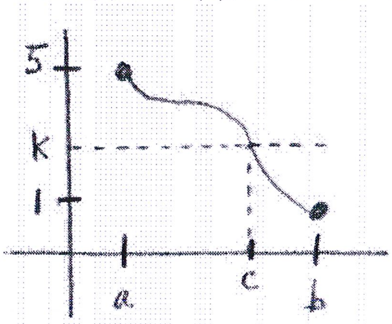
ii) $\lim_{x\to 2} \frac{x^2-4}{x-2} \to \frac{0}{0} \to \lim_{x\to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}} \to 2+2=4$

iii) $\lim_{x\to 2} f(x) \neq f(2)$ therefore removable discontinuity at $x=2$.

Continuity on a closed interval: If a function is continuous on an open interval (a, b) and $\lim_{x\to a^+} f(x) = f(a)$ and $\lim_{x\to b^-} f(x) = f(b)$, then the function is continuous on the closed interval [a, b].



Intermediate Value Theorem: If f is continuous on a closed interval [a, b] and k is any number between f(a) and f(b), then there is at least one number c in [a, b] such that f(c) = k.



*In other words, if a function is continuous, then the graph has to touch all the y-values between the 2 endpoints (at least once)

Example 1: Use the IVT to show that there is a zero in the interval [0, 1] for the function $f(x) = x^3 + 2x - 1$.

$f(c)=0$ ← x-values endpoints of 0 and 1

y-value of 0

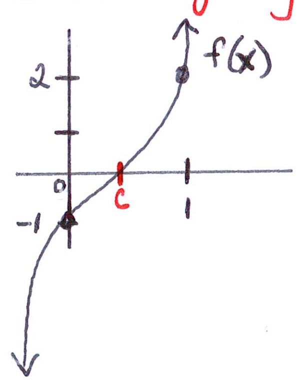
*test endpoints

$*f(x)$ is continuous $[0,1]$ ← Establishing and stating continuity is important!

$f(0) = 0^3 + 2(0) - 1 = -1$ → compare these y-values with our target: $y=0$

$f(1) = 1^3 + 2(1) - 1 = 2$

By IVT, since $f(0) = -1 < 0 \le 2 = f(1)$ then there must be a value of c where $f(c) = 0$.



X-values　y-value　specific x-value of the ordered pair.

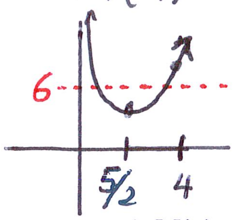
Example 2: Verify the IVT applies to $f(x) = \frac{x^2+x}{x-1}$ on the interval $\left[\frac{5}{2}, 4\right]$ for f(c) = 6 and find c.

* V.A. at $x=1$

* $f(x)$ continuous $[5/2, 4]$

$f(5/2) = \frac{2.5^2+2.5}{2.5-1} = \frac{35}{6} \approx 5.8$

$f(4) = \frac{4^2+4}{4-1} = \frac{20}{3} = 6.7$



a) By IVT, since $5.8 < f(c) = 6 < 6.7$, $f(c)=6$ on interval $[5/2, 4]$

b) To find c, set $f(x)=6$, solve for x.

$\frac{x^2+x}{x-1} = 6 \rightarrow \frac{x^2+x}{x-1} = \frac{6}{1} \rightarrow x^2+x = 6(x-1)$

$\rightarrow x^2+x = 6x-6 \rightarrow x^2+x-6x+6=0$

$x^2-5x+6=0$

$(x-3)(x-2)=0$

$x=3, \cancel{x=2}$

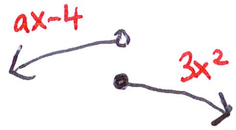
$\boxed{c=3}$

Additional Continuity Practice Problems:

Making a Function Continuous In Exercises 61–66, find the constant a, or the constants a and b, such that the function is continuous on the entire real number line.

Continuity conditions:

1. $f(c)$ is defined
2. $\lim_{x\to c} f(x)$ exists $\quad (\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x))$
3. $\lim_{x\to c} f(x) = f(c)$



61. $f(x) = \begin{cases} 3x^2, & x \ge 1 \\ ax - 4, & x < 1 \end{cases}$

* In this piecewise function, 2 equations define 2 separate graphs (that may or may not be connected)

* step through continuity conditions:

$\boxed{c=1}$ ← find c-value by looking at the restriction defined for the graphs

i) $f(1) = 3(1)^2 = 3$

ii) $\lim_{x\to 1^-} ax-4 = a(1)-4 = \boxed{a-4}$ $\quad$ $\lim_{x\to 1^+} 3x^2 = 3(1)^2 = \boxed{3}$

$a-4=3$

$\boxed{a=7}$

If $a=7$, then $\lim_{x\to 1} f(x) = 3$

iii) $f(1) = \lim_{x\to 1} f(x) = 3$ ✓ $\boxed{f(x) \text{ continuous at } x=1 \text{ if } a=7}$

graph

66. $g(x) = \begin{cases} \frac{x^2 - a^2}{x - a}, & x \ne a \\ 8, & x = a \end{cases}$

point

* In this piecewise function, one equation defines a graph with a hole that may or may not be filled in by the 2nd equation (defined at a point)

$\boxed{c=a}$

i) $g(a)=8$

ii) $\lim_{x\to a} \frac{x^2-a^2}{x-a} \rightarrow \frac{a^2-a^2}{a-a} \rightarrow \frac{0}{0} \rightarrow \lim_{x\to a} \frac{\cancel{(x-a)}(x+a)}{\cancel{(x-a)}}$

$\lim_{x\to a} x+a = a+a = \boxed{2a}$

iii) $\lim_{x\to a} f(x) = g(a) \rightarrow 2a=8 \rightarrow \boxed{a=4}$

$\boxed{f(x) \text{ continuous at } x=a \text{ if } a=4}$