Key

BC Calculus – 10.2 Notes – Geometric Series

Recall: What is a geometric sequence?
A **geometric sequence** is one in which the same number is multiplied to each term to get the next term in the sequence. The number you multiply by is called the common ratio usually denoted by r

n^{th} Term of a Geometric Sequence

The nth term of a geometric sequence with first term a_1 and common ratio r is given by:

$$a_n = a_0 \cdot r^n \quad \text{or} \quad a_n = a_1 \cdot r^{n-1} \quad \text{or} \quad a_n = a_2 \cdot r^{n-2}$$

1. 3, 6, 12, 24, 48, ... $\quad r = \frac{a_n}{a_{n-1}} = \frac{6}{3} = 2$

$a_n = a_0 r^n$	$a_n = a_1 r^{n-1}$	$a_n = a_2 r^{n-2}$
$a_n = \frac{3}{2} \cdot 2^n$	$a_n = 3 \cdot 2^{n-1}$	$a_n = 6 \cdot 2^{n-2}$

2. $25, 5, 1, \frac{1}{5}, \frac{1}{25}, \ldots$ $\quad r = \frac{5}{25} = \frac{1}{5}$

$a_n = a_0 r^n$	$a_n = a_1 r^{n-1}$	$a_n = a_2 r^{n-2}$
$a_n = 125 \cdot \left(\frac{1}{5}\right)^n$	$a_n = 25 \cdot \left(\frac{1}{5}\right)^{n-1}$	$a_n = 5 \cdot \left(\frac{1}{5}\right)^{n-2}$

$$\sum_{n=0}^{\infty} ar^n = a + ar^1 + a \cdot r^2 + a \cdot r^3 + \ldots + a \cdot r^n$$

$a \neq 0$

Geometric Infinite Series Convergence

A geometric series with ratio r diverges when $|r| \geq 1$ If $|r| < 1$ then the series converges to

$$\sum_{n=k}^{\infty} ar^n = \frac{a \cdot r^k}{1-r} \quad \text{or} \quad \boxed{\frac{a_1}{1-r}}$$

first term of the series

Where ar^k is the first term of the series.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{3}{4^n} \rightarrow 3\left(\frac{1}{4}\right)^n$

$a_1 = 3\left(\frac{1}{4}\right)^0 = 3$

$S = \frac{a_1}{1-r} \rightarrow \frac{3}{1 - 1/4} \rightarrow \frac{3}{3/4}$

$3 \cdot 4/3 = \boxed{4}$

2. $\displaystyle\sum_{n=2}^{\infty} \frac{3^{n+1}}{4^n}$ $\quad a_1 = \frac{3^{2+1}}{4^2} = \frac{3^3}{4^2} = \frac{27}{16}$

$\frac{3^n \cdot 3^1}{4^n} \rightarrow 3\left(\frac{3}{4}\right)^n \rightarrow r = \frac{3}{4} < 1$ ✓

$S = \frac{a_1}{1-r} \rightarrow \frac{27/16}{1 - 3/4} = \frac{\frac{27}{16}}{1/4} \rightarrow \frac{27}{16} \cdot \frac{4}{1} = \boxed{\frac{27}{4}}$

3. For what value of r does the infinite series $\sum_{n=0}^{\infty} 17r^n$ equal 23?

$S=23$

$a_1 = 17 \cdot r^0 = 17$

$S = \frac{a_1}{1-r}$

$23 = \frac{17}{1-r}$

$\frac{23}{1} = \frac{17}{1-r}$

$23(1-r) = 17$

$23 - 23r = 17$

$6 = 23r$

$\boxed{r = \frac{6}{23}}$

4. **Calculator active.** If $f(x) = \sum_{n=3}^{\infty} \left(\sin^2\left(\frac{x}{3}\right)\right)^n$, then $f(7) =$

*geometric series

*f(7) is the sum of this infinite geometric series when $x=7$

$a_1 = \left[\sin\left(\frac{7}{3}\right)\right]^{2\cdot 3} = 0.142935$

$r = \left[\sin\left(\frac{7}{3}\right)\right]^2 = 0.52285$

$S = \frac{a_1}{1-r} \rightarrow \frac{0.142935}{1-0.52285}$

$\boxed{S \approx 0.2995}$

10.2 Practice problems:

Find the value of each infinite series.

1. $\sum_{n=1}^{\infty} -\frac{7}{(-3)^n}$

*$S = \frac{a_1}{1-r}$

$a_1 = \frac{-7}{(-3)^1} = \frac{7}{3}$

$r = \left(\frac{-1}{3}\right)^n \rightarrow r = \frac{-1}{3}$

$|r|<1$ so series converge

$\frac{\frac{7}{3}}{1-(-1/3)} \rightarrow \frac{7/3}{4/3}$

$\boxed{S = 7/4}$

2. $\sum_{n=0}^{\infty} \frac{1}{3^n} \rightarrow \left(\frac{1}{3}\right)^n$ $\quad r = 1/3$

$a_1 = \frac{1}{3^0} \rightarrow \frac{1}{1} = 1$

$S = \frac{1}{1-(1/3)} \rightarrow \frac{1}{2/3} \rightarrow \boxed{\frac{3}{2}}$

3. $\sum_{n=0}^{\infty} e^{nx}$ Let x be a real number, with $x < 0$.

$\sum (e^x)^n$

$r = e^x < 1$

$a_1 = e^{0x} = 1$

$\boxed{S = \frac{1}{1-e^x}}$

4. $\sum_{n=1}^{\infty} \left(\frac{e}{\pi}\right)^n$

$r = \frac{e}{\pi} \rightarrow \frac{2.72}{3.14} < 1$

$a_1 = \frac{e}{\pi}$

$S = \frac{\frac{e}{\pi}}{1-\frac{e}{\pi}} \rightarrow \frac{\frac{e}{\pi}}{\frac{\pi-e}{\pi}}$

$\frac{e}{\pi} \cdot \frac{\pi}{\pi-e}$

$\boxed{S = \frac{e}{\pi-e}}$

5. $\sum_{n=1}^{\infty} \frac{3^{n+1}}{5^n}$

$\frac{3^n \cdot 3}{5^n} \to 3\left(\frac{3}{5}\right)^n$

$r = 3/5 < 1$

$a_1 = 3\left(\frac{3}{5}\right)^1 = \frac{9}{5}$

$S = \frac{9/5}{1 - 3/5} \to \frac{9/5}{2/5} \to \boxed{\frac{9}{2}}$

6. $\sum_{n=1}^{\infty} \frac{2^n}{e^{n+1}}$

$\frac{2^n}{e^n \cdot e} \to \frac{1}{e}\left(\frac{2}{e}\right)^n$

$r = \frac{2}{e} < 1$

$a_1 = \frac{2}{e^2}$

$S = \frac{\frac{2}{e^2}}{1 - 2/e} \to \frac{\frac{2}{e^2}}{\frac{e-2}{e}}$

$S = \frac{2}{e^2} \cdot \frac{e}{e-2}$

$\boxed{S = \frac{2}{e(e-2)}}$

7. $\sum_{n=0}^{\infty} (-1)^n \frac{\pi}{e^{n+1}}$

$\to \frac{\pi \cdot (-1)^n}{e \cdot e^n} \to \frac{\pi}{e}\left(\frac{-1}{e}\right)^n$

$r = \frac{1}{e} < 1$

$a_1 = \frac{\pi}{e}$

$S = \frac{\pi/e}{1-(-1/e)}$

$S = \frac{\frac{\pi}{e}}{\frac{e+1}{e}} \to \frac{\pi}{e} \cdot \frac{e}{e+1}$

$\boxed{S = \frac{\pi}{e+1}}$

8. $\sum_{n=0}^{\infty} \left(-\frac{3}{4}\right)^n$

$r = \left|\frac{-3}{4}\right| < 1$

$a_1 = \left(\frac{-3}{4}\right)^0 = 1$

$r = -3/4$

$S = \frac{1}{1-(-3/4)}$

$S = \frac{1}{7/4} = \boxed{\frac{4}{7}}$

9. What is the sum of the infinite series

$25 + -5 + 1 + -\frac{1}{5} + \frac{1}{25} + \cdots$

$r = \frac{-1}{5}$

$a_1 = 25$

$S = \frac{25}{1-(-1/5)}$

$S = \frac{25}{6/5}$

$S = \frac{5}{6} \cdot 25$

$\boxed{S = \frac{125}{6}}$

10. **Calculator active.** If $f(x) = \sum_{n=1}^{\infty} (\sin^2 2x)^n$, then $f(3) =$

$f(3) = \sum_{n=1}^{\infty} \left[\sin^2(6)\right]^n$

$a_1 = \sin^2(6) = 0.078073$

$r = \sin^2(6) = 0.078073$

$S = \frac{0.078073}{1 - 0.078073} \approx \boxed{0.0846}$

11. For what value of a does the infinite series

$$\sum_{n=0}^{\infty} a\left(\frac{2}{3}\right)^n = 14$$

$a_1 = a$

$r = 2/3$

$S = \frac{a_1}{1-r}$

$14 = \frac{a}{1-2/3}$

$14 = \frac{a}{1/3}$

$\boxed{a = 14/3}$

12. Consider the geometric series $\sum_{n=1}^{\infty} a_n$ where $a_n > 0$.

The first term of the series $a_1 = 24$, and the third term $a_3 = 6$. What are possible values for a_2?

$24, a_2, 6$

$r \quad r$

$24 \cdot r^2 = 6$

$r^2 = \frac{6}{24}$

$r^2 = 1/4$

$r = \pm\sqrt{1/4}$

$\boxed{r = 1/2 \text{ or } r = -1/2}$

13. Consider the series $\sum_{n=1}^{\infty} a_n$. If $a_1 = 32$ and $\frac{a_{n+1}}{a_n} = \frac{1}{4}$ for all integers $n \geq 1$, then $\sum_{n=1}^{\infty} a_n =$

$r = 1/4$

$S = \frac{a_1}{1-r} \rightarrow \frac{32}{1-1/4} \rightarrow \frac{32}{3/4}$

$S = 32 \cdot 4/3$

$\boxed{S = \frac{128}{3}}$

14. Use a geometric series to write $0.\overline{2}$ as the ratio of two integers.

$0.\overline{2} = 0.2 + 0.02 + 0.002 + 0.0002$

$r = \frac{0.02}{0.2} = \frac{1}{10}$

$a_1 = 0.2 = \frac{2}{10}$

$S = \frac{\frac{2}{10}}{1 - 1/10} = \frac{\frac{2}{10}}{9/10} = \boxed{\frac{2}{9}}$

10.2 Working with Geometric Series

Test Prep

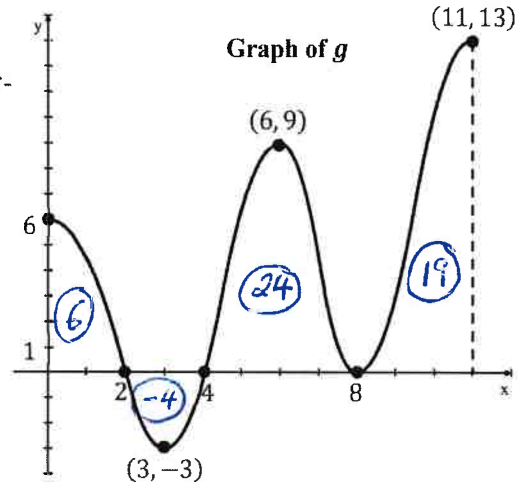
15. If x and y are positive real numbers, which of the following conditions guarantees the infinite series $\sum_{n=0}^{\infty} \frac{x^{n+1}}{y^{2n+1}}$ is geometric and converges?

$\frac{x^n \cdot x^1}{y^{2n} \cdot y^1} \rightarrow \frac{x}{y}\left(\frac{x}{y^2}\right)^n$

$r = \frac{x}{y^2} < 1$ so $y^2 > x$

(A) $x < y$ $\boxed{\text{(B) } x < y^2}$ (C) $x > y^2$ (D) $x > y$

16. The figure to the right shows a portion of the graph of the differentiable function g. Let h be the function defined by $h(x) = \int_4^x g(t)\,dt$. The areas of the regions bounded by the x-axis and the graph of g on the intervals, $[0,2]$, $[2,4]$, $[4,8]$ and $[8,11]$ are $6, 4, 24$, and 19, respectively.



a. Must there exist a value of c, for $2 < c < 4$, such that $h(c) = 3.5$? Justify your answer.

$h(2) = \int_4^2 g(t)dt = 4$

$h(4) = \int_4^4 g(t)dt = 0$

Since $h(4) < 3.5 < h(2)$
By IVT, $h(c) = 3.5$
on interval $[2, 4]$

b. Find the average value of g over the interval, $0 \le x \le 11$. Show the computations that lead to your answer.

Avg. value theorem

$$\frac{1}{b-a}\int_a^b g(x)dx \rightarrow \frac{1}{11-0}\int_0^{11} g(x)dx \rightarrow \frac{1}{11}\left[6-4+24+19\right] \rightarrow \frac{1}{11}(45) = \boxed{\frac{45}{11}}$$

c. Evaluate $\lim_{x\to 8} \frac{h(x)-3x}{x^2-64}$. $\rightarrow \frac{h(8)-24}{8^2-64} \rightarrow \frac{24-24}{64-64} \rightarrow \frac{0}{0}$ Indeterminate Form L'Hopital's Rule

$$\lim_{x\to 8} \frac{h'(x)-3}{2x} \rightarrow \frac{h'(8)-3}{16} \rightarrow \frac{0-3}{16} = \boxed{\frac{-3}{16}}$$

$h'(x) = \frac{d}{dx}\int_4^x g(t)dt \rightarrow g(x)$: $h'(8) = g(8) = 0$

d. Is there a value r such that the series $30 + 30r + 30r^2 + \cdots + 30r^n$ equals the value of $g(6)$?

$g(6) = 9$

$a_1 = 30$

$S = \frac{a_1}{1-r}$

$9 = \frac{30}{1-r}$

$9(1-r) = 30$

$9 - 9r = 30$

$-9r = +21$

$r = \frac{-21}{9} = \frac{-7}{3}$

No, since r would cause the series to diverge.