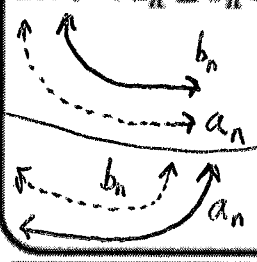
BC Calculus – 10.4 Notes – Comparison Tests for Convergence

Comparison Test or Direct Comparison Test (DCT)

Let $0 < a_n \le b_n$ for all n.



If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges

If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ also diverges

*DCT is a test we can use to determine convergence for more complex series by comparing them with simpler series.

Determine if the following series converge or diverge.

1. $\sum_{n=1}^{\infty} \frac{1}{3+2^n}$ compare with $\frac{1}{2^n}$ (Known converging Geometric Series)

$0 < \frac{1}{3+2^n} < \frac{1}{2^n}$

$\boxed{\text{Series converges by D.C.T.}}$

2. $\sum_{n=1}^{\infty} \frac{1}{4^n-3}$ compare with $\frac{1}{4^n}$

$0 < \frac{1}{4^n} < \frac{1}{4^n-3} \leftarrow$ Inconclusive

*compare with $\frac{1}{3^n}$

$0 < \frac{1}{4^n-3} < \frac{1}{3^n}$

$\boxed{\text{Series converge by DCT}}$

3. $\sum_{n=1}^{\infty} \frac{1}{7n^2+4}$ compare with $\frac{1}{n^2}$ (p-series converges)

$0 < \frac{1}{7n^2+4} < \frac{1}{n^2}$

$\boxed{\text{Series converges}}$

Limit Comparison Test

If $a_n > 0$, $b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ (where L is finite and positive), then

$$\sum_{n=1}^{\infty} a_n \text{ and } \sum_{n=1}^{\infty} b_n$$

either both converge or both diverge

Determine if the following series converge or diverge.

4. $\sum_{n=1}^{\infty} \frac{2n^2-2}{5n^5+3n+1}$ "reduce" numerator and denominator to find a comparison partner $\frac{2n^2}{5n^5} \rightarrow \boxed{\frac{1}{n^3}}$

$$\lim_{n\to\infty} \frac{\frac{2n^2-2}{5n^5+3n+1}}{\frac{1}{n^3}}$$

$$\lim_{n\to\infty} \frac{2n^2-2}{5n^5+3n+1} \cdot \frac{n^3}{1} \rightarrow \frac{2n^5}{5n^5} \rightarrow \frac{2}{5} \rightarrow \boxed{\text{converges by LCT}}$$

5. $\sum_{n=1}^{\infty} \frac{1}{5n^2+5n+5}$ compare with $\frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{1}{5n^2+5n+5} \cdot \frac{n^2}{1} \rightarrow \lim_{n\to\infty} \frac{n^2}{5n^2+5n+5} = \frac{1}{5}$$

$\boxed{\text{Series converge by LCT}}$

6. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n+2}} \rightarrow \frac{1}{\sqrt{n}} \rightarrow \frac{1}{n^{1/2}}$ (diverges → p-series)

$$\lim_{n\to\infty} \frac{1}{\sqrt{3n+2}} \cdot \frac{\sqrt{n}}{1}$$

$$\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{3n+2}} = \frac{1}{3}$$

$\boxed{\text{Series also diverges by LCT}}$

7. $\sum_{n=1}^{\infty} \frac{n^3-7}{2n^5+n^2+n+1} \rightarrow \frac{n^3}{2n^5} \rightarrow \frac{1}{n^2}$ converges (p-series)

$$\lim_{n\to\infty} \frac{n^3-7}{2n^5+n^2+n+1} \cdot \frac{n^2}{1}$$

$$\lim_{n\to\infty} \frac{n^5}{2n^5} \rightarrow \frac{1}{2}$$

$\boxed{\text{Series converges by LCT}}$

8. $\sum_{n=1}^{\infty} \frac{n3^n}{4n^3+2}$ compare with $\frac{n \cdot 3^n}{4n^3} \rightarrow \frac{3^n}{n^2}$

*diverges by n^{th} term test

$$\lim_{n\to\infty} \frac{n \cdot 3^n}{4n^3+2} \cdot \frac{n^2}{3^n}$$

$$\lim_{n\to\infty} \frac{n^3}{4n^3} \rightarrow \frac{1}{4}$$

$\boxed{\text{By LCT, series diverges}}$

9. $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+n}} \rightarrow \frac{n}{n^{3/2}} \rightarrow \frac{1}{n^{1/2}}$ (diverges by p-series)

$$\lim_{n\to\infty} \frac{n}{\sqrt{n^3+n}} \cdot \frac{n^{1/2}}{1}$$

$$\lim_{n\to\infty} \frac{n^{3/2}}{\sqrt{n^3+n}} \rightarrow 1$$

$\boxed{\text{By LCT, series diverges}}$

Comparison Tests for Convergence

Calculus

Practice

1. Which of the following statements about convergence of the series $\sum_{n=1}^{\infty} \frac{1}{\ln(n+2)}$ is true? Logs < Rad < P < E

(A) $\sum_{n=1}^{\infty} \frac{1}{\ln(n+2)}$ converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$ ← Diverges

(B) $\sum_{n=1}^{\infty} \frac{1}{\ln(n+2)}$ converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}$

(C) $\sum_{n=1}^{\infty} \frac{1}{\ln(n+2)}$ diverges by comparison with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}$ ← converges

$\boxed{\text{(D)}}$ $\sum_{n=1}^{\infty} \frac{1}{\ln(n+2)}$ diverges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$ ← diverges $\frac{1}{n} < \frac{1}{\ln(n+2)}$

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2. Which of the following series converges?

$\boxed{\text{(A)}}$ $\sum_{n=1}^{\infty} \frac{3n}{n^3+2}$ compare with $\frac{1}{n^2}$ (converges)

$\lim_{n\to\infty} \frac{3n}{n^3+2} \cdot \frac{n^2}{1} \to \frac{3}{1}$

both converges

(B) $\sum_{n=1}^{\infty} \frac{5n}{2n+1}$ Diverges by nth term test $\lim_{n\to\infty} a_n = \frac{5}{2} \neq 0$

(C) $\sum_{n=1}^{\infty} \frac{7n}{n^2+1}$ compare with $\frac{1}{n}$ (diverges)

$\lim_{n\to\infty} \frac{7n}{n^2+1} \cdot \frac{n}{1} \to \frac{7}{1} = 7$

(Both diverges)

(D) $\sum_{n=1}^{\infty} \frac{5^n}{4^n+1}$ → compare to $\left(\frac{5}{4}\right)^n$ (diverges)

$\lim_{n\to\infty} \frac{5^n}{4^n+1} \cdot \frac{4^n}{5^n} = 1$

both diverge.

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3. Use the Comparison Test to determine whether the series $\sum_{n=1}^{\infty} \frac{1}{2+5^n}$ converges or diverges.

Compare with $\frac{1}{5^n}$ converges by GST

$0 < \frac{1}{2+5^n} < \frac{1}{5^n}$

Both converge by Comparison Test (DCT)

Direct Comparison Test

4. Which of the following series can be used with the Limit Comparison Test to determine convergence of the series $\sum_{n=1}^{\infty} \frac{n^3}{n^4+3}$? → compare with $\frac{1}{n}$ (diverges)

$$\lim_{n\to\infty} \frac{n^3}{n^4+3} \cdot \frac{n}{1} \to \frac{n^4}{n^4+1} = 1$$

(Both Diverge by LCT)

Limit Comparison Test

(A) $\sum_{n=1}^{\infty} \frac{n}{n+3}$

(B) $\sum_{n=1}^{\infty} \frac{1}{n^3+3}$

$\boxed{\text{(C)}}$ $\sum_{n=1}^{\infty} \frac{1}{n}$

(D) $\sum_{n=1}^{\infty} \frac{1}{n^4}$

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5. Consider the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, where $a_n > 0$ and $b_n > 0$ for $n \geq 1$. If $\sum_{n=1}^{\infty} a_n$ diverges which of the following must be true?

*If the smaller series diverge, then the larger series will also diverge.

~~(A)~~ If $a_n \leq b_n$, then $\sum_{n=1}^{\infty} b_n$ converges.

$\boxed{\text{(B)}}$ If $a_n \leq b_n$, then $\sum_{n=1}^{\infty} b_n$ diverges.

~~(C)~~ If $b_n \leq a_n$, then $\sum_{n=1}^{\infty} b_n$ converges.

(D) If $b_n \leq a_n$, then $\sum_{n=1}^{\infty} b_n$ diverges.

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6. Consider the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, where $a_n > 0$ and $b_n > 0$ for $n \geq 1$. If $\sum_{n=1}^{\infty} b_n$ converges which of the following must be true?

If the larger series converge, then the smaller series must also converge.

~~(A)~~ If $a_n \leq b_n$, then $\sum_{n=1}^{\infty} a_n$ diverges.

$\boxed{\text{(B)}}$ If $a_n \leq b_n$, then $\sum_{n=1}^{\infty} a_n$ converges.

~~(C)~~ If $b_n \leq a_n$, then $\sum_{n=1}^{\infty} a_n$ diverges.

(D) If $b_n \leq a_n$, then $\sum_{n=1}^{\infty} a_n$ converges.

7. Let $a > 0, b > 0$, and $c > 0$. Determine whether the series $\sum_{n=0}^{\infty} \frac{1}{an^2 + bn + c}$ converges or diverges.

*compare with $\frac{1}{n^2}$ (converges p-series)

$\lim_{n\to\infty} \frac{1}{an^2+bn+c} \cdot \frac{n^2}{1} \to \frac{1}{a} > 0$, both converges by LCT.

8. Determine the convergence or divergence of the series $\sum_{n=2}^{\infty} \frac{1}{6^n + 6}$. compare with $\frac{1}{6^n}$ (converges by GST)

$\lim_{n\to\infty} \frac{1}{6^n+6} \cdot \frac{6^n}{1} \to 1$ (Both converge by LCT)

OR

$\frac{1}{6^n+6} < \frac{1}{6^n}$ converges by DCT

9. For the series $\sum_{n=1}^{\infty} \frac{n3^n}{2n^4 - 2}$, which of the following could be used with the Limit Comparison Test?

use $\frac{3^n}{n^3}$ (diverges)

(A) $\sum_{n=1}^{\infty} \frac{1}{n^4}$

(B) $\sum_{n=1}^{\infty} \frac{3^n}{n^4}$

(C) $\sum_{n=1}^{\infty} \frac{1}{n^3}$

$\boxed{\text{(D)}}$ $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$

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10. Which of the following can be used with the Comparison Test to determine the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{2 + \sqrt{n}}$?

$\frac{1}{n} < \frac{1}{2+\sqrt{n}}$

(A) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

$\boxed{\text{(B)}}$ $\sum_{n=1}^{\infty} \frac{1}{n}$ (Diverges)

(C) $\sum_{n=1}^{\infty} \frac{1}{n^2}$

(D) $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$

11. Which of the following series diverge?

I. $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n}$ $\frac{n^{1/3}}{n^1}$

$\frac{1}{n^{2/3}}$ diverges by p-series $\to$ $p = 2/3$ $0 < p < 1$

II. $\sum_{n=1}^{\infty} \frac{1}{n^3 - 27}$ ✓ compare with $\frac{1}{n^3}$ (converge) p-series

III. $\sum_{n=1}^{\infty} \frac{1}{4^n + 1}$ Compare with $\left(\frac{1}{4}\right)^n$ converge by GST

$\boxed{\text{(A) I only}}$ (B) II only

(C) I and II only (D) I, II, and III

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12. Consider the series $\sum_{n=2}^{\infty} \frac{1}{n^p \ln n}$, where $p \geq 0$. For what values of p is the series convergent?

Compare with $\frac{1}{n^p}$

$\boxed{\text{To converge, } p > 1}$

13. Determine whether the series $\sum_{n=1}^{\infty} \frac{n-3}{n^3}$ converges or diverges.

Compare with $\frac{1}{n^2}$ (Converge by p-series)

$\lim_{n\to\infty} \frac{n-3}{n^3} \cdot \frac{n^2}{1} \to \frac{n^3 - 3}{n^3} = 1$ converge by Limit Comparison Test. (LCT)

14. Consider the series $1 + \frac{1}{5} + \frac{1}{9} + \frac{1}{13} + \cdots = \sum_{n=1}^{\infty} \frac{1}{4n-3}$. Use the Limit Comparison Test with the series $\sum_{n=1}^{\infty} \frac{1}{4n}$ to determine the convergence of the series.

$\lim_{n\to\infty} \frac{1}{4n-3} \cdot \frac{4n}{1} = 1$ (Both diverge by LCT)

Diverges by harmonic series ($p=1$)

10.6 Comparison Tests for Convergence

Test Prep

15. Consider the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, where $a_n > 0$ and $b_n > 0$ for $n \geq 1$. If $a_n \leq b_n$, then which of the following must be true?

~~(A)~~ If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ diverges.

~~(B)~~ If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges.

$\boxed{\text{(C)}}$ If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

* If the smaller series diverge, then the larger series must also diverge.

(D) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ converges.

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16. Consider the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, where $a_n > 0$ and $b_n > 0$ for $n \geq 1$. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 2$, then which of the following must be true?

I. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ diverges.

II. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ converges.

III. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges.

IV. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

(A) I only (B) II only (C) III only

(D) IV only (E) I and II only $\boxed{\text{(F) III and IV only}}$