10.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Finding a Derivative In Exercises 1–4, find dy/dx.

1. $x = t^2,\ y = 7 - 6t$
2. $x = \sqrt[3]{t},\ y = 4 - t$
3. $x = \sin^2 \theta,\ y = \cos^2 \theta$
4. $x = 2e^{\theta},\ y = e^{-\theta/2}$

Finding Slope and Concavity In Exercises 5–14, find dy/dx and d^2y/dx^2, and find the slope and concavity (if possible) at the given value of the parameter.

Parametric Equations	Parameter
5. $x = 4t,\ y = 3t - 2$	$t = 3$
6. $x = \sqrt{t},\ y = 3t - 1$	$t = 1$
7. $x = t + 1,\ y = t^2 + 3t$	$t = -1$
8. $x = t^2 + 5t + 4,\ y = 4t$	$t = 0$
9. $x = 4\cos\theta,\ y = 4\sin\theta$	$\theta = \dfrac{\pi}{4}$
10. $x = \cos\theta,\ y = 3\sin\theta$	$\theta = 0$
11. $x = 2 + \sec\theta,\ y = 1 + 2\tan\theta$	$\theta = \dfrac{\pi}{6}$
12. $x = \sqrt{t},\ y = \sqrt{t - 1}$	$t = 2$
13. $x = \cos^3\theta,\ y = \sin^3\theta$	$\theta = \dfrac{\pi}{4}$
14. $x = \theta - \sin\theta,\ y = 1 - \cos\theta$	$\theta = \pi$

Finding Equations of Tangent Lines In Exercises 15–18, find an equation of the tangent line at each given point on the curve.

15. $x = 2\cot\theta,\quad y = 2\sin^2\theta,$
$\left(-\frac{2}{\sqrt{3}}, \frac{3}{2}\right), (0, 2), \left(2\sqrt{3}, \frac{1}{2}\right)$

16. $x = 2 - 3\cos\theta,\quad y = 3 + 2\sin\theta,$
$(-1, 3), (2, 5), \left(\frac{4 + 3\sqrt{3}}{2}, 2\right)$

17. $x = t^2 - 4,\quad y = t^2 - 2t,\quad (0, 0), (-3, -1), (-3, 3)$

18. $x = t^4 + 2,\quad y = t^3 + t,\quad (2, 0), (3, -2), (18, 10)$

Finding an Equation of a Tangent Line In Exercises 19–22, (a) use a graphing utility to graph the curve represented by the parametric equations, (b) use a graphing utility to find dx/dt, dy/dt, and dy/dx at the given value of the parameter, (c) find an equation of the tangent line to the curve at the given value of the parameter, and (d) use a graphing utility to graph the curve and the tangent line from part (c).

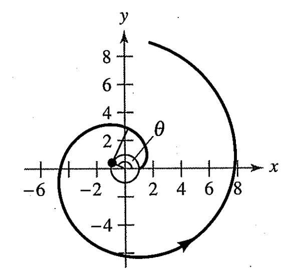
Parametric Equations	Parameter
19. $x = 6t,\ y = t^2 + 4$	$t = 1$
20. $x = t - 2,\ y = \dfrac{1}{t} + 3$	$t = 1$
21. $x = t^2 - t + 2,\ y = t^3 - 3t$	$t = -1$
22. $x = 3t - t^2,\quad y = 2t^{3/2}$	$t = \dfrac{1}{4}$

Finding Equations of Tangent Lines In Exercises 23–26, find the equations of the tangent lines at the point where the curve crosses itself.

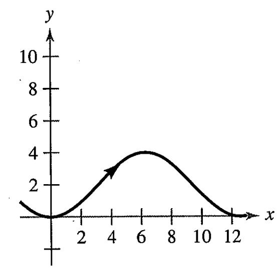
23. $x = 2\sin 2t,\quad y = 3\sin t$
24. $x = 2 - \pi\cos t,\quad y = 2t - \pi\sin t$
25. $x = t^2 - t,\quad y = t^3 - 3t - 1$
26. $x = t^3 - 6t,\quad y = t^2$

Horizontal and Vertical Tangency In Exercises 27 and 28, find all points (if any) of horizontal and vertical tangency to the portion of the curve shown.

27. Involute of a circle:
$x = \cos\theta + \theta\sin\theta$
$y = \sin\theta - \theta\cos\theta$



28. $x = 2\theta$
$y = 2(1 - \cos\theta)$



Horizontal and Vertical Tangency In Exercises 29–38, find all points (if any) of horizontal and vertical tangency to the curve. Use a graphing utility to confirm your results.

29. $x = 4 - t,\quad y = t^2$
30. $x = t + 1,\quad y = t^2 + 3t$
31. $x = t + 4,\quad y = t^3 - 3t$
32. $x = t^2 - t + 2,\quad y = t^3 - 3t$
33. $x = 3\cos\theta,\quad y = 3\sin\theta$
34. $x = \cos\theta,\quad y = 2\sin 2\theta$
35. $x = 5 + 3\cos\theta,\quad y = -2 + \sin\theta$
36. $x = 4\cos^2\theta,\quad y = 2\sin\theta$
37. $x = \sec\theta,\quad y = \tan\theta$
38. $x = \cos^2\theta,\quad y = \cos\theta$

Determining Concavity In Exercises 39–44, determine the open t-intervals on which the curve is concave downward or concave upward.

39. $x = 3t^2,\quad y = t^3 - t$
40. $x = 2 + t^2,\quad y = t^2 + t^3$
41. $x = 2t + \ln t,\quad y = 2t - \ln t$
42. $x = t^2,\quad y = \ln t$
43. $x = \sin t,\quad y = \cos t,\quad 0 < t < \pi$
44. $x = 4\cos t,\quad y = 2\sin t,\quad 0 < t < 2\pi$

Arc Length **In Exercises 45–50, find the arc length of the curve on the given interval.**

Parametric Equations	Interval
45. $x = 3t + 5, \quad y = 7 - 2t$	$-1 \le t \le 3$
46. $x = 6t^2, \quad y = 2t^3$	$1 \le t \le 4$
47. $x = e^{-t}\cos t, \quad y = e^{-t}\sin t$	$0 \le t \le \frac{\pi}{2}$
48. $x = \arcsin t, \quad y = \ln\sqrt{1 - t^2}$	$0 \le t \le \frac{1}{2}$
49. $x = \sqrt{t}, \quad y = 3t - 1$	$0 \le t \le 1$
50. $x = t, \quad y = \frac{t^5}{10} + \frac{1}{6t^3}$	$1 \le t \le 2$

Arc Length **In Exercises 51–54, find the arc length of the curve on the interval $[0, 2\pi]$.**

51. Hypocycloid perimeter: $x = a\cos^3\theta, y = a\sin^3\theta$

52. Circle circumference: $x = a\cos\theta, y = a\sin\theta$

53. Cycloid arch: $x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$

54. Involute of a circle: $x = \cos\theta + \theta\sin\theta, y = \sin\theta - \theta\cos\theta$

55. Path of a Projectile The path of a projectile is modeled by the parametric equations

$$x = (90\cos 30°)t \quad \text{and} \quad y = (90\sin 30°)t - 16t^2$$

where x and y are measured in feet.

(a) Use a graphing utility to graph the path of the projectile.

(b) Use a graphing utility to approximate the range of the projectile.

(c) Use the integration capabilities of a graphing utility to approximate the arc length of the path. Compare this result with the range of the projectile.

56. Path of a Projectile When the projectile in Exercise 55 is launched at an angle θ with the horizontal, its parametric equations are

$$x = (90\cos\theta)t \quad \text{and} \quad y = (90\sin\theta)t - 16t^2.$$

Use a graphing utility to find the angle that maximizes the range of the projectile. What angle maximizes the arc length of the trajectory?

57. Folium of Descartes Consider the parametric equations

$$x = \frac{4t}{1 + t^3} \quad \text{and} \quad y = \frac{4t^2}{1 + t^3}.$$

(a) Use a graphing utility to graph the curve represented by the parametric equations.

(b) Use a graphing utility to find the points of horizontal tangency to the curve.

(c) Use the integration capabilities of a graphing utility to approximate the arc length of the closed loop. (*Hint:* Use symmetry and integrate over the interval $0 \le t \le 1$.)

58. Witch of Agnesi Consider the parametric equations

$$x = 4\cot\theta \quad \text{and} \quad y = 4\sin^2\theta, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.$$

(a) Use a graphing utility to graph the curve represented by the parametric equations.

(b) Use a graphing utility to find the points of horizontal tangency to the curve.

(c) Use the integration capabilities of a graphing utility to approximate the arc length over the interval $\pi/4 \le \theta \le \pi/2$.

59. Writing

(a) Use a graphing utility to graph each set of parametric equations.

$x = t - \sin t, \quad y = 1 - \cos t, \quad 0 \le t \le 2\pi$

$x = 2t - \sin(2t), \quad y = 1 - \cos(2t), \quad 0 \le t \le \pi$

(b) Compare the graphs of the two sets of parametric equations in part (a). When the curve represents the motion of a particle and t is time, what can you infer about the average speeds of the particle on the paths represented by the two sets of parametric equations?

(c) Without graphing the curve, determine the time required for a particle to traverse the same path as in parts (a) and (b) when the path is modeled by

$$x = \tfrac{1}{2}t - \sin\left(\tfrac{1}{2}t\right) \quad \text{and} \quad y = 1 - \cos\left(\tfrac{1}{2}t\right).$$

60. Writing

(a) Each set of parametric equations represents the motion of a particle. Use a graphing utility to graph each set.

First Particle: $x = 3\cos t, \quad y = 4\sin t, \quad 0 \le t \le 2\pi$

Second Particle: $x = 4\sin t, \quad y = 3\cos t, \quad 0 \le t \le 2\pi$

(b) Determine the number of points of intersection.

(c) Will the particles ever be at the same place at the same time? If so, identify the point(s).

(d) Explain what happens when the motion of the second particle is represented by

$$x = 2 + 3\sin t, \quad y = 2 - 4\cos t, \quad 0 \le t \le 2\pi.$$

Surface Area **In Exercises 61–64, write an integral that represents the area of the surface generated by revolving the curve about the x-axis. Use a graphing utility to approximate the integral.**

Parametric Equations	Interval
61. $x = 3t, \quad y = t + 2$	$0 \le t \le 4$
62. $x = \frac{1}{4}t^2, \quad y = t + 3$	$0 \le t \le 3$
63. $x = \cos^2\theta, \quad y = \cos\theta$	$0 \le \theta \le \frac{\pi}{2}$
64. $x = \theta + \sin\theta, \quad y = \theta + \cos\theta$	$0 \le \theta \le \frac{\pi}{2}$

Surface Area **In Exercises 65–70, find the area of the surface generated by revolving the curve about each given axis.**

65. $x = 2t, \quad y = 3t, \quad 0 \le t \le 3$

(a) x-axis (b) y-axis

66. $x = t, \quad y = 4 - 2t, \quad 0 \le t \le 2$

(a) x-axis (b) y-axis

67. $x = 5\cos\theta, \ y = 5\sin\theta, \quad 0 \le \theta \le \frac{\pi}{2}, \quad y$-axis

68. $x = \frac{1}{3}t^3, \ y = t + 1, \quad 1 \le t \le 2, \quad y$-axis

69. $x = a\cos^3\theta, \ y = a\sin^3\theta, \quad 0 \le \theta \le \pi, \quad x$-axis

70. $x = a\cos\theta, \ y = b\sin\theta, \quad 0 \le \theta \le 2\pi$

(a) x-axis (b) y-axis

WRITING ABOUT CONCEPTS

71. Parametric Form of the Derivative Give the parametric form of the derivative.

Mental Math **In Exercises 72 and 73, mentally determine** dy/dx.

72. $x = t, \quad y = 3$

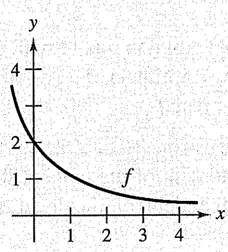
73. $x = t, \quad y = 6t - 5$

74. Arc Length Give the integral formula for arc length in parametric form.

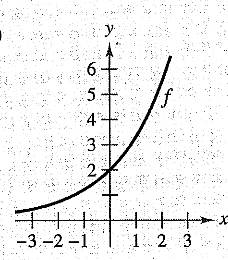
75. Surface Area Give the integral formulas for the areas of the surfaces of revolution formed when a smooth curve C is revolved about (a) the x-axis and (b) the y-axis.

76. HOW DO YOU SEE IT? Using the graph of f, (a) determine whether dy/dt is positive or negative given that dx/dt is negative, and (b) determine whether dx/dt is positive or negative given that dy/dt is positive. Explain your reasoning.

(i)



(ii)



77. Integration by Substitution Use integration by substitution to show that if y is a continuous function of x on the interval $a \le x \le b$, where $x = f(t)$ and $y = g(t)$, then

$$\int_a^b y\,dx = \int_{t_1}^{t_2} g(t)f'(t)\,dt$$

where $f(t_1) = a, f(t_2) = b$, and both g and f' are continuous on $[t_1, t_2]$.

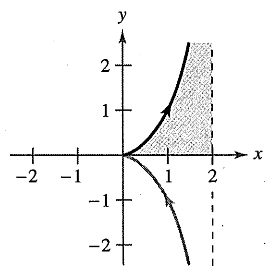
78. Surface Area A portion of a sphere of radius r is removed by cutting out a circular cone with its vertex at the center of the sphere. The vertex of the cone forms an angle of 2θ. Find the surface area removed from the sphere.

Area **In Exercises 79 and 80, find the area of the region. (Use the result of Exercise 77.)**

79. $x = 2\sin^2\theta$

$y = 2\sin^2\theta\tan\theta$

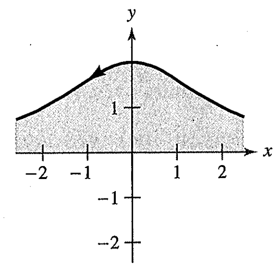
$0 \le \theta < \frac{\pi}{2}$



80. $x = 2\cot\theta$

$y = 2\sin^2\theta$

$0 < \theta < \pi$



Areas of Simple Closed Curves **In Exercises 81–86, use a computer algebra system and the result of Exercise 77 to match the closed curve with its area. (These exercises were based on "The Surveyor's Area Formula" by Bart Braden, *College Mathematics Journal*, September 1986, pp. 335–337, by permission of the author.)**

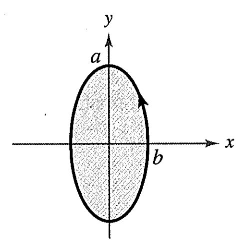
(a) $\frac{8}{3}ab$ **(b)** $\frac{3}{8}\pi a^2$ **(c)** $2\pi a^2$

(d) πab **(e)** $2\pi ab$ **(f)** $6\pi a^2$

81. Ellipse: $(0 \le t \le 2\pi)$

$x = b\cos t$

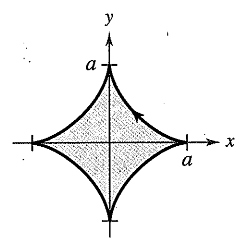
$y = a\sin t$



82. Astroid: $(0 \le t \le 2\pi)$

$x = a\cos^3 t$

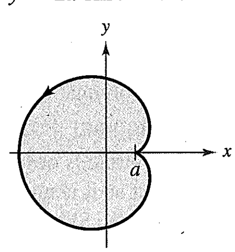
$y = a\sin^3 t$



83. Cardioid: $(0 \le t \le 2\pi)$

$x = 2a\cos t - a\cos 2t$

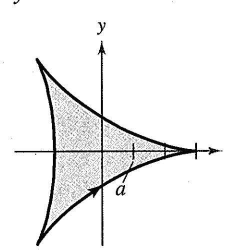
$y = 2a\sin t - a\sin 2t$



84. Deltoid: $(0 \le t \le 2\pi)$

$x = 2a\cos t + a\cos 2t$

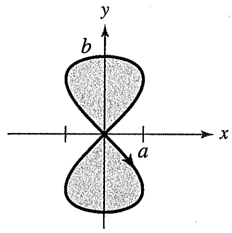
$y = 2a\sin t - a\sin 2t$



85. Hourglass: $(0 \le t \le 2\pi)$

$x = a \sin 2t$

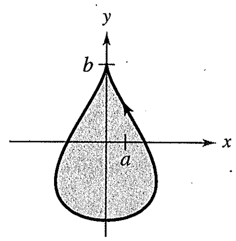
$y = b \sin t$



86. Teardrop: $(0 \le t \le 2\pi)$

$x = 2a \cos t - a \sin 2t$

$y = b \sin t$



Centroid **In Exercises 87 and 88, find the centroid of the region bounded by the graph of the parametric equations and the coordinate axes. (Use the result of Exercise 77.)**

87. $x = \sqrt{t}, \quad y = 4 - t$

88. $x = \sqrt{4 - t}, \quad y = \sqrt{t}$

Volume **In Exercises 89 and 90, find the volume of the solid formed by revolving the region bounded by the graphs of the given equations about the x-axis. (Use the result of Exercise 77.)**

89. $x = 6 \cos \theta, \quad y = 6 \sin \theta$

90. $x = \cos \theta, \quad y = 3 \sin \theta, \quad a > 0$

91. Cycloid Use the parametric equations

$$x = a(\theta - \sin \theta) \quad \text{and} \quad y = a(1 - \cos \theta), a > 0$$

to answer the following.

(a) Find dy/dx and d^2y/dx^2.

(b) Find the equation of the tangent line at the point where $\theta = \pi/6$.

(c) Find all points (if any) of horizontal tangency.

(d) Determine where the curve is concave upward or concave downward.

(e) Find the length of one arc of the curve.

92. Using Parametric Equations Use the parametric equations

$$x = t^2\sqrt{3} \quad \text{and} \quad y = 3t - \frac{1}{3}t^3$$

to answer the following.

(a) Use a graphing utility to graph the curve on the interval $-3 \le t \le 3$.

(b) Find dy/dx and d^2y/dx^2.

(c) Find the equation of the tangent line at the point $\left(\sqrt{3}, \frac{8}{3}\right)$.

(d) Find the length of the curve.

(e) Find the surface area generated by revolving the curve about the x-axis.

93. Involute of a Circle The involute of a circle is described by the endpoint P of a string that is held taut as it is unwound from a spool that does not turn (see figure). Show that a parametric representation of the involute is

$$x = r(\cos \theta + \theta \sin \theta) \quad \text{and} \quad y = r(\sin \theta - \theta \cos \theta).$$

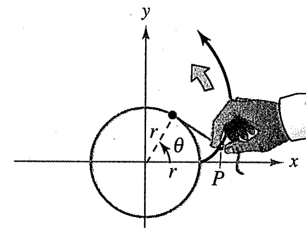


Figure for 93

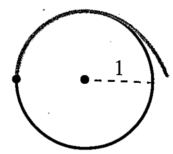


Figure for 94

94. Involute of a Circle The figure shows a piece of string tied to a circle with a radius of one unit. The string is just long enough to reach the opposite side of the circle. Find the area that is covered when the string is unwound counterclockwise.

95. Using Parametric Equations

(a) Use a graphing utility to graph the curve given by

$$x = \frac{1 - t^2}{1 + t^2} \quad \text{and} \quad y = \frac{2t}{1 + t^2}$$

where $-20 \le t \le 20$.

(b) Describe the graph and confirm your result analytically.

(c) Discuss the speed at which the curve is traced as t increases from -20 to 20.

96. Tractrix A person moves from the origin along the positive y-axis pulling a weight at the end of a 12-meter rope. Initially, the weight is located at the point (12, 0).

(a) In Exercise 90 of Section 8.7, it was shown that the path of the weight is modeled by the rectangular equation

$$y = -12 \ln\left(\frac{12 - \sqrt{144 - x^2}}{x}\right) - \sqrt{144 - x^2}$$

where $0 < x \le 12$. Use a graphing utility to graph the rectangular equation.

(b) Use a graphing utility to graph the parametric equations

$$x = 12 \operatorname{sech} \frac{t}{12} \quad \text{and} \quad y = t - 12 \tanh \frac{t}{12}$$

where $t \ge 0$. How does this graph compare with the graph in part (a)? Which graph (if either) do you think is a better representation of the path?

(c) Use the parametric equations for the tractrix to verify that the distance from the y-intercept of the tangent line to the point of tangency is independent of the location of the point of tangency.

True or False? **In Exercises 97 and 98, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

97. If $x = f(t)$ and $y = g(t)$, then $\dfrac{d^2y}{dx^2} = \dfrac{g''(t)}{f''(t)}$.

98. The curve given by $x = t^3$, $y = t^2$ has a horizontal tangent at the origin because $dy/dt = 0$ when $t = 0$.