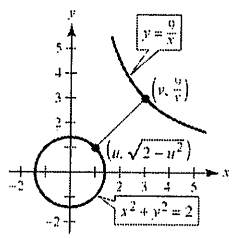
98. Consider circle $x^2 + y^2 = 2$ and hyperbola $y = \dfrac{9}{x}$.

Let $\left(u, \sqrt{2 - u^2}\right)$ and $\left(v, \dfrac{9}{v}\right)$ be points on the circle and hyperbola, respectively. We need to minimize the distance between these 2 points:

$$(\text{Distance})^2 = f(u, v) = (u - v)^2 + \left(\sqrt{2 + u^2} - \frac{9}{v}\right)^2.$$

The tangent lines at $(1, 1)$ and $(3, 3)$ are both perpendicular to $y = x$, and so they are parallel.

The minimum value is $(3 - 1)^2 + (3 - 1)^2 = 8$.



Section 10.2 Plane Curves and Parametric Equations

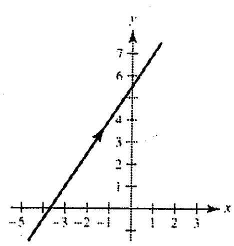
1. $x = 2t - 3$

$y = 3t + 1$

$t = \dfrac{x + 3}{2}$

$y = 3\left(\dfrac{x + 3}{2}\right) + 1 = \dfrac{3}{2}x + \dfrac{11}{2}$

$3x - 2y + 11 = 0$

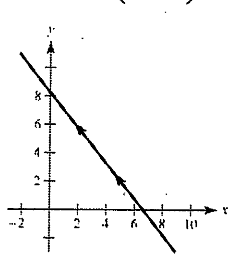


2. $x = 5 - 4t$

$y = 2 + 5t$

$t = \dfrac{5 - x}{4}$

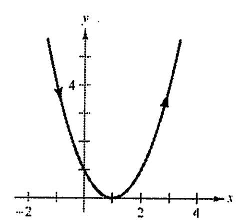
$y = 2 + 5\left(\dfrac{5 - x}{4}\right) = -\dfrac{5}{4}x + \dfrac{33}{4}$



3. $x = t + 1$

$y = t^2$

$y = (x - 1)^2$



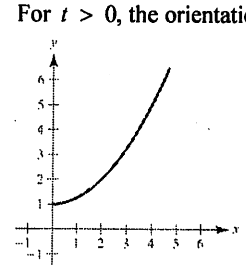
4. $x = 2t^2$

$y = t^4 + 1$

$y = \left(\dfrac{x}{2}\right)^2 + 1 = \dfrac{x^2}{4} + 1, x \geq 0$

For $t < 0$, the orientation is right to left.

For $t > 0$, the orientation is left to right.

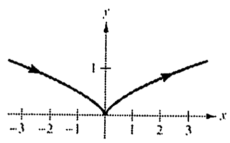


5. $x = t^3$

$y = \frac{1}{2}t^2$

$y = t^3$ implies $t = x^{1/3}$

$y = \frac{1}{2}x^{2/3}$



6. $x = t^2 + t,\ y = t^2 - t$

Subtracting the second equation from the first, you have

$x - y = 2t$ or $t = \dfrac{x - y}{2}.$

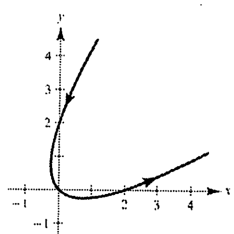
$$y = \frac{(x - y)^2}{4} - \frac{x - y}{2}$$

t	-2	-1	0	1	2
x	2	0	0	2	6
y	6	2	0	0	2

Because the discriminant is

$B^2 - 4AC = (-2)^2 - 4(1)(1) = 0,$

the graph is a rotated parabola.

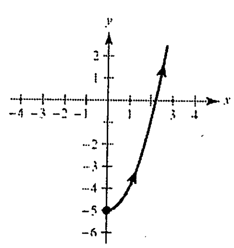


7. $x = \sqrt{t}$

$y = t - 5$

$x^2 = t$

$y = x^2 - 5, x \ge 0$

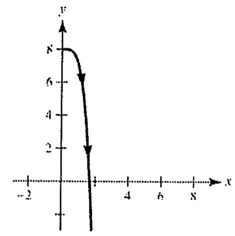


8. $x = \sqrt[4]{t}$

$y = 8 - t$

$x^4 = t$

$y = 8 - x^4, x \ge 0$

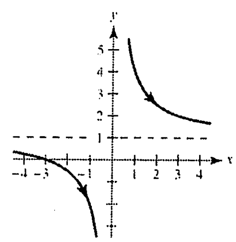


9. $x = t - 3$

$y = \dfrac{t}{t - 3}$

$t = x + 3$

$$y = \frac{x + 3}{(x + 3) - 3} = 1 + \frac{3}{x} = \frac{x + 3}{x}$$

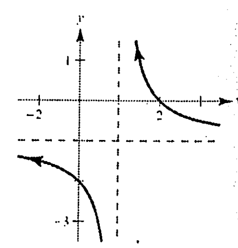


10. $x = 1 + \dfrac{1}{t}$

$y = t - 1$

$x = 1 + \dfrac{1}{t}$ implies $t = \dfrac{1}{x - 1}$

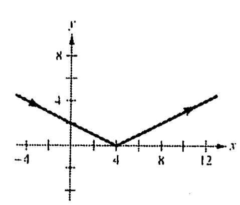
$y = \dfrac{1}{x - 1} - 1$



11. $x = 2t$

$y = |t - 2|$

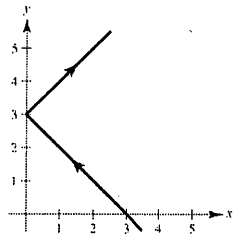
$$y = \left|\frac{x}{2} - 2\right| = \frac{|x - 4|}{2}$$



12. $x = |t - 1|$

$y = t + 2$

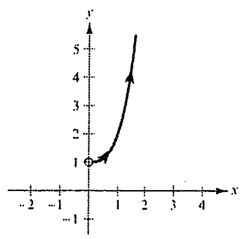
$x = |(y - 2) - 1| = |y - 3|$



13. $x = e^t, x > 0$

$y = e^{3t} + 1$

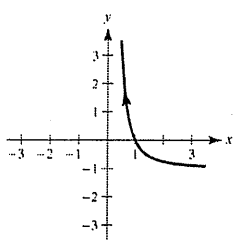
$y = x^3 + 1, x > 0$



14. $x = e^{-t}, x > 0$

$y = e^{2t} - 1$

$y = x^{-2} - 1 = \dfrac{1}{x^2} - 1, x > 0$



15. $x = \sec\theta$

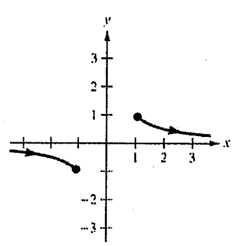
$y = \cos\theta$

$0 \le \theta < \dfrac{\pi}{2}, \dfrac{\pi}{2} < \theta \le \pi$

$xy = 1$

$y = \dfrac{1}{x}$

$|x| \ge 1, |y| \le 1$



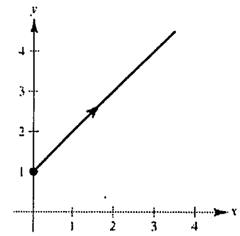
16. $x = \tan^2\theta$

$y = \sec^2\theta$

$\sec^2\theta = \tan^2\theta + 1$

$y = x + 1$

$x \ge 0$

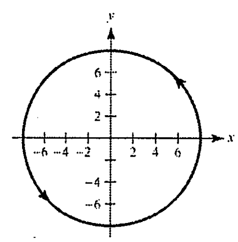


17. $x = 8\cos\theta$

$y = 8\sin\theta$

$x^2 + y^2 = 64\cos^2\theta + 64\sin^2\theta = 64(1) = 64$

Circle



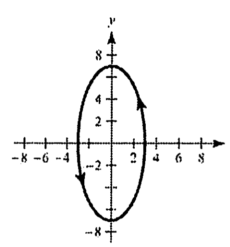
18. $x = 3\cos\theta$

$y = 7\sin\theta$

$\left(\dfrac{x}{3}\right)^2 + \left(\dfrac{y}{7}\right)^2 = \cos^2\theta + \sin^2\theta = 1$

$\dfrac{x^2}{9} + \dfrac{y^2}{49} = 1$

Ellipse



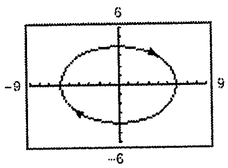
19. $x = 6\sin 2\theta$

$y = 4\cos 2\theta$

$$\left(\frac{x}{6}\right)^2 + \left(\frac{y}{4}\right)^2 = \sin^2 2\theta + \cos^2 2\theta = 1$$

$$\frac{x^2}{36} + \frac{y^2}{16} = 1$$

Ellipse



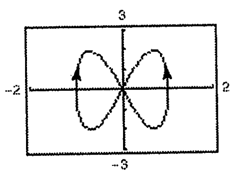
20. $x = \cos\theta$

$y = 2\sin 2\theta$

$y = 4\sin\theta\cos\theta$

$1 - x^2 = \sin^2\theta$

$y = \pm 4x\sqrt{1 - x^2}$



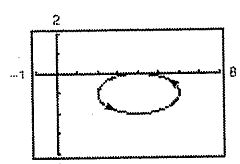
21. $x = 4 + 2\cos\theta$

$y = -1 + \sin\theta$

$$\frac{(x-4)^2}{4} = \cos^2\theta$$

$$\frac{(y+1)^2}{1} = \sin^2\theta$$

$$\frac{(x-4)^2}{4} + \frac{(y+1)^2}{1} = 1$$



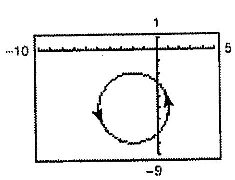
22. $x = -2 + 3\cos\theta$

$y = -5 + 3\sin\theta$

$$(x+2)^2 + (y+5)^2 = 9\cos^2\theta + 9\sin^2\theta = 9$$

$$(x+2)^2 + (y+5)^2 = 9$$

Circle



23. $x = -3 + 4\cos\theta$

$y = 2 + 5\sin\theta$

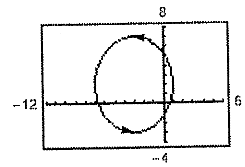
$x + 3 = 4\cos\theta$

$y - 2 = 5\sin\theta$

$$\left(\frac{x+3}{4}\right)^2 + \left(\frac{y-2}{5}\right)^2 = \cos^2\theta + \sin^2\theta = 1$$

$$\frac{(x+3)^2}{16} + \frac{(y-2)^2}{25} = 1$$

Ellipse



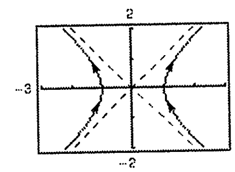
24. $x = \sec\theta$

$y = \tan\theta$

$x^2 = \sec^2\theta$

$y^2 = \tan^2\theta$

$x^2 - y^2 = 1$



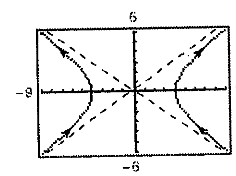
25. $x = 4\sec\theta$

$y = 3\tan\theta$

$$\frac{x^2}{16} = \sec^2\theta$$

$$\frac{y^2}{9} = \tan^2\theta$$

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$



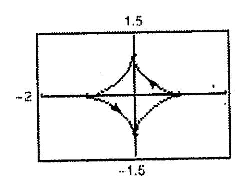
26. $x = \cos^3\theta$

$y = \sin^3\theta$

$x^{2/3} = \cos^2\theta$

$y^{2/3} = \sin^2\theta$

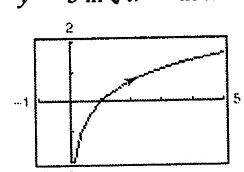
$x^{2/3} + y^{2/3} = 1$



27. $x = t^3$

$y = 3\ln t$

$y = 3\ln\sqrt[3]{x} = \ln x$

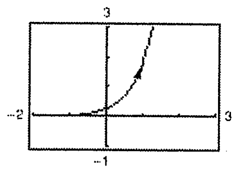


28. $x = \ln 2t$

$y = t^2$

$t = \frac{e^x}{2}$

$y = \frac{e^{2x}}{r} = \frac{1}{4}e^{2x}$



29. $x = e^{-t}$

$y = e^{3t}$

$e^t = \frac{1}{x}$

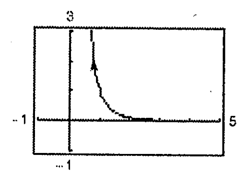
$e^t = \sqrt[3]{y}$

$\sqrt[3]{y} = \frac{1}{x}$

$y = \frac{1}{x^3}$

$x > 0$

$y > 0$



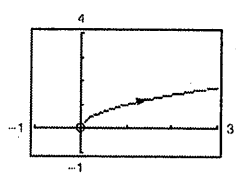
30. $x = e^{2t}$

$y = e^t$

$y^2 = x$

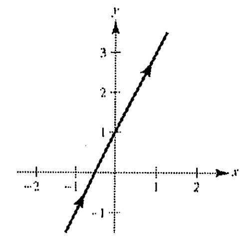
$y > 0$

$y = \sqrt{x}, x > 0$



31. By eliminating the parameters in (a) – (d), you get $y = 2x + 1$. They differ from each other in orientation and in restricted domains. These curves are all smooth except for (b).

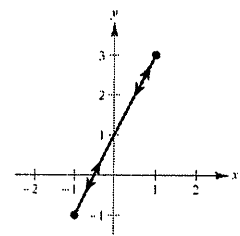
(a) $x = t, y = 2t + 1$



(b) $x = \cos\theta$ $\quad$ $y = 2\cos\theta + 1$

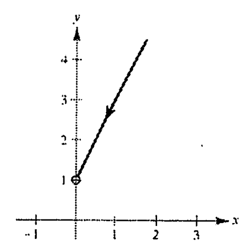
$-1 \le x \le 1$ $\quad$ $-1 \le y \le 3$

$\frac{dx}{d\theta} = \frac{dy}{d\theta} = 0$ when $\theta = 0, \pm\pi, \pm 2\pi, \ldots$.



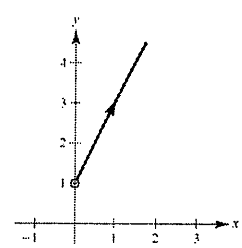
(c) $x = e^{-t}$ $\quad$ $y = 2e^{-t} + 1$

$x > 0$ $\quad$ $y > 1$



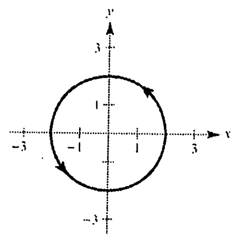
(d) $x = e^t$ $\quad$ $y = 2e^t + 1$

$x > 0$ $\quad$ $y > 1$



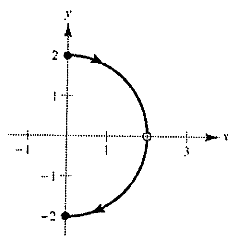
32. By eliminating the parameters in (a) – (d), you get $x^2 + y^2 = 4$. They differ from each other in orientation and in restricted domains. These curves are all smooth.

(a) $x = 2\cos\theta,\ y = 2\sin\theta$



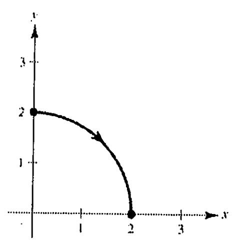
(b) $x = \dfrac{\sqrt{4t^2 - 1}}{|t|} = \sqrt{4 - \dfrac{1}{t^2}}$ $\qquad y = \dfrac{1}{t}$

$x \ge 0,\ x \ne 2$ $\qquad y \ne 0$



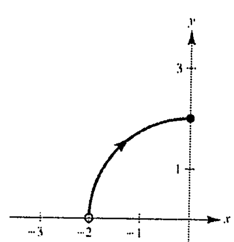
(c) $x = \sqrt{t}$ $\qquad y = \sqrt{4 - t}$

$x \ge 0$ $\qquad y \ge 0$



(d) $x = -\sqrt{4 - e^{2t}}$ $\qquad y = e^t$

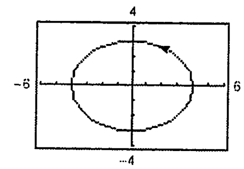
$-2 < x \le 0$ $\qquad y > 0$

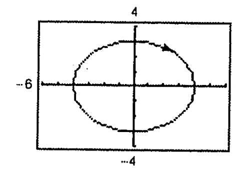


33. The curves are identical on $0 < \theta < \pi$. They are both smooth. They represent $y = 2(1 - x^2)$ for $-1 \le x \le 1$. The orientation is from right to left in part (a) and in part (b).

34. The orientations are reversed. The graphs are the same. They are both smooth.

35. (a)





(b) The orientation of the second curve is reversed.

(c) The orientation will be reversed.

(d) Answers will vary. For example,

$x = 2\sec t \qquad x = 2\sec(-t)$

$y = 5\sin t \qquad y = 5\sin(-t)$

have the same graphs, but their orientations are reversed.

36. The set of points (x, y) corresponding to the rectangular equation of a set of parametric equations does not show the orientation of the curve nor any restriction on the domain of the original parametric equations.

37.

$$x = x_1 + t(x_2 - x_1)$$
$$y = y_1 + t(y_2 - y_1)$$
$$\frac{x - x_1}{x_2 - x_1} = t$$
$$y = y_1 + \left(\frac{x - x_1}{x_2 - x_1}\right)(y_2 - y_1)$$
$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$
$$y - y_1 = m(x - x_1)$$

38.

$$x = h + r\cos\theta$$
$$y = k + r\sin\theta$$
$$\cos\theta = \frac{x - h}{r}$$
$$\sin\theta = \frac{y - k}{r}$$
$$\cos^2\theta + \sin^2\theta = \frac{(x - h)^2}{r^2} + \frac{(y - k)^2}{r^2} = 1$$
$$(x - h)^2 + (y - k)^2 = r^2$$

39.
$$x = h + a\cos\theta$$
$$y = k + b\sin\theta$$
$$\frac{x-h}{a} = \cos\theta$$
$$\frac{y-k}{b} = \sin\theta$$
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

40.
$$x = h + a\sec\theta$$
$$y = k + b\tan\theta$$
$$\frac{x-h}{a} = \sec\theta$$
$$\frac{y-k}{b} = \tan\theta$$
$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

41. From Exercise 37 you have

$x = 4t$

$y = -7t$

Solution not unique

42. From Exercise 37 you have

$x = 1 + 4t$

$y = 4 - 6t.$

Solution not unique

43. From Exercise 38 you have

$x = 3 + 2\cos\theta$

$y = 1 + 2\sin\theta$

Solution not unique

44. From Exercise 38 you have

$x = -6 + 4\cos\theta$

$y = 2 + 4\sin\theta$

45. From Exercise 39 you have

$a = 10, c = 8 \Rightarrow b = 6$

$x = 10\cos\theta$

$y = 6\sin\theta$

Center: $(0, 0)$

Solution not unique

46. From Exercise 39 you have

$a = 5, c = 3 \Rightarrow b = 4$

$x = 4 + 5\cos$

$y = 2 + 4\sin\theta.$

Center: $(4, 2)$

Solution not unique

47. From Exercise 40 you have

$a = 4, c = 5 \Rightarrow b = 3$

$x = 4\sec\theta$

$y = 3\tan\theta.$

Center: $(0, 0)$

Solution not unique

48. From Exercise 40 you have

$a = 1, c = 2 \Rightarrow b = \sqrt{3}$

$x = \sqrt{3}\tan\theta$

$y = \sec\theta.$

Center: $(0, 0)$

Solution not unique

The transverse axis is vertical, so, x and y are interchanged.

49. $y = 6x - 5$

Examples:

$x = t, y = 6t - 5$

$x = t + 1, y = 6t + 1$

50. $y = \dfrac{4}{x-1}$

Examples:

$x = t, y = \dfrac{4}{t-1}$

$x = t + 1, y = \dfrac{4}{t}$

51. $y = x^3$

Example

$x = t, \quad y = t^3$

$x = \sqrt[3]{t}, \quad y = t$

$x = \tan t, \quad y = \tan^3 t$

52. $y = x^2$

Example

$x = t, \quad y = t^2$

$x = t^3, \quad y = t^6$

53. $y = 2x - 5$

At $(3, 1), t = 0$: $\quad x = 3 - t$

$y = 2(3 - t) - 5 = -2t + 1$

or, $x = t + 3$

$y = 2t + 1$

54. $y = 4x + 1$

At $(-2, -7), t = -1$: $\quad x = -1 + t$

$y = 4(-1 + t) + 1 = 4t - 3$

55. $y = x^2$

$t = 4$ at $(4, 16)$: $\quad x = t$

$y = t^2$

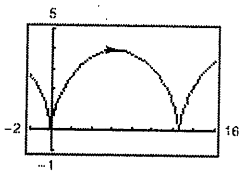
56. $y = 4 - x^2$

$t = 1$ at $(1, 3)$: $\quad x = t$

$y = 4 - t^2$

57. $x = 2(\theta - \sin\theta)$

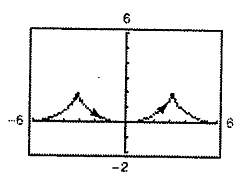
$y = 2(1 - \cos\theta)$



Not smooth at $\theta = 2n\pi$

58. $x = \theta + \sin\theta$

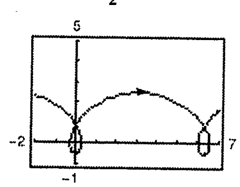
$y = 1 - \cos\theta$



Not smooth at $x = (2n - 1)\pi$

59. $x = \theta - \frac{3}{2}\sin\theta$

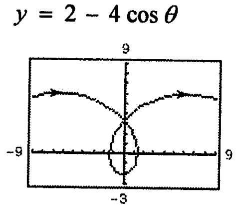
$y = 1 - \frac{3}{2}\cos\theta$



Smooth everywhere

60. $x = 2\theta - 4\sin\theta$

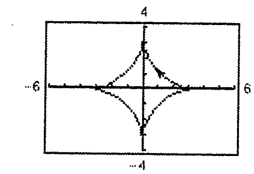
$y = 2 - 4\cos\theta$



Smooth everywhere

61. $x = 3\cos^3\theta$

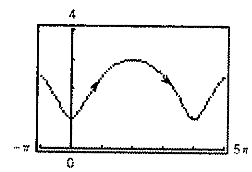
$y = 3\sin^3\theta$



Not smooth at $(x, y) = (\pm 3, 0)$ and $(0, \pm 3)$, or $\theta = \frac{1}{2}n\pi$.

62. $x = 2\theta - \sin\theta$

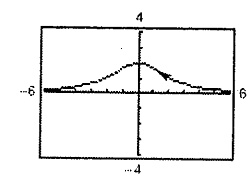
$y = 2 - \cos\theta$



Smooth everywhere

63. $x = 2\cot\theta$

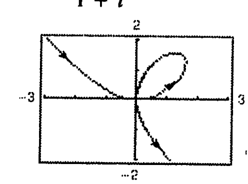
$y = 2\sin^2\theta$



Smooth everywhere

64. $x = \dfrac{3t}{1 + t^3}$

$y = \dfrac{3t^2}{1 + t^3}$



Smooth everywhere

65. If f and g are continuous functions of t on an interval I, then the equations $x = f(t)$ and $y = g(t)$ are called parametric equations and t is the parameter. The set of points (x, y) obtained as t varies over I is the graph. Taken together, the parametric equations and the graph are called a plane curve C.

66. Each point (x, y) in the plane is determined by the plane curve $x = f(t)$, $y = g(t)$. For each t, plot (x, y). As t increases, the curve is traced out in a specific direction called the orientation of the curve.

67. A curve C represented by $x = f(t)$ and $y = g(t)$ on an interval I is called smooth when f' and g' are continuous on I and not simultaneously 0, except possibly at the endpoints of I.

68. The graph matches (a) because $x = t \Rightarrow y = t^2 = x^2$. For (b), you have $y = t \Rightarrow x = t^2 = y^2$, which is not the correct parabola.

69. Matches (d) because $(4, 0)$ is on the graph.

70. Matches (a) because $(0, 2)$ is on the graph.

71. Matches (b) because $(1, 0)$ is on the graph.

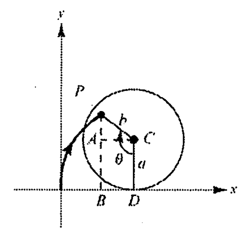
72. Matches (c) because the graph is undefined when $\theta = 0$.

73. When the circle has rolled θ radians, you know that the center is at $(a\theta, a)$.

$$\sin\theta = \sin(180° - \theta) = \frac{|AC|}{b} = \frac{|BD|}{b} \text{ or } |BD| = b\sin\theta$$

$$\cos\theta = -\cos(180° - \theta) = \frac{|AP|}{-b} \text{ or } |AP| = -b\cos\theta$$

So, $x = a\theta - b\sin\theta$ and $y = a - b\cos\theta$.



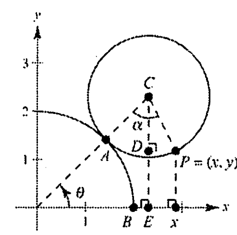
74. Let the circle of radius 1 be centered at C. A is the point of tangency on the line OC. $OA = 2$, $AC = 1$, $OC = 3$. $P = (x, y)$ is the point on the curve being traced out as the angle θ changes $\widehat{AB} = \widehat{AP}$, $\widehat{AB} = 2\theta$ and $\widehat{AP} = \alpha \Rightarrow \alpha = 2\theta$. Form the right triangle ΔCDP. The angle $OCE = (\pi/2) - \theta$ and

$$\angle DCP = \alpha - \left(\frac{\pi}{2} - \theta\right) = \alpha + \theta - \left(\frac{\pi}{2}\right) = 3\theta - \left(\frac{\pi}{2}\right).$$

$$x = OE + Ex = 3\sin\left(\frac{\pi}{2} - \theta\right) + \sin\left(3\theta - \frac{\pi}{2}\right) = 3\cos\theta - \cos 3\theta$$

$$y = EC - CD = 3\sin\theta - \cos\left(3\theta - \frac{\pi}{2}\right) = 3\sin\theta - \sin 3\theta$$

So, $x = 3\cos\theta - \cos 3\theta$, $y = 3\sin\theta - \sin 3\theta$.



75. False

$x = t^2 \Rightarrow x \ge 0$

$y = t^2 \Rightarrow y \ge 0$

The graph of the parametric equations is only a portion of the line $y = x$ when $x \ge 0$.

76. False. Let $x = t^2$ and $y = t$. Then $x = y^2$ and y is not a function of x.

77. True. $y = \cos x$

78. $x = 8\cos t$, $y = 8\sin t$

(a) $\left(\frac{x}{8}\right)^2 + \left(\frac{y}{8}\right)^2 = \cos^2 t + \sin^2 t = 1$

$x^2 + y^2 = 64$ Circle radius 8,

Center: $(0, 0)$ Oriented counterclockwise

(b) Circle of radius 8, but Center: $(3, 6)$

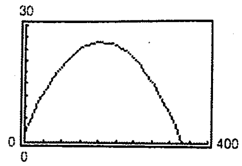
(c) The orientation is reversed.

79. (a) $100 \text{ mi/hr} = \frac{(100)(5280)}{3600} = \frac{440}{3} \text{ ft/sec}$

$$x = (v_0\cos\theta)t = \left(\frac{440}{3}\cos\theta\right)t$$

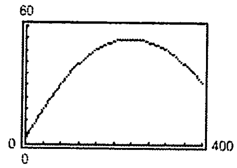
$$y = h + (v_0\sin\theta)t - 16t^2 = 3 + \left(\frac{440}{3}\sin\theta\right)t - 16t^2$$

(b)



It is not a home run when $x = 400, y < 10$.

(c)



Yes, it's a home run when $x = 400, y > 10$.

(d) You need to find the angle θ (and time t) such that

$$x = \left(\frac{440}{3}\cos\theta\right)t = 400$$

$$y = 3 + \left(\frac{440}{3}\sin\theta\right)t - 16t^2 = 10.$$

From the first equation $t = 1200/440\cos\theta$. Substituting into the second equation,

$$10 = 3 + \left(\frac{440}{3}\sin\theta\right)\left(\frac{1200}{440\cos\theta}\right) - 16\left(\frac{1200}{440\cos\theta}\right)^2$$

$$7 = 400\tan\theta - 16\left(\frac{120}{44}\right)^2\sec^2\theta = 400\tan\theta - 16\left(\frac{120}{44}\right)^2\left(\tan^2\theta + 1\right).$$

You now solve the quadratic for $\tan\theta$:

$$16\left(\frac{120}{44}\right)^2\tan^2\theta - 400\tan\theta + 7 + 16\left(\frac{120}{44}\right)^2 = 0.$$

$\tan\theta \approx 0.35185 \Rightarrow \theta \approx 19.4°$

80. (a) $x = (v_0\cos\theta)t$

$y = h + (v_0\sin\theta)t - 16t^2$

$$t = \frac{x}{v_0\cos\theta} \Rightarrow y = h + (v_0\sin\theta)\frac{x}{v_0\cos\theta} - 16\left(\frac{x}{v_0\cos\theta}\right)^2$$

$$y = h + (\tan\theta)x - \frac{16\sec^2\theta}{v_0^2}x^2$$

(b) $y = 5 + x - 0.005x^2 = h + (\tan\theta)x - \dfrac{16\sec^2\theta}{v_0^2}x^2$

$h = 5$, $\tan\theta = 1 \Rightarrow \theta = \dfrac{\pi}{4}$, and

$$0.005 = \frac{16\sec^2(\pi/4)}{v_0^2} = \frac{16}{v_0^2}(2)$$

$$v_0^2 = \frac{32}{0.005} = 6400 \Rightarrow v_0 = 80.$$

So, $x = (80\cos(45°))t$

$y = 5 + (80\sin(45°))t - 16t^2.$