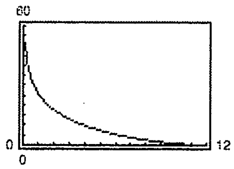
(b) $x = 12 \operatorname{sech} \frac{t}{12}, y = t - 12 \tanh \frac{t}{12}, 0 \le t$



Same as the graph in (a), but has the advantage of showing the position of the object and any given time t.

(c) $\frac{dy}{dx} = \frac{1 - \operatorname{sech}^2(t/12)}{-\operatorname{sech}(t/12)\tanh(t/12)} = -\sinh \frac{t}{12}$

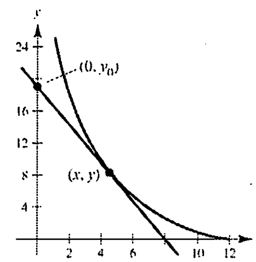
Tangent line: $y - \left(t_0 - 12 \tanh \frac{t_0}{12}\right) = -\sinh \frac{t_0}{12}\left(x - 12 \operatorname{sech} \frac{t_0}{12}\right)$

$$y = t_0 - \left(\sinh \frac{t_0}{12}\right)x$$

y-intercept: $(0, t_0)$

Distance between $(0, t_0)$ and (x, y): $d = \sqrt{\left(12 \operatorname{sech} \frac{t_0}{12}\right)^2 + \left(-12 \tanh \frac{t_0}{12}\right)^2} = 12$

$d = 12$ for any $t \ge 0$.



97. False. $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{g'(t)}{f'(t)}\right]}{f'(t)} = \frac{f'(t)g''(t) - g'(t)f''(t)}{\left[f'(t)\right]^3}$

98. False. Both dx/dt and dy/dt are zero when $t = 0$. By eliminating the parameter, you have $y = x^{2/3}$ which does not have a horizontal tangent at the origin.

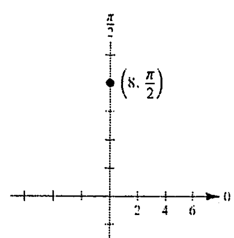
Section 10.4 Polar Coordinates and Polar Graphs

1. $\left(8, \frac{\pi}{2}\right)$

$x = 8 \cos \frac{\pi}{2} = 0$

$y = 8 \sin \frac{\pi}{2} = 8$

$(x, y) = (0, 8)$

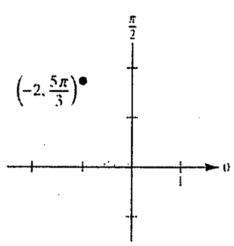


2. $\left(-2, \frac{5\pi}{3}\right)$

$x = -2 \cos \frac{5\pi}{3} = -2\left(\frac{1}{2}\right) = -1$

$y = -2 \sin \frac{5\pi}{3} = -2\left(\frac{-\sqrt{3}}{2}\right) = \sqrt{3}$

$(x, y) = \left(-1, \sqrt{3}\right)$

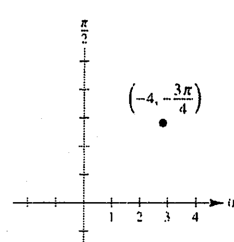


3. $\left(-4, -\frac{3\pi}{4}\right)$

$x = -4 \cos\left(\frac{-3\pi}{4}\right) = -4\left(-\frac{\sqrt{2}}{2}\right) = 2\sqrt{2}$

$y = -4 \sin\left(\frac{-3\pi}{4}\right) = -4\left(-\frac{\sqrt{2}}{2}\right) = 2\sqrt{2}$

$(x, y) = \left(2\sqrt{2}, 2\sqrt{2}\right)$

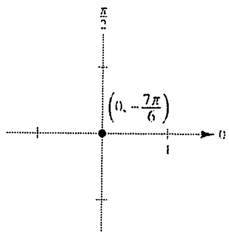


4. $\left(0, -\frac{7\pi}{6}\right)$

$$x = 0\cos\left(-\frac{7\pi}{6}\right) = 0$$

$$y = 0\sin\left(-\frac{7\pi}{6}\right) = 0$$

$$(x, y) = (0, 0)$$

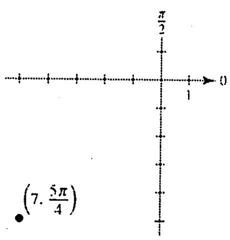


5. $(r, \theta) = \left(7, \frac{5\pi}{4}\right)$

$$x = 7\cos\frac{5\pi}{4} = 7\left(\frac{-\sqrt{2}}{2}\right) = -\frac{7\sqrt{2}}{2}$$

$$y = 7\sin\frac{5\pi}{4} = 7\left(-\frac{\sqrt{2}}{2}\right) = -\frac{7\sqrt{2}}{2}$$

$$(x, y) = \left(-\frac{7\sqrt{2}}{2}, -\frac{7\sqrt{2}}{2}\right)$$

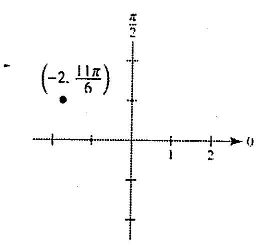


6. $(r, \theta) = \left(-2, \frac{11\pi}{6}\right)$

$$x = -2\cos\left(\frac{11\pi}{6}\right) = -\sqrt{3}$$

$$y = -2\sin\left(\frac{11\pi}{6}\right) = 1$$

$$(x, y) = \left(-\sqrt{3}, 1\right)$$

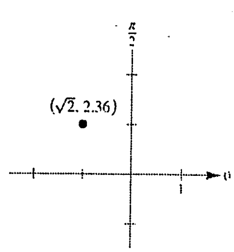


7. $\left(\sqrt{2}, 2.36\right)$

$$x = \sqrt{2}\cos(2.36) \approx -1.004$$

$$y = \sqrt{2}\sin(2.36) \approx 0.996$$

$$(x, y) = (-1.004, 0.996)$$

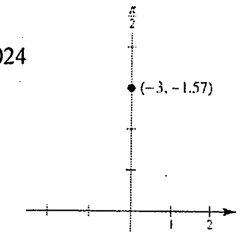


8. $(-3, -1.57)$

$$x = -3\cos(-1.57) \approx -0.0024$$

$$y = -3\sin(-1.57) \approx 3$$

$$(x, y) = (-0.0024, 3)$$

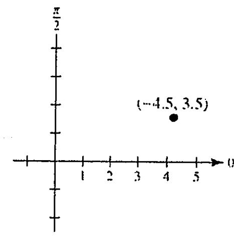


9. $(r, \theta) = (-4.5, 3.5)$

$$x = -4.5\cos 3.5 \approx 4.2141$$

$$y = -4.5\sin 3.5 \approx 1.5785$$

$$(x, y) = (4.2141, 1.5785)$$

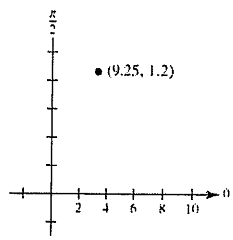


10. $(r, \theta) = (9.25, 1.2)$

$$x = 9.25\cos 1.2 \approx 3.3518$$

$$y = 9.25\sin 1.2 \approx 8.6214$$

$$(x, y) = (3.3518, 8.6214)$$



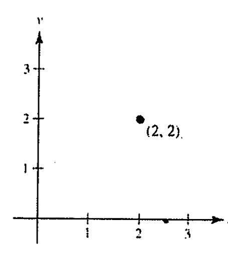
11. $(x, y) = (2, 2)$

$$r = \sqrt{2^2 + 2^2} = 2\sqrt{2}$$

$$\tan\theta = \frac{2}{2} = 1$$

$$\theta = \frac{\pi}{4}, \frac{5\pi}{4}$$

$$\left(2\sqrt{2}, \frac{\pi}{4}\right), \left(-2\sqrt{2}, \frac{5\pi}{4}\right)$$



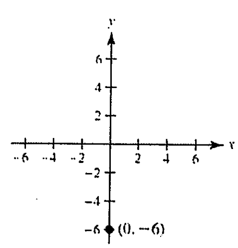
12. $(x, y) = (0, -6)$

$$r = \pm 6$$

$\tan\theta$ undefined

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$\left(6, \frac{3\pi}{2}\right), \left(-6, \frac{\pi}{2}\right)$$



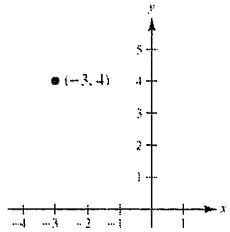
13. $(x, y) = (-3, 4)$

$r = \pm\sqrt{9 + 16} = \pm 5$

$\tan\theta = -\frac{4}{3}$

$\theta \approx 2.214, 5.356, (5, 2.214),$

$(-5, 5.356)$



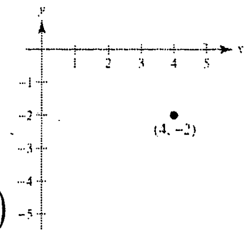
14. $(x, y) = (4, -2)$

$r = \pm\sqrt{16 + 4} = \pm 2\sqrt{5}$

$\tan\theta = -\frac{2}{4} = -\frac{1}{2}$

$\theta \approx -0.464$

$\left(2\sqrt{5}, -0.464\right), \left(-2\sqrt{5}, 2.678\right)$



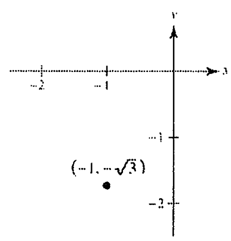
15. $(x, y) = \left(-1, -\sqrt{3}\right)$

$r = \sqrt{4} = 2$

$\tan\theta = \frac{-\sqrt{3}}{-1} = \sqrt{3}$

$\theta = \frac{\pi}{3}, \frac{4\pi}{3}$

$\left(2, \frac{4\pi}{3}\right), \left(-2, \frac{\pi}{3}\right)$

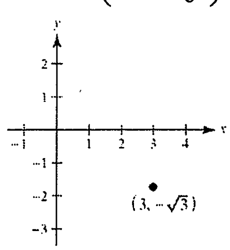


16. $(x, y) = \left(3, -\sqrt{3}\right)$

$r = \sqrt{9 + 3} = 2\sqrt{3}$

$\tan\theta = -\frac{\sqrt{3}}{3}$

$(r, \theta) = \left(2\sqrt{3}, \frac{11\pi}{6}\right) = \left(-2\sqrt{3}, \frac{5\pi}{6}\right)$

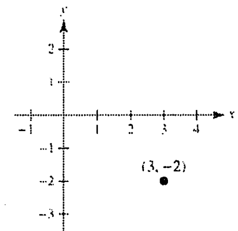


17. $(x, y) = (3, -2)$

$r = \sqrt{3^2 + (-2)^2} = \sqrt{13} \approx 3.6056$

$\tan\theta = -\frac{2}{3} \Rightarrow \theta \approx 5.6952$

$(r, \theta) \approx (3.6056, 5.6952) = (-3.6056, 2.5536)$

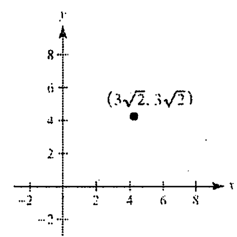


18. $(x, y) = \left(3\sqrt{2}, 3\sqrt{2}\right)$

$r = \sqrt{\left(3\sqrt{2}\right)^2 + \left(3\sqrt{2}\right)^2} = \sqrt{18 + 18} = 6$

$\tan\theta = \frac{3\sqrt{2}}{3\sqrt{2}} = 1 \Rightarrow \theta = \frac{\pi}{4}$

$(r, \theta) = \left(6, \frac{\pi}{4}\right) = \left(-6, \frac{5\pi}{4}\right)$

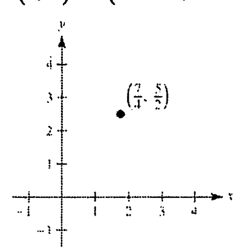


19. $(x, y) = \left(\frac{7}{4}, \frac{5}{2}\right)$

$r = \sqrt{\left(\frac{7}{4}\right)^2 + \left(\frac{5}{2}\right)^2} = \sqrt{\frac{149}{16}} = \frac{\sqrt{149}}{4} \approx 3.0516$

$\tan\theta = \frac{5/2}{7/4} = \frac{10}{7} \Rightarrow \theta \approx 0.9601$

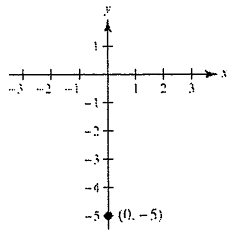
$(r, \theta) \approx (3.0516, 0.9601) \approx (-3.0516, 4.1017)$



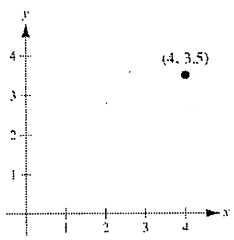
20. $(x, y) = (0, -5)$

$$r = 5, \theta = \frac{3\pi}{2}$$

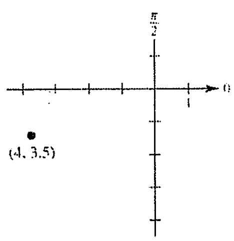
$$(r, \theta) = \left(5, \frac{3\pi}{2}\right) = \left(-5, \frac{\pi}{2}\right)$$



21. (a) $(x, y) = (4, 3.5)$



(b) $(r, \theta) = (4, 3.5)$



22. (a) Moving horizontally, the x-coordinate changes. Moving vertically, the y-coordinate changes.

(b) Both r and θ values change.

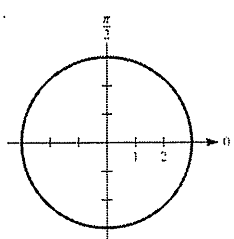
(c) In polar mode, horizontal (or vertical) changes result in changes in both r and θ.

23. $x^2 + y^2 = 9$

$$r^2 = 9$$

$$r = 3$$

Circle



24. $x^2 - y^2 = 9$

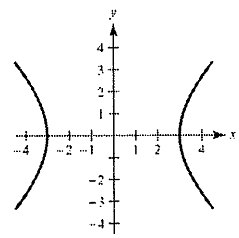
$$(r \cos \theta)^2 - (r \sin \theta)^2 = 9$$

$$r^2(\cos^2 \theta - \sin^2 \theta) = 9$$

$$r^2 \cos 2\theta = 9$$

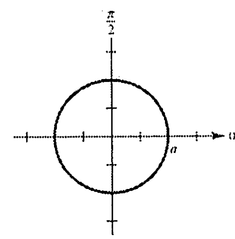
$$r = \frac{3}{\sqrt{\cos 2\theta}}$$

Hyperbola



25. $x^2 + y^2 = a^2$

$$r = a$$

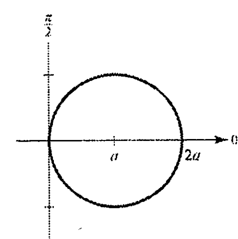


26. $x^2 + y^2 - 2ax = 0$

$$r^2 - 2ar \cos \theta = 0$$

$$r(r - 2a \cos \theta) = 0$$

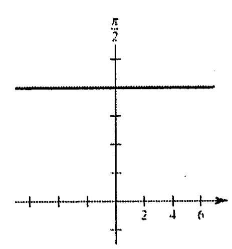
$$r = 2a \cos \theta$$



27. $y = 8$

$$r \sin \theta = 8$$

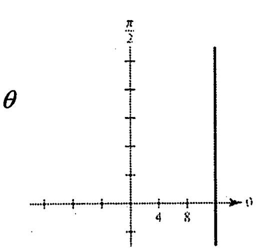
$$r = 8 \csc \theta$$



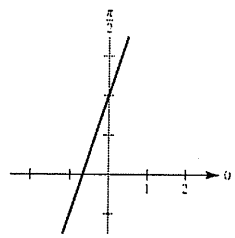
28. $x = 12$

$$r \cos \theta = 12$$

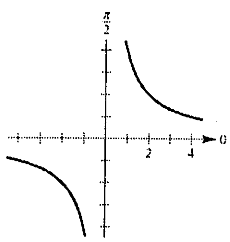
$$r = 12 \sec \theta$$



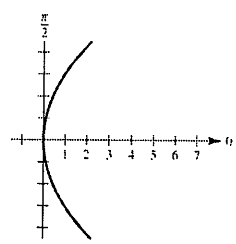
29.
$$\begin{aligned}3x - y + 2 &= 0\\ 3r\cos\theta - r\sin\theta + 2 &= 0\\ r(3\cos\theta - \sin\theta) &= -2\\ r &= \frac{-2}{3\cos\theta - \sin\theta}\end{aligned}$$



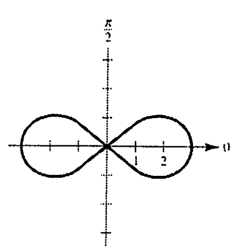
30.
$$\begin{aligned}xy &= 4\\ (r\cos\theta)(r\sin\theta) &= 4\\ r^2 &= 4\sec\theta\csc\theta\\ &= 8\csc 2\theta\end{aligned}$$



31.
$$\begin{aligned}y^2 &= 9x\\ r^2\sin^2\theta &= 9r\cos\theta\\ r &= \frac{9\cos\theta}{\sin^2\theta}\\ r &= 9\csc^2\theta\cos\theta\end{aligned}$$

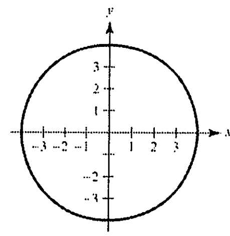


32.
$$\begin{aligned}(x^2 + y^2)^2 - 9(x^2 - y^2) &= 0\\ (r^2)^2 - 9(r^2\cos^2\theta - r^2\sin^2\theta) &= 0\\ r^2[r^2 - 9(\cos 2\theta)] &= 0\\ r^2 &= 9\cos 2\theta\end{aligned}$$



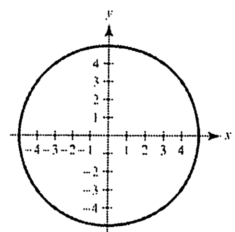
33.
$$\begin{aligned}r &= 4\\ r^2 &= 16\\ x^2 + y^2 &= 16\end{aligned}$$

Circle



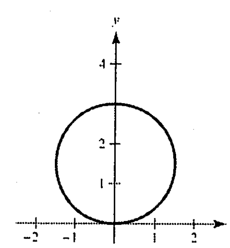
34.
$$\begin{aligned}r &= -5\\ r^2 &= 25\\ x^2 + y^2 &= 25\end{aligned}$$

Circle

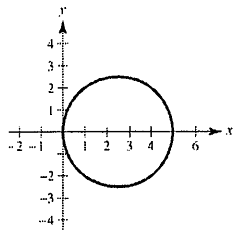


35.
$$\begin{aligned}r &= 3\sin\theta\\ r^2 &= 3r\sin\theta\\ x^2 + y^2 &= 3y\\ x^2 + \left(y^2 - 3y + \tfrac{9}{4}\right) &= \tfrac{9}{4}\\ x^2 + \left(y - \tfrac{3}{2}\right)^2 &= \tfrac{9}{4}\end{aligned}$$

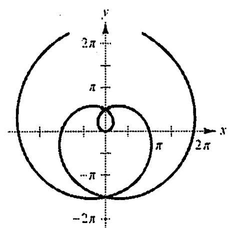
Circle



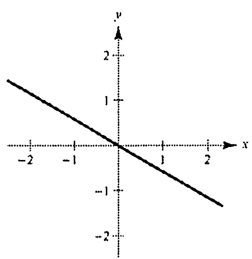
36.
$$\begin{aligned} r &= 5\cos\theta \\ r^2 &= 5r\cos\theta \\ x^2 + y^2 &= 5x \\ x^2 - 5x + \tfrac{25}{4} + y^2 &= \tfrac{25}{4} \\ \left(x - \tfrac{5}{2}\right)^2 + y^2 &= \left(\tfrac{5}{2}\right)^2 \end{aligned}$$



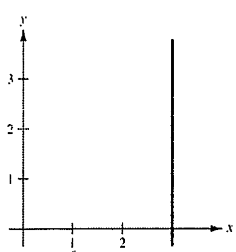
37.
$$\begin{aligned} r &= \theta \\ \tan r &= \tan\theta \\ \tan\sqrt{x^2 + y^2} &= \frac{y}{x} \\ \sqrt{x^2 + y^2} &= \arctan\frac{y}{x} \end{aligned}$$



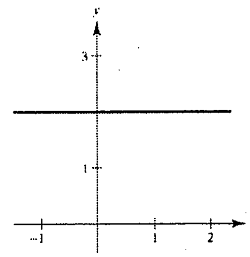
38.
$$\begin{aligned} \theta &= \frac{5\pi}{6} \\ \tan\theta &= \tan\frac{5\pi}{6} \\ \frac{y}{x} &= -\frac{\sqrt{3}}{3} \\ y &= -\frac{\sqrt{3}}{3}x \end{aligned}$$



39.
$$\begin{aligned} r &= 3\sec\theta \\ r\cos\theta &= 3 \\ x &= 3 \\ x - 3 &= 0 \end{aligned}$$

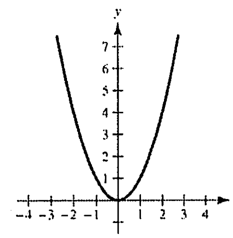


40.
$$\begin{aligned} r &= 2\csc\theta \\ r\sin\theta &= 2 \\ y &= 2 \\ y - 2 &= 0 \end{aligned}$$



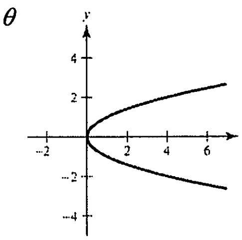
41.
$$\begin{aligned} r &= \sec\theta\tan\theta \\ r\cos\theta &= \tan\theta \\ x &= \frac{y}{x} \\ y &= x^2 \end{aligned}$$

Parabola



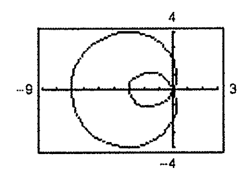
42.
$$\begin{aligned} r &= \cot\theta\csc\theta \\ r\sin\theta &= \cot\theta \\ y &= \frac{x}{y} \\ x &= y^2 \end{aligned}$$

Parabola



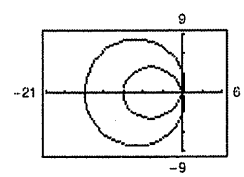
43. $r = 2 - 5\cos\theta$

$0 \le \theta < 2\pi$



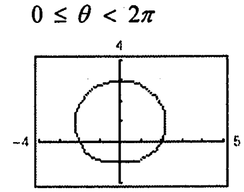
44. $r = 3(1 - 4\cos\theta)$

$0 \le \theta < 2\pi$



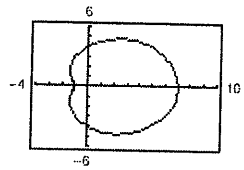
45. $r = 2 + \sin\theta$

$0 \le \theta < 2\pi$



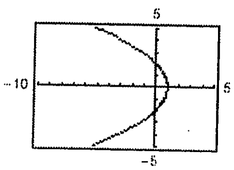
46. $r = 4 + 3\cos\theta$

$0 \le \theta < 2\pi$



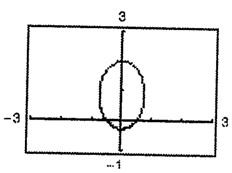
47. $r = \dfrac{2}{1 + \cos\theta}$

Traced out once on $-\pi < \theta < \pi$



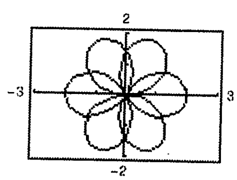
48. $r = \dfrac{2}{4 - 3\sin\theta}$

Traced out once on $0 \le \theta \le 2\pi$



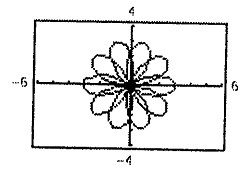
49. $r = 2\cos\left(\dfrac{3\theta}{2}\right)$

$0 \le \theta < 4\pi$



50. $r = 3\sin\left(\dfrac{5\theta}{2}\right)$

$0 \le \theta < 4\pi$

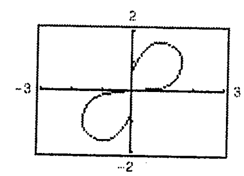


51. $r^2 = 4\sin 2\theta$

$r_1 = 2\sqrt{\sin 2\theta}$

$r_2 = -2\sqrt{\sin 2\theta}$

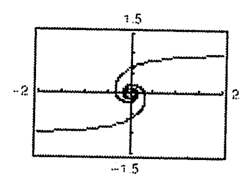
$0 \le \theta < \dfrac{\pi}{2}$



52. $r^2 = \dfrac{1}{\theta}.$

Graph as $r_1 = \dfrac{1}{\sqrt{\theta}}, r_2 = -\dfrac{1}{\sqrt{\theta}}.$

It is traced out once on $[0, \infty)$.



53.

$$\begin{aligned}
r &= 2(h\cos\theta + k\sin\theta)\\
r^2 &= 2r(h\cos\theta + k\sin\theta)\\
r^2 &= 2[h(r\cos\theta) + k(r\sin\theta)]\\
x^2 + y^2 &= 2(hx + ky)\\
x^2 + y^2 - 2hx - 2ky &= 0\\
(x^2 - 2hx + h^2) + (y^2 - 2ky + k^2) &= 0 + h^2 + k^2\\
(x - h)^2 + (y - k)^2 &= h^2 + k^2
\end{aligned}$$

Radius: $\sqrt{h^2 + k^2}$

Center: (h, k)

54. (a) The rectangular coordinates of (r_1, θ_1) are $(r_1 \cos \theta_1, r_1 \sin \theta_1)$. The rectangular coordinates of (r_2, θ_2) are $(r_2 \cos \theta_2, r_2 \sin \theta_2)$.

$$\begin{aligned} d^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\ &= (r_2 \cos \theta_2 - r_1 \cos \theta_1)^2 + (r_2 \sin \theta_2 - r_1 \sin \theta_1)^2 \\ &= r_2^2 \cos^2 \theta_2 - 2r_1r_2 \cos \theta_1 \cos \theta_2 + r_1^2 \cos^2 \theta_1 + r_2^2 \sin^2 \theta_2^2 - 2r_1r_2 \sin \theta_1 \sin \theta_2 + r_1^2 \sin^2 \theta_1 \\ &= r_2^2(\cos^2 \theta_2 + \sin^2 \theta_2) + r_1^2(\cos^2 \theta_1 + \sin^2 \theta_1) - 2r_1r_2(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2) \\ &= r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_1 - \theta_2) \\ d &= \sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_1 - \theta_2)} \end{aligned}$$

(b) If $\theta_1 = \theta_2$, the points lie on the same line passing through the origin. In this case,

$$\begin{aligned} d &= \sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos(0)} \\ &= \sqrt{(r_1 - r_2)^2} = |r_1 - r_2|. \end{aligned}$$

(c) If $\theta_1 - \theta_2 = 90°$, then $\cos(\theta_1 - \theta_2) = 0$ and $d = \sqrt{r_1^2 + r_2^2}$, the Pythagorean Theorem!

(d) Many answers are possible. For example, consider the two points $(r_1, \theta_1) = (1, 0)$ and $(r_2, \theta_2) = \left(2, \frac{\pi}{2}\right)$.

$$d = \sqrt{1 + 2^2 - 2(1)(2) \cos\left(0 - \frac{\pi}{2}\right)} = \sqrt{5}$$

Using $(r_1, \theta_1) = (-1, \pi)$ and $(r_2, \theta_2) = \left[2, \left(\frac{5\pi}{2}\right)\right]$, $d = \sqrt{(-1)^2 + (2)^2 - 2(-1)(2) \cos\left(\pi - \frac{5\pi}{2}\right)} = \sqrt{5}.$

You always obtain the same distance.

55. $\left(1, \frac{5\pi}{6}\right), \left(4, \frac{\pi}{3}\right)$

$$\begin{aligned} d &= \sqrt{1^2 + 4^2 - 2(1)(4) \cos\left(\frac{5\pi}{6} - \frac{\pi}{3}\right)} \\ &= \sqrt{17 - 8 \cos \frac{\pi}{2}} = \sqrt{17} \end{aligned}$$

56. $\left(8, \frac{7\pi}{4}\right), (5, \pi)$

$$\begin{aligned} d &= \sqrt{8^2 + 5^2 - 2(8)(5) \cos\left(\frac{7\pi}{4} - \pi\right)} \\ &= \sqrt{89 - 80 \cos \frac{3\pi}{4}} \\ &= \sqrt{89 - 80\left(-\frac{\sqrt{2}}{2}\right)} \\ &= \sqrt{89 + 40\sqrt{2}} \approx 12.0652 \end{aligned}$$

57. $(2, 0.5), (7, 1.2)$

$$\begin{aligned} d &= \sqrt{2^2 + 7^2 - 2(2)(7) \cos(0.5 - 1.2)} \\ &= \sqrt{53 - 28 \cos(-0.7)} \approx 5.6 \end{aligned}$$

58. $(4, 2.5), (12, 1)$

$$\begin{aligned} d &= \sqrt{4^2 + 12^2 - 2(4)(12) \cos(2.5 - 1)} \\ &= \sqrt{160 - 96 \cos 1.5} \approx 12.3 \end{aligned}$$

59. $r = 2 + 3 \sin \theta$

$$\begin{aligned} \frac{dy}{dx} &= \frac{3 \cos \theta \sin \theta + \cos \theta(2 + 3 \sin \theta)}{3 \cos \theta \cos \theta - \sin \theta(2 + 3 \sin \theta)} \\ &= \frac{2 \cos \theta(3 \sin \theta + 1)}{3 \cos 2\theta - 2 \sin \theta} = \frac{2 \cos \theta(3 \sin \theta + 1)}{6 \cos^2 \theta - 2 \sin \theta - 3} \end{aligned}$$

At $\left(5, \frac{\pi}{2}\right)$, $\frac{dy}{dx} = 0$.

At $(2, \pi)$, $\frac{dy}{dx} = -\frac{2}{3}$.

At $\left(-1, \frac{3\pi}{2}\right)$, $\frac{dy}{dx} = 0$.

60. $r = 2(1 - \sin\theta)$

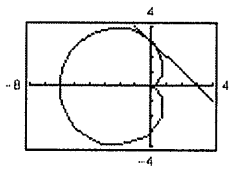
$$\frac{dy}{dx} = \frac{-2\cos\theta\sin\theta + 2\cos\theta(1 - \sin\theta)}{-2\cos\theta\cos\theta - 2\sin\theta(1 - \sin\theta)}$$

At $(2, 0)$, $\dfrac{dy}{dx} = -1$.

At $\left(3, \dfrac{7\pi}{6}\right)$, $\dfrac{dy}{dx}$ is undefined.

At $\left(4, \dfrac{3\pi}{2}\right)$, $\dfrac{dy}{dx} = 0$.

61. (a), (b) $r = 3(1 - \cos\theta)$



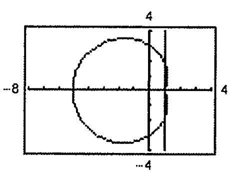
$(r, \theta) = \left(3, \dfrac{\pi}{2}\right) \Rightarrow (x, y) = (0, 3)$

Tangent line: $y - 3 = -1(x - 0)$

$y = -x + 3$

(c) At $\theta = \dfrac{\pi}{2}$, $\dfrac{dy}{dx} = -1.0$.

62. (a), (b) $r = 3 - 2\cos\theta$

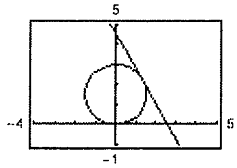


$(r, \theta) = (1, 0) \Rightarrow (x, y) = (1, 0)$

Tangent line: $x = 1$

(c) At $\theta = 0$, $\dfrac{dy}{dx}$ does not exist (vertical tangent).

63. (a), (b) $r = 3\sin\theta$



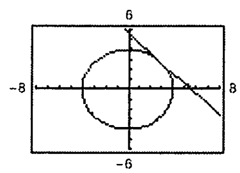
$(r, \theta) = \left(\dfrac{3\sqrt{3}}{2}, \dfrac{\pi}{3}\right) \Rightarrow (x, y) = \left(\dfrac{3\sqrt{3}}{4}, \dfrac{9}{4}\right)$

Tangent line: $y - \dfrac{9}{4} = -\sqrt{3}\left(x - \dfrac{3\sqrt{3}}{4}\right)$

$y = -\sqrt{3}x + \dfrac{9}{2}$

(c) At $\theta = \dfrac{\pi}{3}$, $\dfrac{dy}{dx} = -\sqrt{3} \approx -1.732$.

64. (a), (b) $r = 4$



$(r, \theta) = \left(4, \dfrac{\pi}{4}\right) \Rightarrow (x, y) = \left(2\sqrt{2}, 2\sqrt{2}\right)$

Tangent line: $y - 2\sqrt{2} = -1\left(x - 2\sqrt{2}\right)$

$y = -x + 4\sqrt{2}$

(c) At $\theta = \dfrac{\pi}{4}$, $\dfrac{dy}{dx} = -1$.

65. $r = 1 - \sin\theta$

$$\frac{dy}{d\theta} = (1 - \sin\theta)\cos\theta - \cos\theta\sin\theta$$
$$= \cos\theta(1 - 2\sin\theta) = 0$$

$\cos\theta = 0$ or $\sin\theta = \dfrac{1}{2} \Rightarrow \theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

Horizontal tangents: $\left(2, \dfrac{3\pi}{2}\right), \left(\dfrac{1}{2}, \dfrac{\pi}{6}\right), \left(\dfrac{1}{2}, \dfrac{5\pi}{6}\right)$

$$\frac{dx}{d\theta} = (-1 + \sin\theta)\sin\theta - \cos\theta\cos\theta$$
$$= -\sin\theta + \sin^2\theta + \sin^2\theta - 1$$
$$= 2\sin^2\theta - \sin\theta - 1$$
$$= (2\sin\theta + 1)(\sin\theta - 1) = 0$$

$\sin\theta = 1$ or $\sin\theta = -\dfrac{1}{2} \Rightarrow \theta = \dfrac{\pi}{2}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

Vertical tangents: $\left(\dfrac{3}{2}, \dfrac{7\pi}{6}\right), \left(\dfrac{3}{2}, \dfrac{11\pi}{6}\right)$

66. $r = a\sin\theta$

$$\frac{dy}{d\theta} = a\sin\theta\cos\theta + a\cos\theta\sin\theta$$
$$= 2a\sin\theta\cos\theta = 0$$

$$\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

$$\frac{dx}{d\theta} = -a\sin^2\theta + a\cos^2\theta = a\left(1 - 2\sin^2\theta\right) = 0$$

$$\sin\theta = \pm\frac{1}{\sqrt{2}}, \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

Horizontal tangents: $(0, 0), \left(a, \frac{\pi}{2}\right)$

Vertical tangents: $\left(\frac{a\sqrt{2}}{2}, \frac{\pi}{4}\right), \left(\frac{a\sqrt{2}}{2}, \frac{3\pi}{4}\right)$

67. $r = 2\csc\theta + 3$

$$\frac{dy}{d\theta} = \left(2\csc\theta + 3\right)\cos\theta + \left(-2\csc\theta\cot\theta\right)\sin\theta$$
$$= 3\cos\theta = 0$$

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

Horizontal tangents: $\left(5, \frac{\pi}{2}\right), \left(1, \frac{3\pi}{2}\right)$

68. $r = a\sin\theta\cos^2\theta$

$$\frac{dy}{d\theta} = a\sin\theta\cos^3\theta + \left[-2a\sin^2\theta\cos\theta + a\cos^3\theta\right]\sin\theta$$
$$= 2a\left[\sin\theta\cos^3\theta - \sin^3\theta\cos\theta\right]$$
$$= 2a\sin\theta\cos\theta\left(\cos^2\theta - \sin^2\theta\right) = 0$$

$$\theta = 0, \tan^2\theta = 1, \theta = \frac{\pi}{4}, \frac{3\pi}{4}$$

Horizontal tangents: $\left(\frac{\sqrt{2}a}{4}, \frac{\pi}{4}\right), \left(\frac{\sqrt{2}a}{4}, \frac{3\pi}{4}\right), (0, 0)$

69. $r = 5\sin\theta$

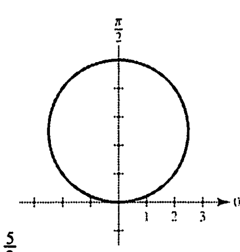
$$r^2 = 5r\sin\theta$$
$$x^2 + y^2 = 5y$$
$$x^2 + \left(y^2 - 5y + \tfrac{25}{4}\right) = \tfrac{25}{4}$$
$$x^2 + \left(y - \tfrac{5}{2}\right)^2 = \tfrac{25}{4}$$

Circle: center: $\left(0, \frac{5}{2}\right)$, radius: $\frac{5}{2}$

Tangent at pole: $\theta = 0$

Note: $f(\theta) = r = 5\sin\theta$

$f(0) = 0, f'(0) \neq 0$

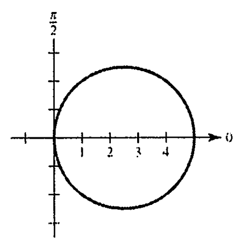


70. $r = 5\cos\theta$

$$r^2 = 5r\cos\theta$$
$$x^2 + y^2 = 5x$$
$$\left(x^2 - 5x + \frac{25}{4}\right) + y^2 = \frac{25}{4}$$
$$\left(x - \frac{5}{2}\right)^2 + y^2 = \frac{25}{4}$$

Circle: center: $\left(\frac{5}{2}, 0\right)$, radius: $\frac{5}{2}$

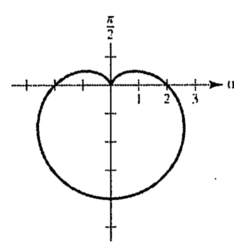
Tangent at pole: $\theta = \frac{\pi}{2}$.



71. $r = 2(1 - \sin\theta)$

Cardioid

Symmetric to y-axis, $\theta = \frac{\pi}{2}$

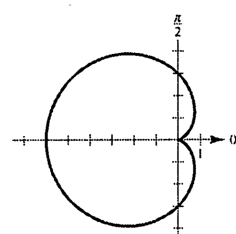


72. $r = 3(1 - \cos\theta)$

Cardioid

Symmetric to polar axis since r is a function of $\cos\theta$.

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	0	$\frac{3}{2}$	3	$\frac{9}{2}$	6

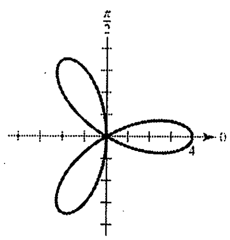


73. $r = 4 \cos 3\theta$

Rose curve with three petals.

Tangents at pole: $(r = 0, r' \neq 0)$:

$$\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$$



74. $r = -\sin(5\theta)$

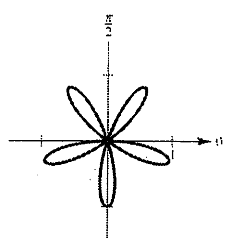
Rose curve with five petals

Symmetric to $\theta = \frac{\pi}{2}$

Relative extrema occur when

$$\frac{dr}{d\theta} = -5\cos(5\theta) = 0 \text{ at } \theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}.$$

Tangents at the pole: $\theta = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$



75. $r = 3 \sin 2\theta$

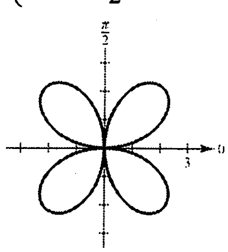
Rose curve with four petals

Symmetric to the polar axis, $\theta = \frac{\pi}{2}$, and pole

Relative extrema: $\left(\pm 3, \frac{\pi}{4}\right), \left(\pm 3, \frac{5\pi}{4}\right)$

Tangents at the pole: $\theta = 0, \frac{\pi}{2}$

$\left(\theta = \pi, \frac{3\pi}{2} \text{ give the same tangents.}\right)$



76. $r = 3 \cos 2\theta$

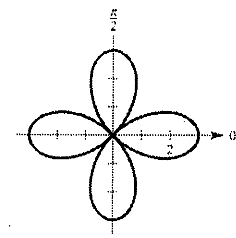
Rose curve with four petals

Symmetric to the polar axis, $\theta = \frac{\pi}{2}$, and pole

Relative extrema: $(3, 0), \left(-3, \frac{\pi}{2}\right), (3, \pi), \left(-3, \frac{3\pi}{2}\right)$

Tangents at the pole: $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$

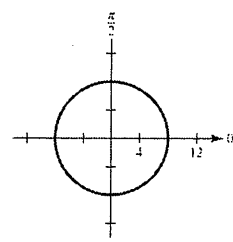
$\theta = \frac{5\pi}{4}$ and $\frac{7\pi}{4}$ given the same tangents.



77. $r = 8$

Circle radius 8

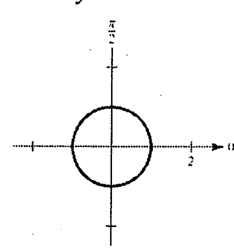
$x^2 + y^2 = 64$



78. $r = 1$

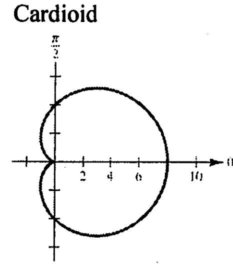
Circle radius 1

$x^2 + y^2 = 1$



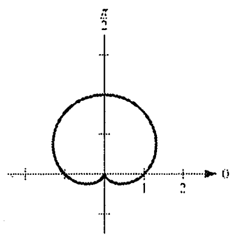
79. $r = 4(1 + \cos\theta)$

Cardioid



80. $r = 1 + \sin\theta$

Cardioid

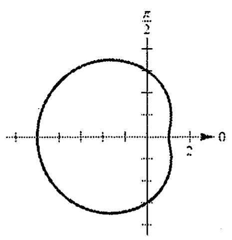


81. $r = 3 - 2\cos\theta$

Limaçon

Symmetric to polar axis

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	1	2	3	4	5

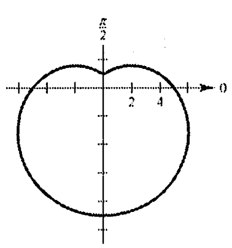


82. $r = 5 - 4\sin\theta$

Limaçon

Symmetric to $\theta = \frac{\pi}{2}$

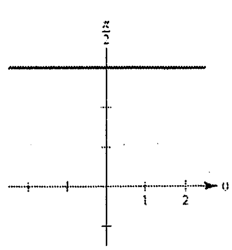
θ	$-\frac{\pi}{2}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$
r	9	7	5	3	1



83.

$$r = 3\csc\theta$$
$$r\sin\theta = 3$$
$$y = 3$$

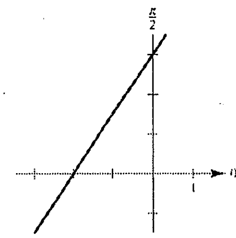
Horizontal line



84.

$$r = \frac{6}{2\sin\theta - 3\cos\theta}$$
$$2r\sin\theta - 3r\cos\theta = 6$$
$$2y - 3x = 6$$

Line



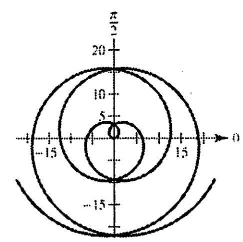
85. $r = 2\theta$

Spiral of Archimedes

Symmetric to $\theta = \frac{\pi}{2}$

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$
r	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π

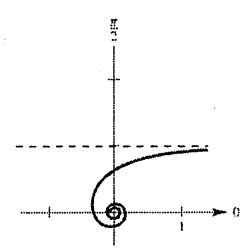
Tangent at the pole: $\theta = 0$



86. $r = \frac{1}{\theta}$

Hyperbolic spiral

θ	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$
r	$\frac{4}{\pi}$	$\frac{2}{\pi}$	$\frac{4}{3\pi}$	$\frac{1}{\pi}$	$\frac{4}{5\pi}$	$\frac{2}{3\pi}$



87. $r^2 = 4\cos(2\theta)$

$r = 2\sqrt{\cos 2\theta}, \quad 0 \le \theta \le 2\pi$

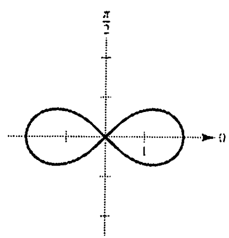
Lemniscate

Symmetric to the polar axis, $\theta = \frac{\pi}{2}$, and pole

Relative extrema: $(\pm 2, 0)$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$
r	± 2	$\pm\sqrt{2}$	0

Tangents at the pole: $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$



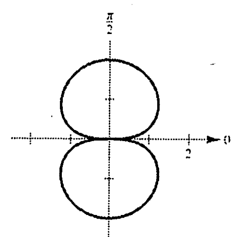
88. $r^2 = 4\sin\theta$

Lemniscate

Symmetric to the polar axis,

$\theta = \frac{\pi}{2}$, and pole

Relative extrema: $\left(\pm 2, \frac{\pi}{2}\right)$



θ	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	π
r	0	$\pm\sqrt{2}$	± 2	$\pm\sqrt{2}$	0

Tangent at the pole: $\theta = 0$

89. Because

$$r = 2 - \sec\theta = 2 - \frac{1}{\cos\theta},$$

the graph has polar axis symmetry and the tangents at the pole are

$$\theta = \frac{\pi}{3}, -\frac{\pi}{3}.$$

Furthermore,

$$r \Rightarrow -\infty \text{ as } \theta \Rightarrow \frac{\pi}{2^-}$$

$$r \Rightarrow \infty \text{ as } \theta \Rightarrow -\frac{\pi}{2^+}.$$

Also,

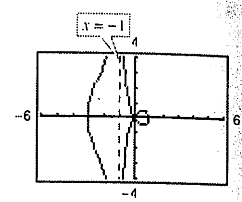
$$r = 2 - \frac{1}{\cos\theta}$$

$$= 2 - \frac{r}{r\cos\theta} = 2 - \frac{r}{x}$$

$$rx = 2x - r$$

$$r = \frac{2x}{1 + x}.$$

So, $r \Rightarrow \pm\infty$ as $x \Rightarrow -1$.



90. Because

$$r = 2 + \csc\theta = 2 + \frac{1}{\sin\theta},$$

the graphs has symmetry with respect to $\theta = \pi/2$. Furthermore,

$$r \Rightarrow \infty \text{ as } \theta \Rightarrow 0^+$$

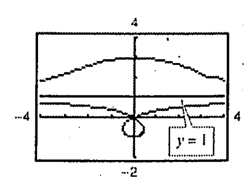
$$r \Rightarrow \infty \text{ as } \theta \Rightarrow \pi^-.$$

Also, $r = 2 + \frac{1}{\sin\theta} = 2 + \frac{r}{\sin\theta} = 2 + \frac{r}{y}$

$$ry = 2y + r$$

$$r = \frac{2y}{y - 1}.$$

So, $r \Rightarrow \pm\infty$ as $y \Rightarrow 1$.



91. $r = \frac{2}{\theta}$

Hyperbolic spiral

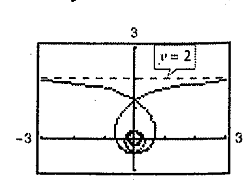
$r \Rightarrow \infty$ as $\theta \Rightarrow 0$

$$r = \frac{2}{\theta} \Rightarrow \theta = \frac{2}{r} = \frac{2\sin\theta}{r\sin\theta} = \frac{2\sin\theta}{y}$$

$$y = \frac{2\sin\theta}{\theta}$$

$$\lim_{\theta\to 0} \frac{2\sin\theta}{\theta} = \lim_{\theta\to 0} \frac{2\cos\theta}{1}$$

$$= 2$$



92. $r = 2\cos 2\theta \sec\theta$

Strophoid

$r \Rightarrow -\infty$ as $\theta \Rightarrow \dfrac{\pi^-}{2}$

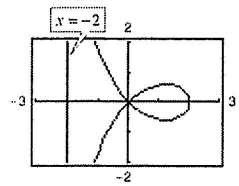
$r \Rightarrow \infty$ as $\theta \Rightarrow \dfrac{-\pi^+}{2}$

$$r = 2\cos 2\theta \sec\theta = 2\left(2\cos^2\theta - 1\right)\sec\theta$$

$$r\cos\theta = 4\cos^2\theta - 2$$

$$x = 4\cos^2\theta - 2$$

$$\lim_{\theta\to\pm\pi/2}\left(4\cos^2\theta - 2\right) = -2$$



93. The rectangular coordinate system consists of all points of the form (x, y) where x is the directed distance from the y-axis to the point, and y is the directed distance from the x-axis to the point.

Every point has a unique representation.

The polar coordinate system uses (r, θ) to designate the location of a point.

r is the directed distance to the origin and θ is the angle the point makes with the positive x-axis, measured counterclockwise.

Points do not have a unique polar representation.

94. $x = r\cos\theta,\ y = r\sin\theta$

$x^2 + y^2 = r^2,\ \tan\theta = \dfrac{y}{x}$

95. Slope of tangent line to graph of $r = f(\theta)$ at (r, θ) is

$$\frac{dy}{dx} = \frac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}.$$

If $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, then $\theta = \alpha$ is tangent at the pole.

96. (a) The graph is a circle, where $a = 2$ is measured along the y-axis. So, the equation of the polar graph is $r = 2\sin\theta$.

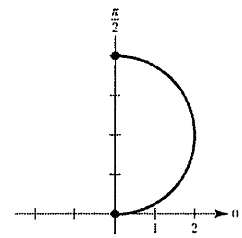
(b) The graph is a rose curve with $n = 3$ petals and $a = 3$. So, the equation of the polar graph is $r = 3\sin 3\theta$.

(c) The graph is a rose curve with $2n = 4$ petals and $a = 4$. So, the equation of the polar graph is $r = 4\cos 2\theta$.

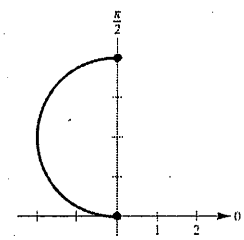
(d) The graph is a lemniscate with $a = 3$, which is measured along the x-axis. So, the equation of the polar graph is $r^2 = 9\cos 2\theta$.

97. $r = 4\sin\theta$

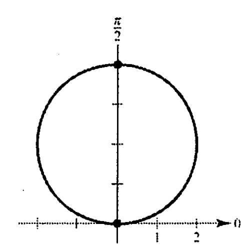
(a) $0 \le \theta \le \dfrac{\pi}{2}$



(b) $\dfrac{\pi}{2} \le \theta \le \pi$

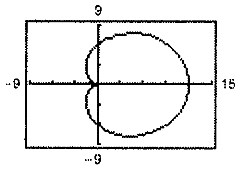


(c) $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$

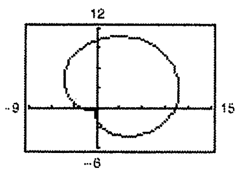


98. $r = 6[1 + \cos(\theta - \phi)]$

(a) $\phi = 0, r = 6[1 + \cos\theta]$



(b) $\theta = \frac{\pi}{4}, r = 6\left[1 + \cos\left(\theta - \frac{\pi}{4}\right)\right]$

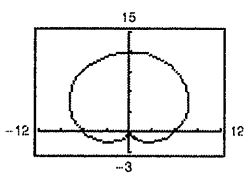


The graph of $r = 6[1 + \cos\theta]$ is rotated through the angle $\pi/4$.

(c) $\theta = \frac{\pi}{2}$

$$r = 6\left[1 + \cos\left(\theta - \frac{\pi}{2}\right)\right]$$

$$= 6\left[1 + \cos\theta\cos\frac{\pi}{2} + \sin\theta\sin\frac{\pi}{2}\right] = 6[1 + \sin\theta]$$



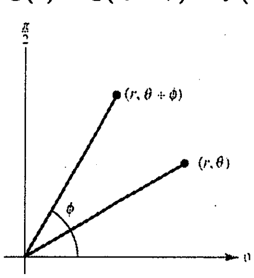
The graph of $r = 6[1 + \cos\theta]$ is rotated through the angle $\pi/2$.

99. Let the curve $r = f(\theta)$ be rotated by ϕ to form the curve $r = g(\theta)$. If (r_1, θ_1) is a point on $r = f(\theta)$, then $(r_1, \theta_1 + \phi)$ is on $r = g(\theta)$. That is,

$g(\theta_1 + \phi) = r_1 = f(\theta_1)$.

Letting $\theta = \theta_1 + \phi$, or $\theta_1 = \theta - \phi$, you see that

$g(\theta) = g(\theta_1 + \phi) = f(\theta_1) = f(\theta - \phi)$.



100. (a) $\sin\left(\theta - \frac{\pi}{2}\right) = \sin\theta\cos\left(\frac{\pi}{2}\right) - \cos\theta\sin\left(\frac{\pi}{2}\right)$

$$= -\cos\theta$$

$$r = f\left[\sin\left(\theta - \frac{\pi}{2}\right)\right]$$

$$= f(-\cos\theta)$$

(b) $\sin(\theta - \pi) = \sin\theta\cos\pi - \cos\theta\sin\pi$

$$= -\sin\theta$$

$$r = f[\sin(\theta - \pi)]$$

$$= f(-\sin\theta)$$

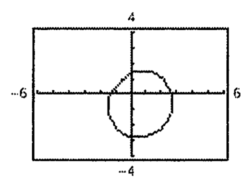
(c) $\sin\left(\theta - \frac{3\pi}{2}\right) = \sin\theta\cos\left(\frac{3\pi}{2}\right) - \cos\theta\sin\left(\frac{3\pi}{2}\right)$

$$= \cos\theta$$

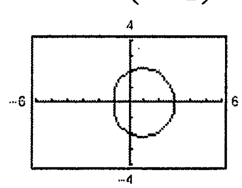
$$r = f\left[\sin\left(\theta - \frac{3\pi}{2}\right)\right] = f(\cos\theta)$$

101. $r = 2 - \sin\theta$

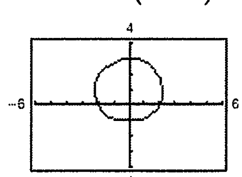
(a) $r = 2 - \sin\left(\theta - \frac{\pi}{4}\right) = 2 - \frac{\sqrt{2}}{2}(\sin\theta - \cos\theta)$



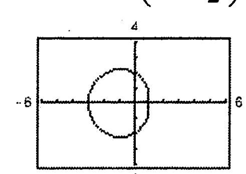
(b) $r = 2 - \sin\left(\theta - \frac{\pi}{2}\right) = 2 - (-\cos\theta) = 2 + \cos\theta$



(c) $r = 2 - \sin(\theta - \pi) = 2 - (-\sin\theta) = 2 + \sin\theta$

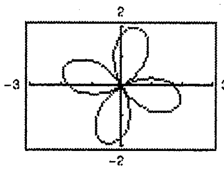


(d) $r = 2 - \sin\left(\theta - \frac{3\pi}{2}\right) = 2 - \cos\theta$

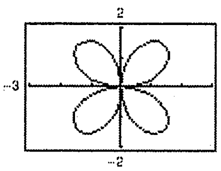


102. $r = 2 \sin 2\theta = 4 \sin \theta \cos \theta$

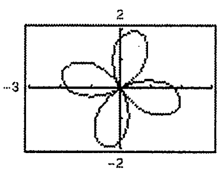
(a) $r = 4 \sin\left(\theta - \dfrac{\pi}{6}\right)\cos\left(\theta - \dfrac{\pi}{6}\right)$



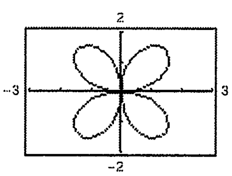
(b) $r = 4 \sin\left(\theta - \dfrac{\pi}{2}\right)\cos\left(\theta - \dfrac{\pi}{2}\right) = -4 \sin \theta \cos \theta$



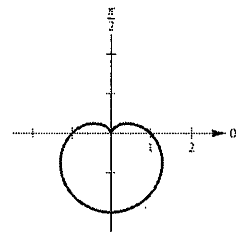
(c) $r = 4 \sin\left(\theta - \dfrac{2\pi}{3}\right)\cos\left(\theta - \dfrac{2\pi}{3}\right)$



(d) $r = 4 \sin(\theta - \pi)\cos(\theta - \pi) = 4 \sin \theta \cos \theta$

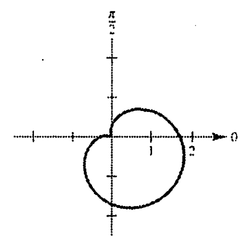


103. (a) $r = 1 - \sin \theta$



(b) $r = 1 - \sin\left(\theta - \dfrac{\pi}{4}\right)$

Rotate the graph of
$r = 1 - \sin \theta$
through the angle $\pi/4$.

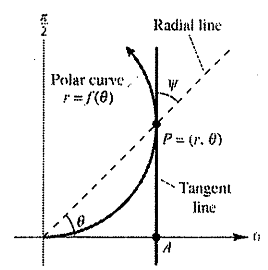


104. By Theorem 9.11, the slope of the tangent line through A and P is

$$\frac{f \cos \theta + f' \sin \theta}{-f \sin \theta + f' \cos \theta}.$$

This is equal to

$$\tan(\theta + \psi) = \frac{\tan \theta + \tan \psi}{1 - \tan \theta \tan \psi} = \frac{\sin \theta + \cos \theta \tan \psi}{\cos \theta - \sin \theta \tan \psi}.$$



Equating the expressions and cross-multiplying, you obtain

$$(f \cos \theta + f' \sin \theta)(\cos \theta - \sin \theta \tan \psi) = (\sin \theta + \cos \theta \tan \psi)(-f \sin \theta + f' \cos \theta)$$

$$f \cos^2 \theta - f \cos \theta \sin \theta \tan \psi + f' \sin \theta \cos \theta - f' \sin^2 \theta \tan \psi = -f \sin^2 \theta - f \sin \theta \cos \theta \tan \psi + f' \sin \theta \cos \theta + f' \cos^2 \theta \tan \psi$$

$$f(\cos^2 \theta + \sin^2 \theta) = f' \tan \psi(\cos^2 \theta + \sin^2 \theta)$$

$$\tan \psi = \frac{f}{f'} = \frac{r}{dr/d\theta}.$$