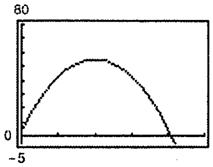
(c)



(d) Maximum height: $y = 55$ (at $x = 100$)

Range: 204.88

Section 10.3 Parametric Equations and Calculus

1. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-6}{2t} = -\frac{3}{t}$

2. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-1}{(1/3)t^{-2/3}} = -3t^{2/3}$

3. $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-2\cos\theta\sin\theta}{2\sin\theta\cos\theta} = -1$

$\left[\text{Note: } x + y = 1 \Rightarrow y = 1 - x \text{ and } \frac{dy}{d\theta} = -1\right]$

4. $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(-1/2)e^{-\theta/2}}{2e^{\theta}} = -\frac{1}{4}e^{-3\theta/2} = \frac{-1}{4e^{3\theta/2}}$

5. $x = 4t,\ y = 3t - 2$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3}{4}$

$\frac{d^2y}{dx^2} = 0$

At $t = 3$, slope is $\frac{3}{4}$. (Line)

Neither concave upward nor downward

6. $x = \sqrt{t},\ y = 3t - 1$

$\frac{dy}{dx} = \frac{3}{1/(2\sqrt{t})} = 6\sqrt{t} = 6$ when $t = 1$.

$\frac{d^2y}{dx^2} = \frac{3/\sqrt{t}}{1/(2\sqrt{t})} = 6$

Concave upward

7. $x = t + 1,\ y = t^2 + 3t$

$\frac{dy}{dx} = \frac{2t + 3}{1} = 1$ when $t = -1$.

$\frac{d^2y}{dx^2} = 2$

Concave upward

8. $x = t^2 + 5t + 4,\ y = 4t$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4}{2t + 5}$

$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{4}{2t + 5}\right]}{dx/dt} = \frac{\frac{-8}{(2t + 5)^2}}{2t + 5} = \frac{-8}{(2t + 5)^3}$

At $t = 0$, $\frac{dy}{dx} = \frac{4}{5}$.

At $t = 0$, $\frac{d^2y}{dx^2} = -\frac{8}{125}$

Concave downward

9. $x = 4\cos\theta,\ y = 4\sin\theta$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{4\cos\theta}{-4\sin\theta} = \frac{-\cos\theta}{\sin\theta} = -\cot\theta$

$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}[-\cot\theta]}{dx/d\theta} = \frac{\csc^2\theta}{-4\sin\theta} = \frac{-1}{4\sin^3\theta} = -\frac{1}{4}\csc^3\theta$

At $\theta = \frac{\pi}{4}$, $\frac{dy}{dx} = -1$.

$\frac{d^2y}{dx^2} = \frac{-1}{4\left(\sqrt{2}/2\right)^3} = \frac{-\sqrt{2}}{2}$

Concave downward

10. $x = \cos\theta,\ y = 3\sin\theta$

$\frac{dy}{dx} = \frac{3\cos\theta}{-\sin\theta} = -3\cot\theta$. $\frac{dy}{dx}$ is undefined when $\theta = 0$.

$\frac{d^2y}{dx^2} = \frac{3\csc^2\theta}{-\sin\theta} = \frac{-3}{\sin^3\theta}$. $\frac{d^2y}{dx^2}$ is undefined when $\theta = 0$.

Neither concave upward nor downward

11. $x = 2 + \sec\theta,\ y = 1 + 2\tan\theta$

$$\frac{dy}{dx} = \frac{2\sec^2\theta}{\sec\theta\tan\theta} = \frac{2\sec\theta}{\tan\theta} = 2\csc\theta = 4 \text{ when } \theta = \frac{\pi}{6}.$$

$$\frac{d^2y}{dx^2} = \frac{d\left[\frac{dy}{dx}\right]}{\frac{dx}{d\theta}} = \frac{-2\csc\theta\cot\theta}{\sec\theta\tan\theta} = -2\cot^3\theta = -6\sqrt{3} \text{ when } \theta = \frac{\pi}{6}.$$

Concave downward

12. $x = \sqrt{t},\ y = \sqrt{t-1}$

$$\frac{dy}{dx} = \frac{1/\left(2\sqrt{t-1}\right)}{1/\left(2\sqrt{t}\right)} = \frac{\sqrt{t}}{\sqrt{t-1}} = \sqrt{2} \text{ when } t = 2.$$

$$\frac{d^2y}{dx^2} = \frac{\left[\sqrt{t-1}/\left(2\sqrt{t}\right) - \sqrt{t}\left(1/2\sqrt{t-1}\right)\right]/(t-1)}{1/\left(2\sqrt{t}\right)} = \frac{-1}{(t-1)^{3/2}} = -1 \text{ when } t = 2.$$

Concave downward

13. $x = \cos^3\theta,\ y = \sin^3\theta$

$$\frac{dy}{dx} = \frac{3\sin^2\theta\cos\theta}{-3\cos^2\theta\sin\theta} = -\tan\theta = -1 \text{ when } \theta = \frac{\pi}{4}.$$

$$\frac{d^2y}{dx^2} = \frac{-\sec^2\theta}{-3\cos^2\theta\sin\theta} = \frac{1}{3\cos^4\theta\sin\theta} = \frac{\sec^4\theta\csc\theta}{3} = \frac{4\sqrt{2}}{3} \text{ when } \theta = \frac{\pi}{4}.$$

Concave upward

14. $x = \theta - \sin\theta,\ y = 1 - \cos\theta$

$$\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta} = 0 \text{ when } \theta = \pi.$$

$$\frac{d^2y}{dx^2} = \frac{\dfrac{\left[(1-\cos\theta)\cos\theta - \sin^2\theta\right]}{(1-\cos\theta)^2}}{(1-\cos\theta)} = \frac{-1}{(1-\cos\theta)^2} = -\frac{1}{4} \text{ when } \theta = \pi.$$

Concave downward

15. $x = 2\cot\theta,\ y = 2\sin^2\theta$

$$\frac{dy}{dx} = \frac{4\sin\theta\cos\theta}{-2\csc^2\theta} = -2\sin^3\theta\cos\theta$$

At $\left(-\frac{2}{\sqrt{3}}, \frac{3}{2}\right)$, $\theta = \frac{2\pi}{3}$, and $\frac{dy}{dx} = \frac{3\sqrt{3}}{8}$.

Tangent line: $y - \frac{3}{2} = \frac{3\sqrt{3}}{8}\left(x + \frac{2}{\sqrt{3}}\right)$

$3\sqrt{3}x - 8y + 18 = 0$

At $(0, 2)$, $\theta = \frac{\pi}{2}$, and $\frac{dy}{dx} = 0$.

Tangent line: $y - 2 = 0$

At $\left(2\sqrt{3}, \frac{1}{2}\right)$, $\theta = \frac{\pi}{6}$, and $\frac{dy}{dx} = -\frac{\sqrt{3}}{8}$.

Tangent line: $y - \frac{1}{2} = -\frac{\sqrt{3}}{8}\left(x - 2\sqrt{3}\right)$

$\sqrt{3}x + 8y - 10 = 0$

16. $x = 2 - 3\cos\theta,\ y = 3 + 2\sin\theta$

$$\frac{dy}{dx} = \frac{2\cos\theta}{3\sin\theta} = \frac{2}{3}\cot\theta$$

At $(-1, 3)$, $\theta = 0$, and $\frac{dy}{dx}$ is undefined.

Tangent line: $x = -1$

At $(2, 5)$, $\theta = \frac{\pi}{2}$, and $\frac{dy}{dx} = 0$.

Tangent line: $y = 5$

At $\left(\frac{4 + 3\sqrt{3}}{2}, 2\right)$, $\theta = \frac{7\pi}{6}$, and $\frac{dy}{dx} = \frac{2\sqrt{3}}{3}$.

Tangent line:

$$y - 2 = \frac{2\sqrt{3}}{3}\left(x - \frac{4 + 3\sqrt{3}}{2}\right)$$

$2\sqrt{3}x - 3y - 4\sqrt{3} - 3 = 0$

17. $x = t^2 - 4$

$y = t^2 - 2t$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t - 2}{2t}$$

At $(0, 0)$, $t = 2$, $\frac{dy}{dx} = \frac{1}{2}$.

Tangent line: $y = \frac{1}{2}x$

$2y - x = 0$

At $(-3, -1)$, $t = 1$, $\frac{dy}{dx} = 0$.

Tangent line: $y = -1$

$y + 1 = 0$

At $(-3, 3)$, $t = -1$, $\frac{dy}{dx} = 2$.

Tangent line: $y - 3 = 2(x + 3)$

$2x - y + 9 = 0$

18. $x = t^4 + 2$

$y = t^3 + t$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 + 1}{4t^3}$$

At $(2, 0)$, $t = 0$, $\frac{dy}{dx}$ undefined.

Tangent line: $x = 2$ (vertical tangent)

At $(3, -2)$, $t = -1$, $\frac{dy}{dx} = -1$.

Tangent line: $y + 2 = -(x - 3)$

$y = -x + 1$

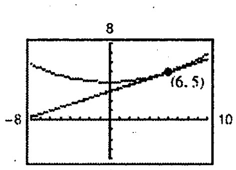
At $(18, 10)$, $t = 2$, $\frac{dy}{dx} = \frac{13}{32}$.

Tangent line: $y - 10 = \frac{13}{32}(x - 18)$

$y = \frac{13}{32}x + \frac{43}{16}$

19. $x = 6t$, $y = t^2 + 4$, $t = 1$

(a), (d)



(b) At $t = 1$, $(x, y) = (6, 5)$, and

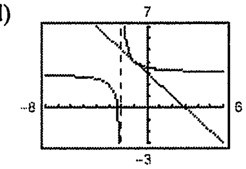
$$\frac{dx}{dt} = 6, \frac{dy}{dt} = 2, \frac{dy}{dx} = \frac{1}{3}.$$

(c) $y - 5 = \frac{1}{3}(x - 6)$

$y = \frac{1}{3}x + 3$

20. $x = t - 2$, $y = \frac{1}{t} + 3$, $t = 1$

(a), (d)



(b) At $t = 1$, $(x, y) = (-1, 4)$, and

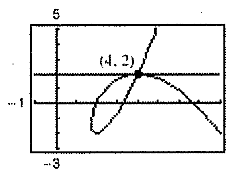
$$\frac{dx}{dt} = 1, \frac{dy}{dt} = -1, \frac{dy}{dx} = -1.$$

(c) $y - 4 = -(x + 1)$

$y = -x + 3$

21. $x = t^2 - t + 2$, $y = t^3 - 3t$, $t = -1$

(a), (d)



(b) At $t = -1$, $(x, y) = (4, 2)$, and

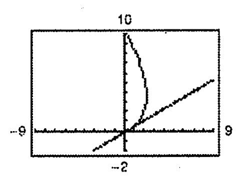
$$\frac{dx}{dt} = -3, \frac{dy}{dt} = 0, \frac{dy}{dx} = 0.$$

(c) $\frac{dy}{dx} = 0$. At $(4, 2)$, $y - 2 = 0(x - 4)$

$y = 2$.

22. $x = 3t - t^2$, $y = 2t^{3/2}$, $t = \frac{1}{4}$

(a), (d)



(b) At $t = \frac{1}{4}$, $(x, y) = \left(\frac{11}{16}, \frac{1}{4}\right)$, and

$$\frac{dx}{dt} = \frac{5}{2}, \frac{dy}{dt} = \frac{3}{2}, \frac{dy}{dx} = \frac{3/2}{5/2} = \frac{3}{5}.$$

(c) $\frac{dy}{dx} = \frac{3}{5}$. At $\left(\frac{11}{16}, \frac{1}{4}\right)$, $y - \frac{1}{4} = \frac{3}{5}\left(x - \frac{11}{16}\right)$

$y = \frac{3}{5}x - \frac{13}{80}$.

23. $x = 2\sin 2t,\ y = 3\sin t$ crosses itself at the origin, $(x, y) = (0, 0)$.

At this point, $t = 0$ or $t = \pi$.

$$\frac{dy}{dx} = \frac{3\cos t}{4\cos 2t}$$

At $t = 0$: $\frac{dy}{dx} = \frac{3}{4}$ and $y = \frac{3}{4}x$. Tangent Line

At $t = \pi$, $\frac{dy}{dx} = -\frac{3}{4}$ and $y = -\frac{3}{4}x$. Tangent Line

24. $x = 2 - \pi\cos t,\ y = 2t - \pi\sin t$ crosses itself at a point on the x-axis: $(2, 0)$. The corresponding t-values are $t = \pm\pi/2$.

$$\frac{dy}{dt} = 2 - \pi\cos t,\ \frac{dx}{dt} = \pi\sin t,\ \frac{dy}{dx} = \frac{2 - \pi\cos t}{\pi\sin t}$$

At $t = \frac{\pi}{2}$: $\frac{dy}{dx} = \frac{2}{\pi}$.

Tangent line: $y - 0 = \frac{2}{\pi}(x - 2)$

$$y = \frac{2}{\pi}x - \frac{4}{\pi}$$

At $t = -\frac{\pi}{2}$: $\frac{dy}{dx} = -\frac{2}{\pi}$.

Tangent line: $y - 0 = -\frac{2}{\pi}(x - 2)$

$$y = -\frac{2}{\pi}x + \frac{4}{\pi}$$

25. $x = t^2 - t,\ y = t^3 - 3t - 1$ crosses itself at the point $(x, y) = (2, 1)$.

At this point, $t = -1$ or $t = 2$.

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2t - 1}$$

At $t = -1$, $\frac{dy}{dx} = 0$ and $y = 1$. Tangent Line

At $t = 2$, $\frac{dy}{dt} = \frac{9}{3} = 3$ and $y - 1 = 3(x - 2)$ or $y = 3x - 5$.

Tangent Line

26. $x = t^3 - 6t,\ y = t^2$ crosses itself at $(0, 6)$. The corresponding t-values are $t = \pm\sqrt{6}$.

$$\frac{dy}{dx} = \frac{2t}{3t^2 - 6}$$

At $t = \sqrt{6}$, $\frac{dy}{dx} = \frac{2\sqrt{6}}{12} = \frac{\sqrt{6}}{6}$.

Tangent line: $y - 6 = \frac{\sqrt{6}}{6}(x - 0)$

$$y = \frac{\sqrt{6}}{6}x + 6$$

At $t = -\sqrt{6}$, $\frac{dy}{dx} = -\frac{2\sqrt{6}}{12} = -\frac{\sqrt{6}}{6}$.

Tangent line: $y = -\frac{\sqrt{6}}{6}x + 6$

27. $x = \cos\theta + \theta\sin\theta,\ y = \sin\theta - \theta\cos\theta$

Horizontal tangents: $\frac{dy}{d\theta} = \theta\sin\theta = 0$ when $\theta = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

Points: $(-1, [2n - 1]\pi), (1, 2n\pi)$ where n is an integer.

Points shown: $(1, 0), (-1, \pi), (1, -2\pi)$

Vertical tangents: $\frac{dx}{d\theta} = \theta\cos\theta = 0$ when $\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

Note: $\theta = 0$ corresponds to the cusp at $(x, y) = (1, 0)$.

$$\frac{dy}{dx} = \frac{\theta\sin\theta}{\theta\cos\theta} = \tan\theta = 0 \text{ at } \theta = 0$$

Points: $\left(\frac{(-1)^{n+1}(2n - 1)\pi}{2}, (-1)^{n+1}\right)$

Points shown: $\left(\frac{\pi}{2}, 1\right), \left(-\frac{3\pi}{2}, -1\right), \left(\frac{5\pi}{2}, 1\right)$

28. $x = 2\theta,\ y = 2(1 - \cos\theta)$

Horizontal tangents: $\frac{dy}{d\theta} = 2\sin\theta = 0$ when $\theta = 0, \pm\pi, \pm 2\pi, \ldots$

Points: $(4n\pi, 0), (2[2n - 1]\pi, 4)$ where n is an integer

Points shown: $(0, 0), (2\pi, 4), (4\pi, 0)$

Vertical tangents: $\frac{dx}{d\theta} = 2 \neq 0$; none

29. $x = 4 - t, y = t^2$

Horizontal tangents: $\frac{dy}{dt} = 2t = 0$ when $t = 0$.

Point: $(4, 0)$

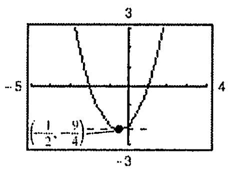
Vertical tangents: $\frac{dx}{dt} = -1 \neq 0$ None

30. $x = t + 1, y = t^2 + 3t$

Horizontal tangents: $\frac{dy}{dt} = 2t + 3 = 0$ when $t = -\frac{3}{2}$

Point: $\left(-\frac{1}{2}, -\frac{9}{4}\right)$

Vertical tangents: $\frac{dx}{dt} = 1 \neq 0$; none



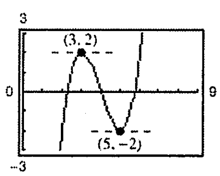
31. $x = t + 4, y = t^3 - 3t$

Horizontal tangents:

$\frac{dy}{dt} = 3t^2 - 3 = 3(t - 1)(t + 1) = 0 \Rightarrow t = \pm 1$

Points: $(5, -2), (3, 2)$

Vertical tangents: $\frac{dx}{dt} = 1 \neq 0$ None



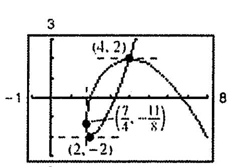
32. $x = t^2 - t + 2, y = t^3 - 3t$

Horizontal tangents: $\frac{dy}{dt} = 3t^2 - 3 = 0$ when $t = \pm 1$.

Points: $(2, -2), (4, 2)$

Vertical tangents: $\frac{dx}{dt} = 2t - 1 = 0$ when $t = \frac{1}{2}$.

Point: $\left(\frac{7}{4}, -\frac{11}{8}\right)$



33. $x = 3\cos\theta, y = 3\sin\theta$

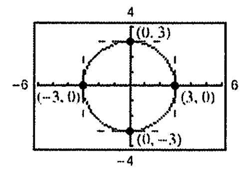
Horizontal tangents: $\frac{dy}{d\theta} = 3\cos\theta = 0$ when

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

Points: $(0, 3), (0, -3)$

Vertical tangents: $\frac{dx}{d\theta} = -3\sin\theta = 0$ when $\theta = 0, \pi$.

Points: $(3, 0), (-3, 0)$



34. $x = \cos\theta, y = 2\sin 2\theta$

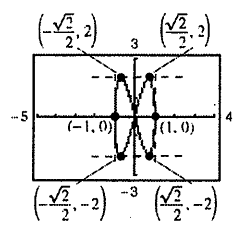
Horizontal tangents: $\frac{dy}{d\theta} = 4\cos 2\theta = 0$ when

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

Points: $\left(\frac{\sqrt{2}}{2}, 2\right), \left(-\frac{\sqrt{2}}{2}, -2\right), \left(-\frac{\sqrt{2}}{2}, 2\right), \left(\frac{\sqrt{2}}{2}, -2\right)$

Vertical tangents: $\frac{dx}{d\theta} = -\sin\theta = 0$ when $\theta = 0, \pi$.

Points: $(1, 0), (-1, 0)$



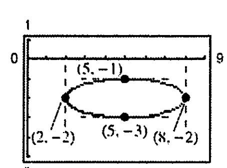
35. $x = 5 + 3\cos\theta, y = -2 + \sin\theta$

Horizontal tangents: $\frac{dy}{dt} = \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}$

Points: $(5, -1), (5, -3)$

Vertical tangents: $\frac{dx}{dt} = -3\sin\theta = 0 \Rightarrow \theta = 0, \pi$

Points: $(8, -2), (2, -2)$



36. $x = 4\cos^2\theta,\ y = 2\sin\theta$

Horizontal tangents: $\dfrac{dy}{d\theta} = 2\cos\theta = 0$ when

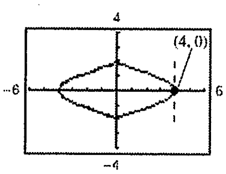
$\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}.$

Because $dx/d\theta = 0$ at $\pi/2$ and $3\pi/2$, exclude them.

Vertical tangents: $\dfrac{dx}{d\theta} = -8\cos\theta\sin\theta = 0$ when

$\theta = 0, \pi.$

Point: $(4, 0)$



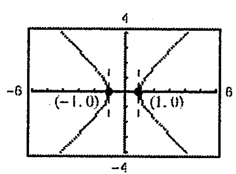
37. $x = \sec\theta,\ y = \tan\theta$

Horizontal tangents: $\dfrac{dy}{d\theta} = \sec^2\theta \neq 0$; None

Vertical tangents: $\dfrac{dx}{d\theta} = \sec\theta\tan\theta = 0$ when

$x = 0, \pi.$

Points: $(1, 0), (-1, 0)$



38. $x = \cos^2\theta,\ y = \cos\theta$

Horizontal tangents: $\dfrac{dy}{d\theta} = -\sin\theta = 0$ when $x = 0, \pi.$

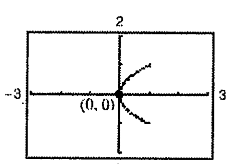
Since $dx/d\theta = 0$ at these values, exclude them.

Vertical tangents: $\dfrac{dx}{d\theta} = -2\cos\theta\sin\theta = 0$ when

$\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}.$

(Exclude $0, \pi$.)

Point: $(0, 0)$



39. $x = 3t^2,\ y = t^3 - t$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 1}{6t} = \frac{t}{2} - \frac{1}{6t}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{t}{2} - \frac{1}{6t}\right]}{dx/dt} = \frac{\frac{1}{2} + \frac{1}{6t^2}}{6t} = \frac{6t^2 + 2}{36t^3}$$

Concave upward for $t > 0$

Concave downward for $t < 0$

40. $x = 2 + t^2,\ y = t^2 + t^3$

$$\frac{dy}{dx} = \frac{2t + 3t^2}{2t} = 1 + \frac{3}{2}t$$

$$\frac{d^2y}{dx^2} = \frac{3/2}{2t} = \frac{3}{4t}$$

Concave upward for $t > 0$

Concave downward for $t < 0$

41. $x = 2t + \ln t,\ y = 2t - \ln t,\ t > 0$

$$\frac{dy}{dx} = \frac{2 - (1/t)}{2 + (1/t)} = \frac{2t - 1}{2t + 1}$$

$$\frac{d^2y}{dx^2} = \left[\frac{(2t + 1)2 - (2t - 1)2}{(2t + 1)^2}\right] \Big/ \left(2 + \frac{1}{t}\right)$$

$$= \frac{4}{(2t + 1)^2} \cdot \frac{t}{2t + 1} = \frac{4t}{(2t + 1)^3}$$

Because $t > 0,\ \dfrac{d^2y}{dx^2} > 0$

Concave upward for $t > 0$

42. $x = t^2,\ y = \ln t,\ t > 0$

$$\frac{dy}{dx} = \frac{1/t}{2t} = \frac{1}{2t^2}$$

$$\frac{d^2y}{dx^2} = -\frac{1/t^3}{2t} = -\frac{1}{2t^4}$$

Because $t > 0,\ \dfrac{d^2y}{dx^2} < 0$

Concave downward for $t > 0$

43. $x = \sin t,\ y = \cos t,\ 0 < t < \pi$

$$\frac{dy}{dx} = -\frac{\sin t}{\cos t} = -\tan t$$

$$\frac{d^2y}{dx^2} = -\frac{\sec^2 t}{\cos t} = -\frac{1}{\cos^3 t}$$

Concave upward on $\pi/2 < t < \pi$

Concave downward on $0 < t < \pi/2$

44. $x = 4\cos t,\ y = 2\sin t,\ 0 < t < 2\pi$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos t}{-4\sin t} = -\frac{1}{2}\cot t$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[-\frac{1}{2}\cot t\right]}{dx/dt} = \frac{\frac{1}{2}\csc^2 t}{-4\sin t} = \frac{-1}{8\sin^3 t}$$

Concave upward on $\pi < t < 2\pi$

Concave downward on $0 < t < \pi$

45. $x = 3t + 5,\ y = 7 - 2t,\ -1 \le t \le 3$

$$\frac{dx}{dt} = 3,\ \frac{dy}{dt} = -2$$

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_{-1}^3 \sqrt{9 + 4}\,dt$$

$$\left[\sqrt{13}\,t\right]_{-1}^3 = 4\sqrt{13} \approx 14.422$$

46. $x = 6t^2,\ y = 2t^3,\ 1 \le t \le 4$

$$\frac{dx}{dt} = 12t,\ \frac{dy}{dt} = 6t^2$$

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_1^4 \sqrt{144t^2 + 36t^4}\,dt$$

$$= \int_1^4 6t\sqrt{4 + t^2}\,dt$$

$$= \left[2\left(4 + t^2\right)^{3/2}\right]_1^4$$

$$= 2\left(20^{3/2} - 5^{3/2}\right)$$

$$= 70\sqrt{5} \approx 156.525$$

47. $x = e^{-t}\cos t,\ y = e^{-t}\sin t,\ 0 \le t \le \frac{\pi}{2}$

$$\frac{dx}{dt} = -e^{-t}(\sin t + \cos t),\ \frac{dy}{dt} = e^{-t}(\cos t - \sin t)$$

$$s = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_0^{\pi/2} \sqrt{2e^{-2t}}\,dt = -\sqrt{2}\int_0^{\pi/2} e^{-t}(-1)\,dt$$

$$= \left[-\sqrt{2}e^{-t}\right]_0^{\pi/2}$$

$$= \sqrt{2}\left(1 - e^{-\pi/2}\right) \approx 1.12$$

48. $x = \arcsin t,\ y = \ln\sqrt{1 - t^2},\ 0 \le t \le \frac{1}{2}$

$$\frac{dx}{dt} = \frac{1}{\sqrt{1 - t^2}},\ \frac{dy}{dt} = \frac{1}{2}\left(\frac{-2t}{1 - t^2}\right) = \frac{t}{1 - t^2}$$

$$s = \int_0^{1/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_0^{1/2} \sqrt{\frac{1}{\left(1 - t^2\right)^2}}\,dt = \int_0^{1/2} \frac{1}{1 - t^2}\,dt$$

$$= \left[-\frac{1}{2}\ln\left|\frac{t - 1}{t + 1}\right|\right]_0^{1/2}$$

$$= -\frac{1}{2}\ln\left(\frac{1}{3}\right) = \frac{1}{2}\ln(3) \approx 0.549$$

49. $x = \sqrt{t},\ y = 3t - 1,\ \frac{dx}{dt} = \frac{1}{2\sqrt{t}},\ \frac{dy}{dt} = 3$

$$s = \int_0^1 \sqrt{\frac{1}{4t} + 9}\,dt = \frac{1}{2}\int_0^1 \frac{\sqrt{1 + 36t}}{\sqrt{t}}\,dt$$

$$= \frac{1}{6}\int_0^6 \sqrt{1 + u^2}\,du$$

$$= \frac{1}{12}\left[\ln\left(\sqrt{1 + u^2} + u\right) + u\sqrt{1 + u^2}\right]_0^6$$

$$= \frac{1}{12}\left[\ln\left(\sqrt{37} + 6\right) + 6\sqrt{37}\right] \approx 3.249$$

$$u = 6\sqrt{t},\ du = \frac{3}{\sqrt{t}}\,dt$$

50. $x = t,\ y = \frac{t^5}{10} + \frac{1}{6t^3},\ \frac{dx}{dt} = 1,\ \frac{dy}{dt} = \frac{t^4}{2} - \frac{1}{2t^4}$

$$S = \int_1^2 \sqrt{1 + \left(\frac{t^4}{2} - \frac{1}{2t^4}\right)^2}\,dt$$

$$= \int_1^2 \sqrt{\left(\frac{t^4}{2} + \frac{1}{2t^4}\right)^2}\,dt$$

$$= \int_1^2 \left(\frac{t^4}{2} + \frac{1}{2t^4}\right)dt = \left[\frac{t^5}{10} - \frac{1}{6t^3}\right]_1^2 = \frac{779}{240}$$

51. $x = a\cos^3\theta,\ y = a\sin^3\theta,\ \frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta,$

$$\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$s = 4\int_0^{\pi/2} \sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\,d\theta$$

$$= 12a\int_0^{\pi/2} \sin\theta\cos\theta\sqrt{\cos^2\theta + \sin^2\theta}\,d\theta$$

$$= 6a\int_0^{\pi/2} \sin 2\theta\,d\theta = \left[-3a\cos 2\theta\right]_0^{\pi/2} = 6a$$

52. $x = a\cos\theta, y = a\sin\theta, \dfrac{dx}{d\theta} = -a\sin\theta,$

$$\frac{dy}{d\theta} = a\cos\theta$$

$$S = 4\int_0^{\pi/2}\sqrt{a^2\sin^2\theta + a^2\cos^2\theta}\,d\theta$$

$$= 4a\int_0^{\pi/2}d\theta = \left[4a\theta\right]_0^{\pi/2} = 2\pi a$$

53. $x = a(\theta - \sin\theta), y = a(1 - \cos\theta),$

$$\frac{dx}{d\theta} = a(1 - \cos\theta), \frac{dy}{d\theta} = a\sin\theta$$

$$s = 2\int_0^{\pi}\sqrt{a^2(1 - \cos\theta)^2 + a^2\sin^2\theta}\,d\theta$$

$$= 2\sqrt{2}a\int_0^{\pi}\sqrt{1 - \cos\theta}\,d\theta$$

$$= 2\sqrt{2}a\int_0^{\pi}\frac{\sin\theta}{\sqrt{1 + \cos\theta}}\,d\theta$$

$$= \left[-4\sqrt{2}a\sqrt{1 + \cos\theta}\right]_0^{\pi} = 8a$$

54. $x = \cos\theta + \theta\sin\theta, y = \sin\theta - \theta\cos\theta,$

$$\frac{dx}{d\theta} = \theta\cos\theta$$

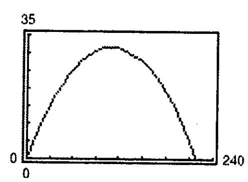
$$\frac{dy}{d\theta} = \theta\sin\theta$$

$$S = \int_0^{2\pi}\sqrt{\theta^2\cos^2\theta + \theta^2\sin^2\theta}\,d\theta$$

$$= \int_0^{2\pi}\theta\,d\theta = \left[\frac{\theta^2}{2}\right]_0^{2\pi} = 2\pi^2$$

55. $x = (90\cos 30^\circ)t, y = (90\sin 30^\circ)t - 16t^2$

(a)



(b) Range: 219.2 ft, $\left(t = \dfrac{45}{16}\right)$

(c) $\dfrac{dx}{dt} = 90\cos 30^\circ, \dfrac{dy}{dt} = 90\sin 30^\circ - 32t$

$y = 0$ for $t = \dfrac{45}{16}$.

$$s = \int_0^{45/16}\sqrt{(90\cos 30^\circ)^2 + (90\sin 30^\circ - 32t)^2}\,dt$$

≈ 230.8 ft

56. $y = 0 \Rightarrow (90\sin\theta)t = 16t^2 \Rightarrow t = 0, \dfrac{90}{16}\sin\theta$

$$x = (90\cos\theta)t = (90\cos\theta)\frac{90}{16}\sin\theta$$

$$= \frac{90^2}{16}\sin\theta\cos\theta = \frac{90^2}{32}\sin 2\theta$$

$$x'(\theta) = \frac{90^2}{32}2\cos 2\theta = 0 \Rightarrow \theta = \frac{\pi}{4}$$

By the First Derivative Test, $\theta = \dfrac{\pi}{4}(45^\circ)$ maximizes the range ($x = 253.125$ feet).

To maximize the arc length, you have

$\dfrac{dx}{dt} = 90\cos\theta, \dfrac{dy}{dt} = 90\sin\theta - 32t.$

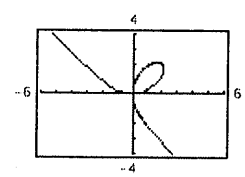
$$s = \int_0^{(90/16)\sin\theta}\sqrt{(90\cos\theta)^2 + (90\sin\theta - 32t)^2}\,dt$$

$$= \frac{2025}{8}\sin\theta + \frac{2025}{16}\cos^2\theta\ln\left[\frac{1 + \sin\theta}{1 - \sin\theta}\right]$$

Using a graphing utility, we see that s is a maximum of approximately 303.67 feet at $\theta \approx 0.9855(56.5^\circ)$.

57. $x = \dfrac{4t}{1 + t^3}, y = \dfrac{4t^2}{1 + t^3}$

(a) $x^3 + y^3 = 4xy$



(b) $$\frac{dy}{dt} = \frac{(1 + t^3)(8t) - 4t^2(3t^2)}{(1 + t^3)^2}$$

$$= \frac{4t(2 - t^3)}{(1 + t^3)^2} = 0 \text{ when } t = 0 \text{ or } t = \sqrt[3]{2}.$$

Points: $(0, 0), \left(\dfrac{4\sqrt[3]{2}}{3}, \dfrac{4\sqrt[3]{4}}{3}\right) \approx (1.6799, 2.1165)$

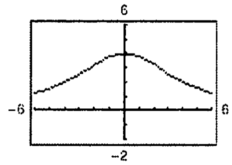
(c) $$s = 2\int_0^1\sqrt{\left[\frac{4(1 - 2t^3)}{(1 + t^3)^2}\right]^2 + \left[\frac{4t(2 - t^3)}{(1 + t^3)^2}\right]^2}\,dt$$

$$= 2\int_0^1\sqrt{\frac{16}{(1 + t^3)^4}\left[t^8 + 4t^6 - 4t^5 - 4t^3 + 4t^2 + 1\right]}\,dt$$

$$= 8\int_0^1\frac{\sqrt{t^8 + 4t^6 - 4t^5 - 4t^3 + 4t^2 + 1}}{(1 + t^3)^2}\,dt \approx 6.557$$

58. $x = 4\cot\theta = \dfrac{4}{\tan\theta}, y = 4\sin^2\theta, -\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$

(a)



(b) $\dfrac{dy}{d\theta} = 8\sin\theta\cdot\cos\theta$

$\dfrac{dx}{d\theta} = -4\csc^2\theta$

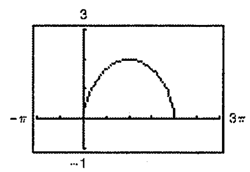
$\dfrac{dy}{d\theta} = 0$ for $\theta = 0, \pm\dfrac{\pi}{2}$

Horizontal tangent at $(x, y) = (0, 4)\left(\theta = \pm\dfrac{\pi}{2}\right)$

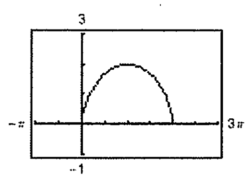
(Function is not defined at $\theta = 0$)

(c) Arc length over $\dfrac{\pi}{4} \le t \le \dfrac{\pi}{2}$: 4.5183

59. (a) $x = t - \sin t$
$y = 1 - \cos t$
$0 \le t \le 2\pi$



$x = 2t - \sin(2t)$
$y = 1 - \cos(2t)$
$0 \le t \le \pi$

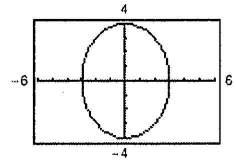


(b) The average speed of the particle on the second path is twice the average speed of a particle on the first path.

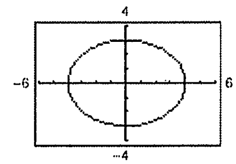
(c) $x = \frac{1}{2}t - \sin\left(\frac{1}{2}t\right)$
$y = 1 - \cos\left(\frac{1}{2}t\right)$

The time required for the particle to traverse the same path is $t = 4\pi$.

60. (a) First particle: $x = 3\cos t, y = 4\sin t, 0 \le t \le 2\pi$



Second particle: $x = 4\sin t, y = 3\cos t$, $0 \le t \le 2\pi$



(b) There are 4 points of intersection.

(c) Suppose at time t that

$3\cos t = 4\sin t$ and $4\sin t = 3\cos t$

$\tan t = \frac{3}{4}$ and $\tan t = \frac{3}{4}$.

Yes, the particles are at the same place at the same time for $\tan t = \frac{3}{4}$. $t \approx 0.6435, 3.7851$. The intersection points are $(2.4, 2.4)$ and $(-2.4, -2.4)$

(d) The curves intersect twice, but not at the same time.

61. $x = 3t, \dfrac{dx}{dt} = 3$

$y = t + 2, \dfrac{dy}{dt} = 1$

$$S = 2\pi\int_0^4 (t + 2)\sqrt{3^2 + 1^2}\,dt$$
$$= 2\pi\sqrt{10}\left[\frac{t^2}{2} + 2t\right]_0^4$$
$$= 2\pi\sqrt{10}[8 + 8] = 32\sqrt{10}\pi \approx 317.9068$$

62. $x = \dfrac{1}{4}t^2 \quad \dfrac{dx}{dt} = \dfrac{t}{2}$

$y = t + 3, \dfrac{dy}{dt} = 1$

$$S = 2\pi\int_0^3 (t + 3)\sqrt{\left(\frac{t}{2}\right)^2 + 1}\,dt$$
$$= 2\pi\int_0^3 (t + 3)\sqrt{\frac{t^2}{4} + 1}\,dt$$
$$\approx 114.1999$$

63. $x = \cos^2\theta, \dfrac{dx}{d\theta} = -2\cos\theta\sin\theta$

$y = \cos\theta, \dfrac{dy}{d\theta} = -\sin\theta$

$$S = 2\pi\int_0^{\pi/2}\cos\theta\sqrt{4\cos^2\theta\sin^2\theta + \sin^2\theta}\,d\theta$$

$$= 2\pi\int_0^{\pi/2}\cos\theta\sin\theta\sqrt{4\cos^2\theta + 1}\,d\theta$$

$$= \frac{\left(5\sqrt{5} - 1\right)\pi}{6}$$

$$\approx 5.3304$$

64. $x = \theta + \sin\theta, \dfrac{dx}{d\theta} = 1 + \cos\theta$

$y = \theta + \cos\theta, \dfrac{dy}{d\theta} = 1 - \sin\theta$

$$S = 2\pi\int_0^{\pi/2}(\theta + \cos\theta)\sqrt{(1 + \cos\theta)^2 + (1 - \sin\theta)^2}\,d\theta$$

$$= 2\pi\int_0^{\pi/2}(\theta + \cos\theta)\sqrt{3 + 2\cos\theta - 2\sin\theta}\,d\theta$$

$$\approx 23.2433$$

65. $x = 2t, \dfrac{dx}{dt} = 2$

$y = 3t, \dfrac{dy}{dt} = 3$

(a) $S = 2\pi\int_0^3 3t\sqrt{4 + 9}\,dt$

$$= 6\sqrt{13}\pi\left[\frac{t^2}{2}\right]_0^3 = 6\sqrt{13}\pi\left(\frac{9}{2}\right) = 27\sqrt{13}\pi$$

(b) $S = 2\pi\int_0^3 2t\sqrt{4 + 9}\,dt$

$$= 4\sqrt{13}\pi\left[\frac{t^2}{2}\right]_0^3 = 4\sqrt{13}\pi\left(\frac{9}{2}\right) = 18\sqrt{13}\pi$$

66. $x = t, y = 4 - 2t, \dfrac{dx}{dt} = 1, \dfrac{dy}{dt} = -2$

(a) $S = 2\pi\int_0^2(4 - 2t)\sqrt{1 + 4}\,dt$

$$= \left[2\sqrt{5}\pi\left(4t - t^2\right)\right]_0^2 = 8\pi\sqrt{5}$$

(b) $S = 2\pi\int_0^2 t\sqrt{1 + 4}\,dt = \left[\sqrt{5}\pi t^2\right]_0^2 = 4\pi\sqrt{5}$

67. $x = 5\cos\theta\ \dfrac{dx}{d\theta} = -5\sin\theta$

$y = 5\sin\theta\ \dfrac{dy}{d\theta} = 5\cos\theta$

$$S = 2\pi\int_0^{\pi/2}5\cos\theta\sqrt{25\sin^2\theta + 25\cos^2\theta}\,d\theta$$

$$= 10\pi\int_0^{\pi/2}5\cos\theta\,d\theta$$

$$= 50\pi\left[\sin\theta\right]_0^{\pi/2} = 50\pi$$

[**Note:** This is the surface area of a hemisphere of radius 5]

68. $x = \dfrac{1}{3}t^3, y = t + 1, 1 \le t \le 2,$ y-axis

$\dfrac{dx}{dt} = t^2, \dfrac{dy}{dt} = 1$

$$S = 2\pi\int_1^2\frac{1}{3}t^3\sqrt{t^4 + 1}\,dt = \frac{\pi}{9}\left[\left(x^4 + 1\right)^{3/2}\right]_1^2$$

$$= \frac{\pi}{9}\left(17^{3/2} - 2^{3/2}\right) \approx 23.48$$

69. $x = a\cos^3\theta, y = a\sin^3\theta, \dfrac{dx}{d\theta} = -3a\cos^2\theta\sin\theta, \ \dfrac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$

$$S = 4\pi\int_0^{\pi/2}a\sin^3\theta\sqrt{9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta}\,d\theta$$

$$= 12a^2\pi\int_0^{\pi/2}\sin^4\theta\cos\theta\,d\theta = \frac{12\pi a^2}{5}\left[\sin^5\theta\right]_0^{\pi/2} = \frac{12}{5}\pi a^2$$

70. $x = a\cos\theta,\ y = b\sin\theta,\ \dfrac{dx}{d\theta} = -a\sin\theta,\ \dfrac{dy}{d\theta} = b\cos\theta$

(a) $$S = 4\pi\int_0^{\pi/2} b\sin\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\,d\theta$$

$$= 4\pi\int_0^{\pi/2} ab\sin\theta\sqrt{1 - \left(\frac{a^2 - b^2}{a^2}\right)\cos^2\theta}\,d\theta = \frac{-4ab\pi}{e}\int_0^{\pi/2}(-e\sin\theta)\sqrt{1 - e^2\cos^2\theta}\,d\theta$$

$$= \frac{-2ab\pi}{e}\left[e\cos\theta\sqrt{1 - e^2\cos^2\theta} + \arcsin(e\cos\theta)\right]_0^{\pi/2} = \frac{2ab\pi}{e}\left[e\sqrt{1 - e^2} + \arcsin(e)\right]$$

$$= 2\pi b^2 + \left(\frac{2\pi a^2 b}{\sqrt{a^2 - b^2}}\right)\arcsin\left(\frac{\sqrt{a^2 - b^2}}{a}\right) = 2\pi b^2 + 2\pi\left(\frac{ab}{e}\right)\arcsin(e)$$

$$\left(e = \frac{\sqrt{a^2 - b^2}}{a} = \frac{c}{a}\text{: eccentricity}\right)$$

(b) $$S = 4\pi\int_0^{\pi/2} a\cos\theta\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\,d\theta$$

$$= 4\pi\int_0^{\pi/2} a\cos\theta\sqrt{b^2 + c^2\sin^2\theta}\,d\theta = \frac{4a\pi}{c}\int_0^{\pi/2} c\cos\theta\sqrt{b^2 + c^2\sin^2\theta}\,d\theta$$

$$= \frac{2a\pi}{c}\left[c\sin\theta\sqrt{b^2 + c^2\sin^2\theta} + b^2\ln\left|c\sin\theta + \sqrt{b^2 + c^2\sin^2\theta}\right|\right]_0^{\pi/2}$$

$$= \frac{2a\pi}{c}\left[c\sqrt{b^2 + c^2} + b^2\ln\left|c + \sqrt{b^2 + c^2}\right| - b^2\ln b\right]$$

$$= 2\pi a^2 + \frac{2\pi ab^2}{\sqrt{a^2 - b^2}}\ln\left|\frac{a + \sqrt{a^2 - b^2}}{b}\right| = 2\pi a^2 + \left(\frac{\pi b^2}{e}\right)\ln\left|\frac{1 + e}{1 - e}\right|$$

71. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$

See Theorem 10.7.

72. $x = t,\ y = 3 \Rightarrow \dfrac{dy}{dx} = 0$

73. $x = t,\ y = 6t - 5 \Rightarrow \dfrac{dy}{dx} = \dfrac{6}{1} = 6$

74. $$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

See Theorem 10.8.

75. (a) $$S = 2\pi\int_a^b g(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

(b) $$S = 2\pi\int_a^b f(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

76. (i) (a) $\dfrac{dx}{dt} < 0$ and $\dfrac{dy}{dx} < 0$ from the graph.

So, $\dfrac{dy}{dt} > 0$ because $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$.

(b) $\dfrac{dy}{dt} > 0$ and $\dfrac{dy}{dx} < 0$ from the graph.

So, $\dfrac{dx}{dt} < 0$ because $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$.

(ii) (a) $\dfrac{dx}{dt} < 0$ and $\dfrac{dy}{dx} > 0$ from the graph.

So, $\dfrac{dy}{dt} < 0$ because $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$.

(b) $\dfrac{dy}{dt} > 0$ and $\dfrac{dy}{dx} > 0$ from the graph.

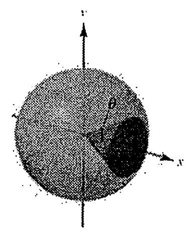
So, $\dfrac{dx}{dt} > 0$ because $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$.

77. Let y be a continuous function of x on $a \le x \le b$. Suppose that $x = f(t)$, $y = g(t)$, and $f(t_1) = a$, $f(t_2) = b$. Then using integration by substitution, $dx = f'(t)\,dt$ and

$$\int_a^b y\,dx = \int_{t_1}^{t_2} g(t)f'(t)\,dt.$$

78. $x = r\cos\phi,\ y = r\sin\phi$

$$S = 2\pi\int_0^{\theta} r\sin\phi\sqrt{r^2\sin^2\phi + r^2\cos^2\phi}\,d\phi$$
$$= 2\pi r^2\int_0^{\theta}\sin\phi\,d\phi$$
$$= \left[-2\pi r^2\cos\phi\right]_0^{\theta}$$
$$= 2\pi r^2(1 - \cos\theta)$$

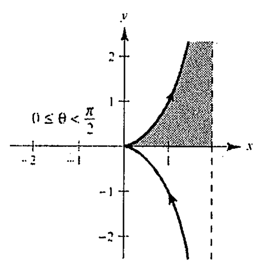


79. $x = 2\sin^2\theta$

$y = 2\sin^2\theta\tan\theta$

$$\frac{dx}{d\theta} = 4\sin\theta\cos\theta$$

$$A = \int_0^{\pi/2} 2\sin^2\theta\tan\theta(4\sin\theta\cos\theta)\,d\theta$$
$$= 8\int_0^{\pi/2}\sin^4\theta\,d\theta$$
$$= 8\left[\frac{-\sin^3\theta\cos\theta}{4} - \frac{3}{8}\sin\theta\cos\theta + \frac{3}{8}\theta\right]_0^{\pi/2} = \frac{3\pi}{2}$$



80. $x = 2\cot\theta,\ y = 2\sin^2\theta,\ \frac{dx}{d\theta} = -2\csc^2\theta$

$$A = 2\int_{\pi/2}^{0}\left(2\sin^2\theta\right)\left(-2\csc^2\theta\right)d\theta$$
$$= -8\int_{\pi/2}^{0}d\theta = \left[-8\theta\right]_{\pi/2}^{0} = 4\pi$$

81. πab is area of ellipse (d).

82. $\frac{3}{8}\pi a^2$ is area of asteroid (b).

83. $6\pi a^2$ is area of cardioid (f).

84. $2\pi a^2$ is area of deltoid (c).

85. $\frac{8}{3}ab$ is area of hourglass (a).

86. $2\pi ab$ is area of teardrop (e).

87. $x = \sqrt{t},\ y = 4 - t,\ 0 < t < 4$

$$A = \int_0^2 y\,dx = \int_0^4 (4 - t)\frac{1}{2\sqrt{t}}\,dt = \frac{1}{2}\int_0^4\left(4t^{-1/2} - t^{1/2}\right)dt = \left[\frac{1}{2}\left(8\sqrt{t} - \frac{2}{3}t\sqrt{t}\right)\right]_0^4 = \frac{16}{3}$$

$$\bar{x} = \frac{1}{A}\int_0^2 yx\,dx = \frac{3}{16}\int_0^4 (4 - t)\sqrt{t}\left(\frac{1}{2\sqrt{t}}\right)dt = \frac{3}{32}\int_0^4 (4 - t)\,dt = \left[\frac{3}{32}\left(4t - \frac{t^2}{2}\right)\right]_0^4 = \frac{3}{4}$$

$$\bar{y} = \frac{1}{A}\int_0^2\frac{y^2}{2}\,dx = \frac{3}{32}\int_0^4 (4 - t)^2\frac{1}{2\sqrt{t}}\,dt = \frac{3}{64}\int_0^4\left(16t^{-1/2} - 8t^{1/2} + t^{3/2}\right)dt = \frac{3}{64}\left[32\sqrt{t} - \frac{16}{3}t\sqrt{t} + \frac{2}{5}t^2\sqrt{t}\right]_0^4 = \frac{8}{5}$$

$$(\bar{x}, \bar{y}) = \left(\frac{3}{4}, \frac{8}{5}\right)$$

88. $x = \sqrt{4-t},\ y = \sqrt{t},\ \frac{dx}{dt} = -\frac{1}{2\sqrt{4-t}},\ 0 \le t \le 4$

$$A = \int_4^0 \sqrt{t}\left(-\frac{1}{2\sqrt{4-t}}\right)dt = \int_0^2 \sqrt{4-u^2}\,du = \frac{1}{2}\left[u\sqrt{4-u^2} + 4\arcsin\frac{u}{2}\right]_0^2 = \pi$$

Let $u = \sqrt{4-t}$, then $du = -1/\left(2\sqrt{4-t}\right)dt$ and $\sqrt{t} = \sqrt{4-u^2}$.

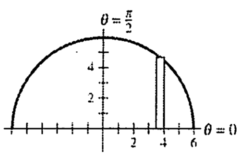
$$\bar{x} = \frac{1}{\pi}\int_4^0 \sqrt{4-t}\sqrt{t}\left(-\frac{1}{2\sqrt{4-t}}\right)dt = -\frac{1}{2\pi}\int_4^0 \sqrt{t}\,dt = \left[-\frac{1}{2\pi}\frac{2}{3}t^{3/2}\right]_4^0 = \frac{8}{3\pi}$$

$$\bar{y} = \frac{1}{2\pi}\int_4^0 \left(\sqrt{t}\right)^2\left(-\frac{1}{2\sqrt{4-t}}\right)dt = -\frac{1}{4\pi}\int_4^0 \frac{t}{\sqrt{4-t}}\,dt = -\frac{1}{4\pi}\left[\frac{-2(8+t)}{3}\sqrt{4-t}\right]_4^0 = \frac{8}{3\pi}$$

$$(\bar{x}, \bar{y}) = \left(\frac{8}{3\pi}, \frac{8}{3\pi}\right)$$

89. $x = 6\cos\theta,\ y = 6\sin\theta,\ \frac{dx}{d\theta} = -6\sin\theta\,d\theta$

$$\begin{aligned} V &= 2\pi\int_{\pi/2}^0 (6\sin\theta)^2(-6\sin\theta)\,d\theta \\ &= -432\pi\int_{\pi/2}^0 \sin^3\theta\,d\theta \\ &= -432\pi\int_{\pi/2}^0 \left(1-\cos^2\theta\right)\sin\theta\,d\theta \\ &= -432\pi\left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_{\pi/2}^0 \\ &= -432\pi\left(-1+\frac{1}{3}\right) = 288\pi \end{aligned}$$



Note: Volume of sphere is $\frac{4}{3}\pi\left(6^3\right) = 288\pi$.

90. $x = \cos\theta,\ y = 3\sin\theta,\ \frac{dx}{d\theta} = -\sin\theta$

$$\begin{aligned} V &= 2\pi\int_{\pi/2}^0 (3\sin\theta)^2(-\sin\theta)\,d\theta \\ &= -18\pi\int_{\pi/2}^0 \sin^3\theta\,d\theta \\ &= -18\pi\left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_{\pi/2}^0 = 12\pi \end{aligned}$$

91. $x = a(\theta - \sin\theta),\ y = a(1-\cos\theta)$

(a) $\frac{dy}{d\theta} = a\sin\theta,\ \frac{dx}{d\theta} = a(1-\cos\theta)$

$$\frac{dy}{dx} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{\sin\theta}{1-\cos\theta}$$

$$\frac{d^2y}{dx^2} = \left[\frac{(1-\cos\theta)\cos\theta - \sin\theta(\sin\theta)}{(1-\cos\theta)^2}\right]\Big/\left[a(1-\cos\theta)\right] = \frac{\cos\theta - 1}{a(1-\cos\theta)^3} = \frac{-1}{a(\cos\theta-1)^2}$$

(b) At $\theta = \frac{\pi}{6}$, $x = a\left(\frac{\pi}{6} - \frac{1}{2}\right)$, $y = a\left(1 - \frac{\sqrt{3}}{2}\right)$, $\frac{dy}{dx} = \frac{1/2}{1-\sqrt{3}/2} = 2+\sqrt{3}$.

Tangent line: $y - a\left(1-\frac{\sqrt{3}}{2}\right) = \left(2+\sqrt{3}\right)\left(x - a\left(\frac{\pi}{6}-\frac{1}{2}\right)\right)$

(c) $\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta} = 0 \Rightarrow \sin\theta = 0,\ 1-\cos\theta \ne 0$

Points of horizontal tangency: $(x, y) = \left(a(2n+1)\pi, 2a\right)$

(d) Concave downward on all open θ-intervals:

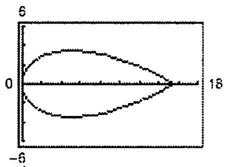
$\ldots, (-2\pi, 0), (0, 2\pi), (2\pi, 4\pi), \ldots$

(e) $s = \int_0^{2\pi} \sqrt{a^2 \sin^2 \theta + a^2(1 - \cos\theta)^2}\, d\theta$

$$= a\int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta = a\int_0^{2\pi} \sqrt{4\sin^2 \frac{\theta}{2}}\, d\theta = 2a\int_0^{2\pi} \sin\frac{\theta}{2}\, d\theta = \left[-4a\cos\left(\frac{\theta}{2}\right)\right]_0^{2\pi} = 8a$$

92. $x = t^2\sqrt{3},\ y = 3t - \frac{1}{3}t^3$

(a)



(b) $\frac{dx}{dt} = 2\sqrt{3}t, \frac{dy}{dt} = 3 - t^2, \frac{dy}{dx} = \frac{3 - t^2}{2\sqrt{3}t}$

$$\frac{d^2y}{dx^2} = \left[\frac{2\sqrt{3}(t)(-2t) - (3 - t^2)2\sqrt{3}}{12t^2}\right] \Big/ \left[2\sqrt{3}t\right] = \frac{-2\sqrt{3}t^2 - 6\sqrt{3}}{(12t^2)(2\sqrt{3}t)} = -\frac{t^2 + 3}{12t^3}$$

(c) $(x, y) = \left(\sqrt{3}, \frac{8}{3}\right)$ at $t = 1$. $\frac{dy}{dx} = \frac{2}{2\sqrt{3}} = \frac{\sqrt{3}}{3}$

$$y - \frac{8}{3} = \frac{\sqrt{3}}{3}\left(x - \sqrt{3}\right)$$

$$y = \frac{\sqrt{3}}{3}x + \frac{5}{3}$$

(d) $s = \int_{-3}^{3} \sqrt{12t^2 + (3 - t)^2}\, dt = \int_{-3}^{3} \sqrt{t^4 - 6t^2 + 9 + 12t^2}\, dt = \int_{-3}^{3} \sqrt{(t^2 + 3)^2}\, dt = \int_{-3}^{3} (t^3 + 3)\, dt = 36$

(e) $S = 2\pi\int_0^3 \left(3t - \frac{1}{3}t^3\right)(t^3 + 3)\, dt = 81\pi$

93. $x = t + u = r\cos\theta + r\theta\sin\theta$

$= r(\cos\theta + \theta\sin\theta)$

$y = v - w = r\sin\theta - r\theta\cos\theta$

$= r(\sin\theta - \theta\cos\theta)$