Key

BC Calculus – 10.1 Notes – Convergent & Divergent Infinite Series

Recall: Writing terms of a sequence.

$a_n = \{1 + (-2)^n\}$

$-1, 5, -7, 17, -31$

$a_1 = -1$ $\quad a_4 = 17$

$a_2 = 5$ $\quad a_5 = -31$

$a_3 = -7$

Sequence: A collection of numbers that are in one-to-one correspondence with positive integers.

$-2 \qquad 4 \qquad -\frac{26}{6} \qquad \frac{80}{24} \qquad -\frac{242}{120}$

* values either always increase or always decrease

Monotonic Sequences never decreases or never increases	Bounded Sequences
$a_1 \le a_2 \le a_3 \le \cdots \le a_n$ or $a_1 \ge a_2 \ge a_3 \ge \cdots \ge a_n$	$a_n \le M$ (upper bound / above) $a_n \ge N$ (lower bound / below) $\{a_n\}$ bounded if both are true

Infinite Series: when you take sequence of numbers and adding up infinite number of terms.

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n$$

* The sum may add up to infinity, but not always.

Partial Sum: The sum of the first "n" terms

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$

a_n vs S_n:

a_n is an expression that gives the n^{th} term in a sequence

S_n is an expression that gives the sum of the first "n" terms of the sequence.

1. Use the following sequence 2, 4, 6, 8, 10 to find a_4 and S_4.

$a_4 = 8$

$S_4 = 2+4+6+8 = 20$

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n$$

Convergent and Divergent Series

For the infinite series $\sum_{n=1}^{\infty} a_n$, the n^{th} partial sum is $S_n = a_1 + a_2 + a_3 + \cdots + a_n$.

If the sequence of the partial sum $\{S_n\}$ converges to S, then the series $\sum_{n=1}^{\infty} a_n$ converges. The limit S is called the sum of the series.

a finite value

Likewise, if $\{S_n\}$ diverges then the series also diverges

2. **Does the series converge or diverge?** $\sum_{n=1}^{\infty} \frac{1}{2^n}$

$S_1 = \frac{1}{2}$

$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$

$S_4 = \frac{7}{8} + \frac{1}{16} = \frac{15}{16}$

* An easy guess is that the partial sums are getting closer to 1. (converging to 1)

* The sum converges to 1 since the sequence of the partial sums approaches 1.

* $S_n = 1 - \frac{1}{2^n}$ $\quad \lim_{n\to\infty} S_n \Rightarrow \lim_{n\to\infty}\left(1 - \frac{1}{2^n}\right) = \boxed{1}$

3. Use a calculator to find the partial sum S_n of the series $\sum_{n=1}^{\infty} \frac{10}{n(n+2)}$ for $n = 200, 1000$.

$S_{200} = 7.4504$

*select Math → 0

$S_{1000} = 7.49$

$\lim_{n\to\infty} S_n = 7.5$

4. **Does the series converge or diverge?** $\sum_{n=1}^{\infty} n$ $\quad 1+2+3+4+5+6+\ldots$

$S_1 = 1$

$S_2 = 3$

$S_3 = 6$

$S_4 = 10$

$S_5 = 10 + 5 = 15$

$\lim_{n\to\infty} S_n = \infty$ so series $\boxed{\text{diverge}}$

10.1 Convergent and Divergent Infinite Series

Calculus

Practice

1. Given the infinite series $\sum_{n=1}^{\infty} (-1)^n$, find the sequence of partial sums S_1, S_2, S_3, S_4, and S_5.

$S_1 = (-1)^1 = -1$

$S_2 = S_1 + (-1)^2 = 0$

$S_3 = S_2 + -1 = -1$

$S_4 = S_3 + (-1)^4 = 0$

$S_5 = S_4 + (-1)^5 = -1$

2. Find the sequence of partial sums S_1, S_2, S_3, S_4, and S_5 for the infinite series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \cdots$.

$S_1 = \boxed{1}$

$S_2 = 1 + \frac{1}{2} = \boxed{\frac{3}{2}}$

$S_3 = S_2 + 1/4 = \frac{3}{2} + 1/4 = \frac{6}{4} + \frac{1}{4} = \boxed{7/4}$

$S_4 = S_3 + 1/6 \to \frac{7}{4} + \frac{1}{6} \to \frac{42+4}{24} = \frac{46}{24} = \boxed{\frac{23}{12}}$

$S_5 = S_4 + 1/8 \to \frac{23}{12} + \frac{1}{8}$

$S_5 = \frac{46}{24} + \frac{3}{24} = \boxed{\frac{49}{24}}$

3. If the infinite series $\sum_{n=1}^{\infty} a^n$ has nth partial sum $S_n = (-1)^{n+1}$ for $n \geq 1$, what is the sum of the series?

$\lim_{n\to\infty} S_n \to \lim_{n\to\infty} (-1)^{n+1} = -1, 1, -1, 1, -1, \ldots$ $\boxed{\text{diverges}}$

4. The infinite series $\sum_{n=1}^{\infty} a^n$ has nth partial sum $S_n = \frac{n}{4n+1}$ for $n \geq 1$. What is the sum of the series?

$\lim_{n\to\infty} \frac{n}{4n+1} = \boxed{\frac{1}{4}}$

5. Use a calculator to find the partial sum S_n of the series $\sum_{n=1}^{\infty} \frac{6}{n(n+3)}$ for $n = 100, 500, 1000$.

*select Math → 0

$\sum_{x=1}^{100} \left(\frac{6}{x(x+3)}\right)$

$S_{100} \approx 3.6078$

$S_{500} \approx 3.6547$

$S_{1000} \approx 3.6606$

6. Show that the sequence with the given nth term $a_n = 1 + 2n$ is monotonic.

$a_1 = 3 \quad a_2 = 5 \quad a_3 = 7 \quad a_4 = 9, \ldots$

a_n is $\underbrace{\text{monotonic}}_{}$ because $a_1 \leq a_2 \leq a_3 \leq \ldots \leq a_n$

pattern that is always increasing or decreasing

7. What is the nth partial sum of the infinite series $\sum_{n=1}^{\infty} \frac{1}{2^{n+1}}$?

$\frac{1}{2}\left[S_n = \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \ldots \frac{1}{2^{n+1}}\right]$

$\frac{1}{2}S_n = \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \ldots \frac{1}{2^{n+2}}$

$S_n - \frac{1}{2}S_n = \frac{1}{2^2} - \frac{1}{2^{n+2}}$

$2\left[\frac{1}{2}S_n = \frac{1}{2^2} - \frac{1}{2^{n+2}}\right]$

$\boxed{S_n = \frac{1}{2} - \frac{1}{2^{n+1}}}$

8. Which of the following could be the nth partial sum for the infinite series $\sum_{n=1}^{\infty} \frac{1}{4^n}$?

$$\frac{1}{4}\left[S_n = \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^4} + \dots + \frac{1}{4^n}\right]$$

$$\frac{1}{4}S_n = \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^4} + \dots + \frac{1}{4^{n+1}}$$

$$S_n - \frac{1}{4}S_n = \frac{1}{4} - \frac{1}{4^{n+1}}$$

$$\left[\frac{3}{4}S_n = \frac{1}{4} - \frac{1}{4^{n+1}}\right]4$$

$$3S_n = 1 - \frac{1}{4^n}$$

$$\boxed{S_n = \frac{1}{3}\left[1 - \frac{1}{4^n}\right]}$$

(A) $S_n = \frac{1}{3}\left(1 + \frac{1}{4^n}\right)$ (B) $S_n = \frac{1}{3}\left(1 - \frac{1}{4^{n+1}}\right)$ (C) $\boxed{S_n = \frac{1}{3}\left(1 - \frac{1}{4^n}\right)}$ (D) $S_n = \frac{1}{4}\left(1 - \frac{1}{3^n}\right)$

9. If the infinite series $\sum_{n=1}^{\infty} a_n$ is convergent and has a sum of $\frac{7}{8}$, which of the following could be the nth partial sum?

$$\lim_{n\to\infty} S_n = \frac{7}{8}$$

(A) $S_n = \frac{7n+1}{8n^2+1} \to 0$

(B) $S_n = \frac{7n^2+1}{8n+1} \to \infty$

(C) $S_n = 2\left(\frac{7}{8} - \frac{1}{n+2} - \frac{1}{n+3}\right) \to \frac{14}{8}$

(D) $\boxed{S_n = \left(\frac{7}{8} - \frac{1}{n+2} - \frac{1}{n+3}\right)} \to \frac{7}{8}$

10. Which of the following sequences with the given nth term is bounded and monotonic? always increasing or always decreasing

(A) bounded, not monotonic → $a_n = 2 + (-1)^n$

(B) not bounded, monotonic → $a_n = \frac{n^2}{n+1}$

(C) bounded, monotonic → $\boxed{a_n = \frac{3n}{n+2}}$

(D) bounded, not monotonic → $a_n = \frac{\cos n}{n}$