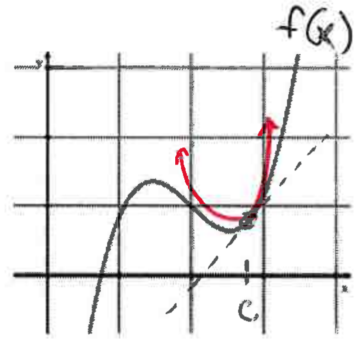
Key

BC Calculus – 10.10a Notes - Finding Taylor Polynomial Approximations of Functions

Taylor polynomials are created to help us approximate other functions. Why would we do this? Because polynomials are easy to work with in calculus (i.e., taking a derivative or integral).

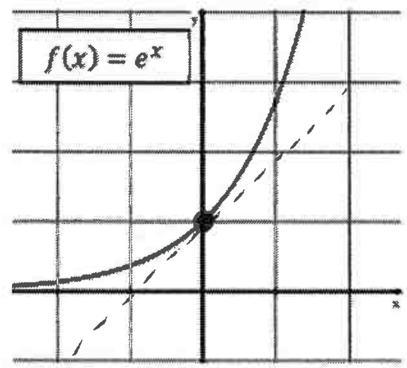
A **Maclaurin polynomial** is a special type of Taylor polynomial.



To start, we choose an x-value to center our polynomial approximation. Let's call that $x = c$. Our approximation will have the same y-value as the original function at $x = c$.

$$f(c) = p(c)$$

We expand the approximation about $x = c$. Another way of saying this: "the functions are **centered at $x = c$.**"



Explore with an example: $f(x) = e^x$. Let $c = 0$.

We know $f(0) = 1$. We need $p(0) = 1$

We want to make the graphs have a similar shape at the point $x = c$. They should have the same slope. That leads us to

$$f'(c) = p'(c)$$

In this example, that means $f'(0) = p'(0)$

The polynomial approximation will look like this:

$$y - y_1 = m(x - x_1)$$

$$p(x) - f(c) = f'(c)(x - c)$$ ← point-slope form

Or rewritten:

$* \; f'(x) = e^x$

$f'(0) = e^0 = 1$

$$p(x) = f(c) + f'(c)(x - c)$$
$$p(x) = f(0) + f'(0)(x - 0)$$
$$p(x) = 1 + 1x$$

$$\boxed{p(x) = 1 + x}$$

This is called a $\boxed{\text{first order}}$ approximation. It works for a small interval around our point of center.

To improve the approximation, make the second derivatives agree at $x = c$.

We want $f(c) = p(c)$, $f'(c) = p'(c)$, $f''(c) = p''(c)$.
For our example this is $f(0) = p(0)$, $f'(0) = p'(0)$, $f''(0) = p''(0)$

If we worked through a similar process, we'd end up with the following:

Second-order approximation: $p(x) = 1 + x + \frac{1}{2}x^2$

If we are centered at $x = 0$, then we have the following pattern:

$$p_n(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n$$

N[th] Taylor Polynomial

If $f(x)$ is a differentiable function, then an approximation of f centered about $x = c$ can be modeled by

$$p_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!} + \frac{f'''(c)(x-c)^3}{3!} + \dots + \dots \boxed{\frac{f^n(c)(x-c)^n}{n!}}$$

where n is the order of the approximation.

Maclaurin Polynomial

A Maclaurin polynomial is a Taylor polynomial centered about $x = 0$. It can be modeled by

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \dots + \boxed{\frac{f^n(0)x^n}{n!}}$$

where n is the order of the approximation.

centered at $x=0$

1. Find the third-degree Maclaurin polynomial for $f(x) = e^{2x}$

$f(x) = e^{2x}$	$f(0) = 1$
$f'(x) = 2e^{2x}$	$f'(0) = 2$
$f''(x) = 2e^{2x} \cdot 2$	$f''(0) = 4$
$f'''(x) = 8e^{2x}$	$f'''(0) = 8$

$$P_3(x) = f(0) + f'(0) \cdot x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!}$$

$$P_3(x) = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!}$$

$$\boxed{P_3(x) = 1 + 2x + 2x^2 + \frac{4}{3}x^3}$$

Evaluate at $f(0.2)$ and $p_3(0.2)$

$f(0.2) = e^{2(0.2)} \approx 1.4918$

$P_3(0.2) \approx 1.490\overline{6}$

this is a Maclaurin polynomial approximation of the actual y-value for $f(0.2)$

2. Find a fourth-degree Taylor Polynomial for $f(x) = \ln x$ centered at $x = 1$.

$f(x) = \ln x$	$f(1) = 0$
$f'(x) = \frac{1}{x} = x^{-1}$	$f'(1) = 1$
$f''(x) = -1x^{-2}$	$f''(1) = -1$
$f'''(x) = +2x^{-3}$	$f^3(1) = 2$
$f^4(x) = -6x^{-4}$	$f^4(1) = -6$

$$P_4(x) = f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2!} + \frac{f'''(1)(x-1)^3}{3!} + \frac{f^4(1)(x-1)^4}{4!}$$

$$P_4(x) = 0 + 1(x-1) + \frac{-1(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} + \frac{-6(x-1)^4}{4!}$$

$$P_4(x) = x - 1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$$

Evaluate at $f(1.1)$ and $p_4(1.1)$

$f(1.1) = 0.0953101798$

$P_4(1.1) = 0.0953083333$

Coefficients of a Taylor Polynomial

The coefficient of the nth degree term in a Taylor polynomial for a function f cenetered at $x = c$ is

$$\to \frac{f^n(c)}{n!}$$

3. Let f be a function with third derivative $f'''(x) = (8x+2)^{\frac{3}{2}}$. What is the coefficient of $(x-2)^4$ in the fourth-degree Taylor Polynomial for f about $x = 2$.

$$\frac{f^4(2)}{4!}$$

$$f^4(x) = \frac{3}{2}(8x+2)^{1/2}(8)$$

$$f^4(2) = \frac{3}{2}(18)^{1/2} \cdot 8$$

$$f^4(2) = 12 \cdot \sqrt{9 \cdot 2}$$

$$f^4(2) = 36\sqrt{2}$$

$$\frac{36\sqrt{2}}{4!} \to \frac{36\sqrt{2}}{4 \cdot 3 \cdot 2 \cdot 1}$$

$$\boxed{\frac{3\sqrt{2}}{2}}$$

Practice Problems

$c = 0$

1. Find the fourth-degree Maclaurin Polynomial for e^{4x}.

$$P_4(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^4(0)}{4!}x^4$$

$f(x) = e^{4x}$ $\quad f(0) = e^0 = 1$

$f'(x) = 4e^{4x}$ $\quad f'(0) = 4$

$f''(x) = 4^2e^{4x}$ $\quad f''(0) = 4^2$

$f'''(x) = 4^3e^{4x}$ $\quad f'''(0) = 4^3$

$f^4(x) = 4^4e^{4x}$ $\quad f^4(0) = 4^4$

$$P_4(x) = 1 + 4x + \frac{4^2}{2!}x^2 + \frac{4^3}{3!}x^3 + \frac{4^4}{4!}x^4$$

$$\boxed{P_4(x) = 1 + 4x + 8x^2 + \frac{32}{3}x^3 + \frac{32}{3}x^4}$$

$c = 0$

2. Find the fifth-degree Maclaurin Polynomial for the function $f(x) = \sin x$.

$$P_5(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^4(0)}{4!}x^4 + \frac{f^5(0)}{5!}x^5$$

$f = \sin(x)$ $\quad f(0) = 0$

$f' = \cos x$ $\quad f'(0) = 1$

$f'' = -\sin x$ $\quad f''(0) = 0$

$f''' = -\cos x$ $\quad f'''(0) = -1$

$f^4 = \sin x$ $\quad f^4(0) = 0$

$f^5 = \cos x$ $\quad f^5(0) = 1$

$$0 + 1x + \frac{0}{2!}x^2 - \frac{1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5$$

$$\boxed{P_5(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5}$$

$$* \; P_n(x) = \frac{f^n(c)}{n!}(x-c)^n$$

$c=1$

3. Find the third-degree Taylor Polynomial for $f(x) = \ln(2x)$ about $x = 1$.

$$P_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

$f = \ln(2x)$ $f(1) = \ln 2$

$f' = \frac{2}{2x} \rightarrow \frac{1}{x} = x^{-1}$ $f'(1) = 1$

$f'' = -1x^{-2}$ $f''(1) = -1$

$f''' = 2x^{-3}$ $f'''(1) = 2$

$$P_3(x) = \ln 2 + 1(x-1) + -\tfrac{1}{2}(x-1)^2 + \frac{2}{3!}(x-1)^3$$

$$\boxed{P_3(x) = \ln 2 + (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3}$$

$c=0$

4. Find the third-degree Taylor Polynomial about $x = 0$ for $\ln(1-x)$.

$$P_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$$

$f = \ln(1-x)$ $f(0) = \ln 1 = 0$

$f' = \frac{-1}{1-x} = -(1-x)^{-1}$ $f'(0) = -1$

$f'' = 1(1-x)^{-2}(-1)$ $f''(0) = -1$

$f''' = 2(1-x)^{-3}(-1)$ $f'''(0) = -2$

$$P_3(x) = 0 + -1(x-0) - \frac{1}{2!}(x-0)^2 - \frac{2}{3!}(x-0)^3$$

$$P_3(x) = -1(x-0) - \tfrac{1}{2}(x-0)^2 - \tfrac{1}{3}(x-0)^3$$

$$\boxed{P_3(x) = -1x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3}$$

5. Find the third-degree Taylor Polynomial about $x = 4$ of $\sqrt{x}$.

$c=4$

$$P_3(x) = f(4) + f'(4)(x-4) + \frac{f''(4)(x-4)^2}{2!} + \frac{f'''(4)}{3!}(x-4)^3$$

$f = x^{1/2}$ $f(4) = \sqrt{4} = 2$

$f' = \tfrac{1}{2}x^{-1/2}$ $f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

$f'' = \frac{-1}{4}x^{-3/2}$ $f''(4) = \frac{-1}{4(4)^{3/2}} = \frac{-1}{4(8)} = \frac{-1}{32}$

$f''' = \frac{3}{8}x^{-5/2} \rightarrow f'''(4) = \frac{3}{8(4)^{5/2}} \rightarrow \frac{3}{8(2)^5} \rightarrow \frac{3}{256}$

$$P_3(x) = 2 + \tfrac{1}{4}(x-4) - \tfrac{1}{32}\cdot\tfrac{1}{2}(x-4)^2 + \tfrac{3}{256}\cdot\tfrac{1}{3!}(x-4)^3$$

$$\boxed{P_3(x) = 2 + \tfrac{1}{4}(x-4) - \tfrac{1}{64}(x-4)^2 + \tfrac{1}{512}(x-4)^3}$$

6. The function f has derivatives of all orders for all real numbers with $f(1) = -1$, $f'(1) = 4$, $f''(1) = 6$, and $f'''(1) = 12$. Using the third-degree polynomial for f about $x = 1$, what is the approximation of $f(1.1)$?

$c=1$

$$f(x) = f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2!} + \frac{f'''(1)(x-1)^3}{3!}$$

$$f(x) = -1 + 4(x-1) + \frac{6}{2!}(x-1)^2 + \frac{12}{3!}(x-1)^3$$

$$f(x) = -1 + 4(x-1) + 3(x-1)^2 + 2(x-1)^3$$

$$f(1.1) = -1 + 4(0.1) + 3(0.1)^2 + 2(0.1)^3$$

$$\boxed{f(1.1) = -0.568}$$

$f(0)=3$ $\quad f'(0)=4$

7. A function f has a Maclaurin series given by $3 + 4x + 2x^2 + \frac{1}{3}x^3 + \cdots$, and the series converges to $f(x)$ for all real numbers x. If g is the function defined by $g(x) = e^{f(x)}$, what is the coefficient of x in the Maclaurin series for g? $(c=0)$

$$P_n(x) = g(0) + g'(0)(x-0) + \frac{g''(0)}{2}(x-0)^2$$

$$g'(x) = e^{f(x)} \cdot f'(x)$$

$$g'(0) = e^{f(0)} \cdot f'(0) \rightarrow e^3 \cdot 4 \rightarrow \boxed{4e^3}$$

8. Let f be a function with third derivative $f'''(x) = (7x+2)^{\frac{3}{2}}$. What is the coefficient of $(x-2)^4$ in the fourth-degree Taylor Polynomial for f about $x = 2$?

$$f^4(x) = \frac{3}{2}(7x+2)^{1/2}(7)$$

$$f^4(2) = \frac{3}{2}(16)^{1/2}(7)$$

$$= \frac{3}{2}(4)(7) = 42$$

$$\frac{f^4(2)}{4!}(x-2)^4$$

$$\frac{42}{4!} \rightarrow \frac{42}{24} = \boxed{\frac{7}{4}}$$

9. Let $P(x) = 4x^2 - 6x^3 + 8x^4 + 4x^5$ be the fifth-degree Taylor Polynomial for the function f about $x = 0$. What is the value of $f'''(0)$? $c=0$

$$\frac{f'''(0)}{3!}(x-0)^3 \qquad \frac{f'''(0)}{3!} = -6 \qquad f'''(0) = -6 \cdot 3!$$

$$\boxed{f'''(0) = -36}$$

10. Let P be the second-degree Taylor Polynomial for $f(x) = e^{-3x}$ about $x = 3$. What is the slope of the line tangent to the graph of P at $x = 3$?

at $x=3$, $P'(3) = f'(3)$

$$f'(x) = e^{-3x}(-3)$$

$$f'(3) = e^{-3(3)} \cdot -3$$

$$\boxed{f'(3) = -3e^{-9} = \frac{-3}{e^9}}$$

11. Let f be a function with $f(4) = 2$, $f'(4) = -1$, $f''(4) = 6$, and $f'''(4) = 12$. What is the third-degree Taylor Polynomial for f about $x = 4$? $c=4$

$$P_3(x) = f(4) + f'(4)(x-4) + \frac{f''(4)}{2!}(x-4)^2 + \frac{f'''(4)}{3!}(x-4)^3$$

$$P_3(x) = 2 - 1(x-4) + \frac{6}{2}(x-4)^2 + \frac{12}{3!}(x-4)^3$$

$$\boxed{P_3(x) = 2 - 1(x-4) + 3(x-4) + 2(x-4)^3}$$

12. Let f be a function that has derivates of all orders for all real numbers. Assume $f(1) = 3, f'(1) = -2$, $f''(1) = 2$, and $f'''(1) = 4$. Use a second-degree Taylor Polynomial to approximate $f(0.7)$.

$P_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2$

$P_2(x) = 3 - 2(x-1) + \frac{2}{2!}(x-1)^2$

$f(0.7) \approx P_2(0.7) =$

$3 - 2(-0.3) + (-0.3)^2 = \boxed{3.69}$

13. The function f has derivatives of all orders for all real numbers with $f(0) = 4, f'(0) = -3, f''(0) = 3$, and $f'''(0) = 2$. Let g be the function given by $g(x) = \int_0^x f(t)\,dt$. Find the third-degree Taylor Polynomial for g about $x = 0$. $c = 0$

$P_3(x) = g(0) + g'(0)(x-0) + \frac{g''(0)}{2!}(x-0)^2 + \frac{g'''(0)}{3!}(x-0)^3$

$g(x) = \int_0^x f(t)dt$ $\quad g(0) = 0$

$g'(x) = f(x)$ $\quad g'(0) = f(0) = 4$

$g''(x) = f'(x)$ $\quad g''(0) = f'(0) = -3$

$g'''(x) = f''(x)$ $\quad g'''(0) = f''(0) = 3$

$P_3(x) = 0 + 4x - \frac{3}{2}x^2 + \frac{3}{3!}x^3$

$\boxed{P_3(x) = 4x - \frac{3}{2}x^2 + \frac{1}{2}x^3}$

14.

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$
-4	1	-2	-4	2

Selected values for $f(x)$ and its first three derivatives are shown in the table above. What is the approximation for the value of $f(-3)$ about $x = -4$ obtained using the third-degree Taylor Polynomial for f.

$P_3(x) = f(-4) + f'(-4)(x+4) + \frac{f''(-4)}{2!}(x+4)^2 + \frac{f'''(-4)}{3!}(x+4)^3$

$P_3(x) = 1 - 2(x+4) - \frac{4}{2}(x+4)^2 + \frac{2}{3!}(x+4)^3$

$f(-3) \approx P_3(-3) = 1 - 2(1) - 2(1)^2 + \frac{2}{6}(1)^3 \boxed{= \frac{-8}{3} \approx -2.667}$

~~10.3~~ Taylor Polynomial Approximations

Test Prep

15. Which of the following polynomial approximations is the best for $\cos(3x)$ near $x = 0$?

$f(x) = \cos(3x)$ $\quad f(0) = \cos 0 = 1$

$f'(x) = -\sin(3x)\cdot 3$ $\quad f'(0) = 0$

$f''(x) = -3\cos(3x)\cdot 3$ $\quad f''(0) = -9$

$P(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2$

$P(x) = 1 + 0(x) - \frac{9}{2}x^2$

(A) $1 + \frac{3}{2}x$ $\quad$ (B) $1 - \frac{9}{2}x^2$ $\quad$ (C) $1 + x$ $\quad$ (D) $1 - \frac{9}{2}x + x^2$

16. Consider the logistic differential equation $\frac{dy}{dt} = \frac{y}{6}(4 - y)$. Let $y = f(t)$ be the particular solution to the differential equation with $f(0) = 6$.

a. Write the second-degree Taylor Polynomial for f about $t = 0$.

$P_2(t) = f(0) + f'(0)(t-0) + \frac{f''(0)}{2!}(t-0)^2$

$f(0) = 6$

$f'(0) = \frac{6}{6}(4-6) = -2$

$f''(x) = \frac{1}{6}\left(\frac{dy}{dt}\right)(4-y) + \frac{y}{6}(-1)\left(\frac{dy}{dt}\right)$

$f''(0) = \frac{1}{6}(-2)(4-6) + \frac{6}{6}(-1)(-2)$

$= \frac{2}{3} + 2 = 8/3$

$P_2(t) = 6 - 2(t-0) + \frac{8}{3}\cdot\frac{1}{2}(t)^2$

$\boxed{P_2(t) = 6 - 2t + \frac{4}{3}t^2}$

b. Use the results from part a to approximate $f(1)$.

$f(1) \approx 6 - 2(1) + \frac{4}{3}(1) = \boxed{\frac{16}{3}}$

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t (seconds)	0	4	10
$x'(t)$ meters per second	5.0	5.8	4.0

17. The position of a particle moving along a straight line is modeled by $x(t)$. Selected values of $x'(t)$ are shown in the table above and the position of the particle at time $t = 10$ is $x(10) = 8$.

a. Approximate $x''(8)$ using the average rate of change of $x'(t)$ over the interval $4 \le t \le 10$. Show computations that lead to your answer.

$\frac{x'(10) - x'(4)}{10 - 4} = \frac{4.0 - 5.8}{10 - 4} = \frac{-1.8}{6} = \boxed{-0.3}$

b. Using correct units, explain the meaning of $x''(8)$ in the context of the problem.

$x''(8)$ is the rate at which velocity is changing (which is acceleration) at $t = 8$ with units of meters/sec^2

c. Use a right Riemann sum with two subintervals to approximate $\int_0^{10} |x'(t)|\, dt$. ← Absolute value

$\int_0^{10} |x'(t)|\, dt \approx 4(5.8) + 6(4) = \boxed{47.2}$

d. Let s be a function such that the third derivative of s with respect to t is $(t-3)^7$. Write the fourth-degree term of the fourth-degree Taylor Polynomial for s about $t = 1$.

$f^4(x) = 7(t-3)^6$

$f^4(1) = 7(1-3)^6 = 7(-2)^6 = 448$

4th term is $\frac{f^4(1)}{4!}(t-1)^4 \rightarrow \frac{448}{4!}(t-1)^4$

$\boxed{\frac{56}{3}(t-1)^4}$