BC Calculus Ch. 10.3b Notes Calculus Vectors: Motion Along a Curve

Displacement & Distance Traveled

Suppose a particle moves along a path in the plane so that its velocity at any time t is $\vec{v}(t) = (x'(t), y'(t))$, then the **displacement** from $t = a$ to $t = b$ is given by the vector

$$\left\langle \int_a^b x'(t)\,dt, \int_a^b y'(t)\,dt \right\rangle.$$

The preceding vector is added to the position at time $t = a$ to get the **position** at time $t = b$.

The **distance traveled** from $t = a$ to $t = b$ is the arc length

$$\int_a^b |\vec{v}(t)|\,dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt.$$

Example 1:
(Calculator) An object moving along a curve in the xy-plane has position $(x(t), y(t))$ at time t with $\frac{dx}{dt} = t\sin(t)$, $\frac{dy}{dt} = \cos(t^2)$. At time $t = 2$, the object is at the position $(1, 4)$.

a) Find the acceleration vector for the object at $t = 2$.

b) Write the equation of the tangent line to the curve at the point where t = 2

c) Find the speed of the object at $t = 2$.

d) Find the displacement of the object from $t = 2$ to $t = 7$.

e) Find the distance traveled by the object from $t = 2$ to $t = 7$.

f) Find the position of the object at time $t = 1$.

Example 2:

An object moving along a curve in the xy-plane has position $(x(t), y(t))$ at time $t \geq 0$ with $\frac{dx}{dt} = 2\sin(t^2)$. The derivative $\frac{dy}{dt}$ is not explicitly given. At time $t = 2$, the object is at position $(3,5)$

a) Find the x-coordinate of thé position of the object at time $t = 4$.

b) At time $t = 2$, the value of $\frac{dy}{dt}$ is -6. Write an equation for the line tangent to the curve at the point $(x(2), y(2))$.

c) Find the magnitude of the velocity vector at time $t = 2$.

d) For $t \geq 3$, the line tangent to the curve at $(x(t), y(t))$ has a slope of $2t - 1$. Find the acceleration vector of the object at time $t = 4$.

BC Calculus Ch. 10.3b Notes Calculus Vectors: Motion Along a Curve

Key

Displacement & Distance Traveled

Suppose a particle moves along a path in the plane so that its velocity at any time t is $\vec{v}(t) = (x'(t), y'(t))$, then the **displacement** from $t = a$ to $t = b$ is given by the vector $\left(\int_a^b x'(t)\,dt, \int_a^b y'(t)\,dt\right)$. Horizontal and vertical displacement are found independently.

$$\text{Displacement} = \int_a^b v(t)\,dt$$

$$\text{Distance} = \int_a^b \text{speed} = \int_a^b |v(t)|\,dt$$

The preceding vector is added to the position at time $t = a$ to get the **position** at time $t = b$.

The **distance traveled** from $t = a$ to $t = b$ is the arc length

$$\int_a^b |\vec{v}(t)|\,dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt.$$

Example 1: parametric/vector

(Calculator) An object moving along a curve in the xy-plane has position $(x(t), y(t))$ at time t with $\frac{dx}{dt} = t\sin(t)$, $\frac{dy}{dt} = \cos(t^2)$. At time $t = 2$, the object is at the position $(1, 4)$.

a) Find the acceleration vector for the object at $t = 2$.

$$\vec{a}(2) = \langle x''(2), y''(2) \rangle$$

$$\vec{a}(2) = \langle 0.077, 3.027 \rangle$$

b) Write the equation of the tangent line to the curve at the point where t = 2

point $(1, 4)$ slope $m = \frac{y'(2)}{x'(2)} = \frac{\cos(2)^2}{2\sin(2)}$

$$\boxed{y - 4 = \frac{\cos 4}{2\sin 2}(x - 1) \quad \text{or} \quad y = 4 - 0.359(x - 1)}$$

c) Find the speed of the object at $t = 2$.

$$\|\vec{v}(2)\| = \sqrt{(x'(2))^2 + (y'(2))^2}$$

speed or norm

$$\boxed{= \sqrt{(2\sin 2)^2 + (\cos 4)^2} \text{ or } = 1.932}$$

d) Find the displacement of the object from $t = 2$ to $t = 7$.

Integrate the velocity

Displacement of x: $\int_2^7 x'(t)\,dt = \boxed{-6.361}$

Displacement of y: $\int_2^7 y'(t)\,dt = \boxed{0.096}$

e) Find the distance traveled by the object from $t = 2$ to $t = 7$.

*Arc Length

$$L = \int_2^7 \sqrt{x'(t)^2 + y'(t)^2}\,dt = \boxed{13.229}$$

f) Find the position of the object at time $t = 1$.

final position = initial position + displacement

$$x(1) = x(2) + \int_2^1 x'(t)\,dt = -0.440$$

$$y(1) = y(2) + \int_2^1 y'(t)\,dt = 4.443$$

$$\boxed{(x(1), y(1)) = (-0.440, 4.443)}$$

Example 2:

An object moving along a curve in the xy-plane has position $(x(t), y(t))$ at time $t \geq 0$ with $\frac{dx}{dt} = 2\sin(t^2)$. The derivative $\frac{dy}{dt}$ is not explicitly given. At time $t = 2$, the object is at position $(3, 5)$

(net accumulation) net change in position

final position = initial position + displacement

a) Find the x-coordinate of the position of the object at time $t = 4$.

$$x(4) = x(2) + \int_2^4 x'(t)\,dt$$

$$= 3 + -0.11528 = \boxed{2.885}$$

b) At time $t = 2$, the value of $\frac{dy}{dt}$ is -6. Write an equation for the line tangent to the curve at the point $(x(2), y(2))$.

point: $(3, 5)$

$$\text{slope } m = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-6}{2\sin(2)^2} = \frac{-6}{2\sin 4}$$

$$\boxed{y - 5 = \frac{-6}{2\sin 4}(x - 3)}$$

$$y - 5 = 3.964(x - 3)$$

c) Find the magnitude of the velocity vector at time $t = 2$.

* Find speed

$$\vec{v}(t) = \|\vec{v}(2)\| = \sqrt{(x'(2))^2 + (y'(2))^2}$$

$$= \boxed{\sqrt{(2\sin 4)^2 + (-6)^2} \text{ or } 6.188}$$

d) For $t \geq 3$, the line tangent to the curve at $(x(t), y(t))$ has a slope of $2t - 1$. Find the acceleration vector of the object at time $t = 4$.

Find $\vec{a}(4) = \langle x''(4), y''(4) \rangle$

$$\text{slope} = \frac{dy}{dx} = 2t - 1$$

so...

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = 2t - 1$$

$$\frac{\frac{dy}{dt}}{2\sin(t^2)} = 2t - 1$$

$$\frac{dy}{dt} = (2t - 1)(2\sin(t^2))$$

$$\left.\frac{d^2x}{dt^2}\right|_{t=4} = -15.322$$

$$\left.\frac{d^2y}{dt^2}\right|_{t=4} = -108.408$$

$$\boxed{\vec{a}(4) = \langle -15.322, -108.408 \rangle}$$