Section 10.5 Area and Arc Length in Polar Coordinates

1. $A = \frac{1}{2}\int_{\alpha}^{\beta}\left[f(\theta)\right]^2\,d\theta$

$= \frac{1}{2}\int_0^{\pi/2}\left[4\sin\theta\right]^2\,d\theta = 8\int_0^{\pi/2}\sin^2\theta\,d\theta$

2. $A = \frac{1}{2}\int_{\alpha}^{\beta}\left[f(\theta)\right]^2\,d\theta = \frac{1}{2}\int_{3\pi/4}^{5\pi/4}(\cos 2\theta)^2\,d\theta$

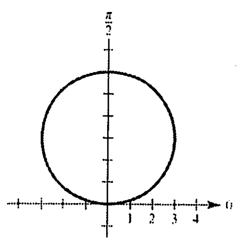
3. $A = \frac{1}{2}\int_{\alpha}^{\beta}\left[f(\theta)\right]^2\,d\theta = \frac{1}{2}\int_{\pi/2}^{3\pi/2}\left[3 - 2\sin\theta\right]^2\,d\theta$

4. $A = \frac{1}{2}\int_{\alpha}^{\beta}\left[f(\theta)\right]^2\,d\theta = \frac{1}{2}\int_0^{\pi/2}\left[1 - \cos 2\theta\right]^2\,d\theta$

5. $A = \frac{1}{2}\int_0^{\pi}\left[6\sin\theta\right]^2\,d\theta$

$= 18\int_0^{\pi}\frac{1 - \cos 2\theta}{2}\,d\theta = 9\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi} = 9\pi$

Note: $r = 6\sin\theta$ is circle of radius 3, $0 \le \theta \le \pi$.

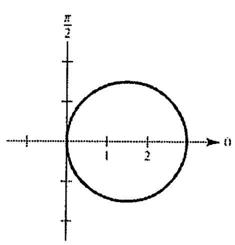


6. $A = \frac{1}{2}\int_0^{\pi}\left[3\cos\theta\right]^2\,d\theta$

$= \frac{9}{2}\int_0^{\pi}\frac{1 + \cos 2\theta}{2}\,d\theta$

$= \frac{9}{4}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi} = \frac{9}{4}\pi$

Note: $r = 3\cos\theta$ is circle of radius $\frac{3}{2}$, $0 \le \theta \le \pi$.



7. $A = 2\left[\frac{1}{2}\int_0^{\pi/6}(2\cos 3\theta)^2\,d\theta\right] = 2\left[\theta + \frac{1}{6}\sin 6\theta\right]_0^{\pi/6} = \frac{\pi}{3}$

8. $A = \frac{1}{2}\int_0^{\pi/3}\left[4\sin 3\theta\right]^2\,d\theta$

$= 8\int_0^{\pi/3}\sin^2 3\theta\,d\theta$

$= 8\int_0^{\pi/3}\frac{1 - \cos 6\theta}{2}\,d\theta$

$= 4\left[\theta - \frac{\sin 6\theta}{6}\right]_0^{\pi/3}$

$= 4\left[\frac{\pi}{3}\right] = \frac{4\pi}{3}$

9. $A = \frac{1}{2}\int_0^{\pi/2}\left[\sin 2\theta\right]^2\,d\theta$

$= \frac{1}{2}\int_0^{\pi/2}\frac{1 - \cos 4\theta}{2}\,d\theta$

$= \frac{1}{4}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/2}$

$= \frac{1}{4}\left[\frac{\pi}{2}\right] = \frac{\pi}{8}$

10. $A = 2\left[\frac{1}{2}\int_0^{\pi/10}(\cos 5\theta)^2\,d\theta\right]$

$= \frac{1}{2}\left[\theta + \frac{1}{10}\sin(10\theta)\right]_0^{\pi/10} = \frac{\pi}{20}$

11. $A = 2\left[\frac{1}{2}\int_{-\pi/2}^{\pi/2}(1 - \sin\theta)^2\,d\theta\right]$

$= \left[\frac{3}{2}\theta + 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_{-\pi/2}^{\pi/2} = \frac{3\pi}{2}$

12. $A = 2\left[\frac{1}{2}\int_0^{\pi/2}(1 - \sin\theta)^2\,d\theta\right]$

$= \left[\frac{3}{2}\theta + 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_0^{\pi/2} = \frac{3\pi - 8}{4}$

13. $A = \frac{1}{2}\int_0^{2\pi}\left[5 + 2\sin\theta\right]^2\,d\theta$

$= \frac{1}{2}\int_0^{2\pi}\left[25 + 20\sin\theta + 4\sin^2\theta\right]d\theta$

$= \frac{1}{2}\int_0^{2\pi}\left[25 + 20\sin\theta + 2(1 - \cos 2\theta)\right]d\theta$

$= \frac{1}{2}\left[27\theta - 20\cos\theta - \sin 2\theta\right]_0^{2\pi}$

$= \frac{1}{2}\left[27(2\pi)\right] = 27\pi$

14. $A = \frac{1}{2}\int_0^{2\pi} [4 - 4\cos\theta]^2\, d\theta$

$= 8\int_0^{2\pi} (1 - \cos\theta)^2\, d\theta$

$= 8\int_0^{2\pi} \left(1 - 2\cos\theta + \cos^2\theta\right) d\theta$

$= 8\int_0^{2\pi} \left(1 - 2\cos\theta + \frac{1 + \cos 2\theta}{2}\right) d\theta$

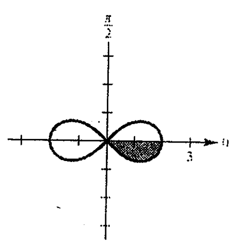
$= 8\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi}$

$= 8\left[\frac{3}{2}(2\pi)\right] = 24\pi$

15. On the interval $-\frac{\pi}{4} \le \theta \le 0$, $r = 2\sqrt{\cos 2\theta}$ traces out one-half of one leaf of the lemniscate. So,

$A = 4\frac{1}{2}\int_{-\pi/4}^{0} 4\cos 2\theta\, d\theta$

$= 8\left[\frac{\sin 2\theta}{2}\right]_{-\pi/4}^{0} = 8\left[\frac{1}{2}\right] = 4.$



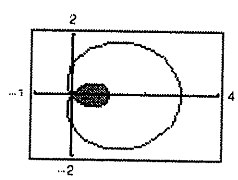
16. On the interval $0 \le \theta \le \pi/2$, $r = \sqrt{6\sin 2\theta}$ traces out half of the lemniscate. So

$A = 2 \cdot \frac{1}{2}\int_0^{\pi/2} 6\sin 2\theta\, d\theta$

$= 6\left[\frac{-\cos 2\theta}{2}\right]_0^{\pi/2} = 6\left[\frac{1}{2} + \frac{1}{2}\right] = 6.$

17. $A = \left[2\frac{1}{2}\int_{2\pi/3}^{\pi} (1 + 2\cos\theta)^2\, d\theta\right]$

$= [3\theta + 4\sin\theta + \sin 2\theta]_{2\pi/3}^{\pi} = \frac{2\pi - 3\sqrt{3}}{2}$



18. Half of the inner loop of $r = 2 - 4\cos\theta$ is traced out on the interval $0 \le \theta \le \frac{\pi}{3}$. So

$A = 2 \cdot \frac{1}{2}\int_0^{\pi/3} (2 - 4\cos\theta)^2\, d\theta$

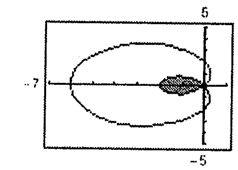
$= \int_0^{\pi/3} \left[4 - 16\cos\theta + 16\cos^2\theta\right] d\theta$

$= \int_0^{\pi/3} \left[4 - 16\cos\theta + 8[1 + \cos 2\theta]\right] d\theta$

$= [12\theta - 16\sin\theta + 4\sin 2\theta]_0^{\pi/3}$

$= 12(\pi/3) - 16\left(\sqrt{3}/2\right) + 4\left(\sqrt{3}/2\right)$

$= 4\pi - 6\sqrt{3}.$



19. The inner loop of $r = 1 + 2\sin\theta$ is traced out on the interval $\frac{7\pi}{6} \le \theta \le \frac{11\pi}{6}$. So,

$A = \frac{1}{2}\int_{7\pi/6}^{11\pi/6} [1 + 2\sin\theta]^2\, d\theta$

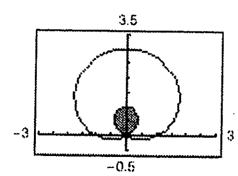
$= \frac{1}{2}\int_{7\pi/6}^{11\pi/6} \left[1 + 4\sin\theta + 4\sin^2\theta\right] d\theta$

$= \frac{1}{2}\int_{7\pi/6}^{11\pi/6} \left[1 + 4\sin\theta + 2(1 - \cos 2\theta)\right] d\theta$

$= \frac{1}{2}[3\theta - 4\cos\theta - \sin 2\theta]_{7\pi/6}^{11\pi/6}$

$= \frac{1}{2}\left[\left(\frac{11\pi}{2} - 2\sqrt{3} + \frac{\sqrt{3}}{2}\right) - \left(\frac{7\pi}{2} + 2\sqrt{3} - \frac{\sqrt{3}}{2}\right)\right]$

$= \frac{1}{2}\left[2\pi - 3\sqrt{3}\right].$

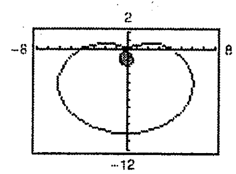


20. $A = 2\left[\dfrac{1}{2}\displaystyle\int_{\arcsin(2/3)}^{\pi/2} (4 - 6\sin\theta)^2\, d\theta\right]$

$$= \int_{\arcsin(2/3)}^{\pi/2} \left[16 - 48\sin\theta + 36\sin^2\theta\right] d\theta$$

$$= \int_{\arcsin(2/3)}^{\pi/2} \left[16 - 48\sin\theta + 36\left(\frac{1 - \cos 2\theta}{2}\right)\right] d\theta$$

$$= \left[34\theta + 48\cos\theta - 9\sin 2\theta\right]_{\arcsin(2/3)}^{\pi/2} \approx 1.7635$$



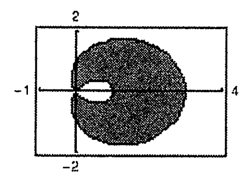
21. The area inside the outer loop is

$$2\left[\frac{1}{2}\int_0^{2\pi/3} (1 + 2\cos\theta)^2\, d\theta\right] = \left[3\theta + 4\sin\theta + \sin 2\theta\right]_0^{2\pi/3}$$

$$= \frac{4\pi + 3\sqrt{3}}{2}.$$

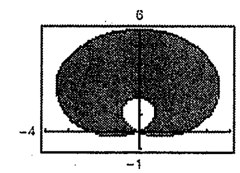
From the result of Exercise 17, the area between the loops is

$$A = \left(\frac{4\pi + 3\sqrt{3}}{2}\right) - \left(\frac{2\pi - 3\sqrt{3}}{2}\right) = \pi + 3\sqrt{3}.$$



22. Four times the area in Exercise 21, $A = 4\left(\pi + 3\sqrt{3}\right)$. More specifically, you see that the area inside the outer loop is

$$2\left[\frac{1}{2}\int_{-\pi/6}^{\pi/2} \left(2(1 + 2\sin\theta)\right)^2 d\theta\right] = \int_{-\pi/6}^{\pi/2} \left(4 + 16\sin\theta + 16\sin^2\theta\right) d\theta = 8\pi + 6\sqrt{3}.$$



The area inside the inner loop is

$$2\frac{1}{2}\left[\int_{7\pi/6}^{3\pi/2} \left(2(1 + 2\sin\theta)\right)^2 d\theta\right] = 4\pi - 6\sqrt{3}.$$

So, the area between the loops is $\left(8\pi + 6\sqrt{3}\right) - \left(4\pi - 6\sqrt{3}\right) = 4\pi + 12\sqrt{3}$.

23. The area inside the outer loop is

$$A = 2\cdot\frac{1}{2}\int_{5\pi/6}^{3\pi/2} [3 - 6\sin\theta]^2\, d\theta$$

$$= \int_{5\pi/6}^{3\pi/2} \left[9 - 36\sin\theta + 36\sin^2\theta\right] d\theta$$

$$= \int_{5\pi/6}^{3\pi/2} \left[9 - 36\sin\theta + 18(1 - \cos 2\theta)\right] d\theta$$

$$= \left[27\theta + 36\cos\theta - 9\sin 2\theta\right]_{5\pi/6}^{3\pi/2} = \left[\frac{81\pi}{2} - \left(\frac{45\pi}{2} - 18\sqrt{3} + \frac{9\sqrt{3}}{2}\right)\right] = 18\pi + \frac{27\sqrt{3}}{2}.$$

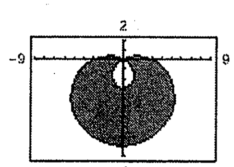
The area inside the inner loop is

$$A = 2\cdot\frac{1}{2}\int_{\pi/6}^{\pi/2} [3 - 6\sin\theta]^2\, d\theta$$

$$= \left[27\theta + 36\cos\theta - 9\sin 2\theta\right]_{\pi/6}^{\pi/2} = \left[\frac{27\pi}{2} - \left(\frac{9\pi}{2} + 18\sqrt{3} - \frac{9\sqrt{3}}{2}\right)\right] = 9\pi - \frac{27\sqrt{3}}{2}.$$

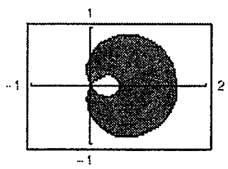
Finally, the area between the loops is

$$\left[18\pi + \frac{27\sqrt{3}}{2}\right] - \left[9\pi - \frac{27\sqrt{3}}{2}\right] = 9\pi + 27\sqrt{3}.$$



24. The area inside the outer loop is

$$A = 2 \cdot \frac{1}{2}\int_0^{2\pi/3}\left[\frac{1}{2} + \cos\theta\right]^2 d\theta$$
$$= \int_0^{2\pi/3}\left[\frac{1}{4} + \cos\theta + \frac{1+\cos 2\theta}{2}\right] d\theta$$
$$= \left[\frac{3}{4}\theta + \sin\theta + \frac{\sin 2\theta}{4}\right]_0^{2\pi/3}$$
$$= \frac{3}{4}\left(\frac{2\pi}{3}\right) + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{8}$$
$$= \frac{\pi}{2} + \frac{3\sqrt{3}}{8}.$$



The area inside the inner loop is

$$A = 2 \cdot \frac{1}{2}\int_{2\pi/3}^{\pi}\left[\frac{1}{2} + \cos\theta\right]^2 d\theta$$
$$= \left[\frac{3}{4}\theta + \sin\theta + \frac{\sin 2\theta}{4}\right]_{2\pi/3}^{\pi}$$
$$= \frac{3}{4}\pi - \left(\frac{\pi}{2} + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{8}\right) = \frac{\pi}{4} - \frac{3\sqrt{3}}{8}$$

Finally, the area between the loops is

$$\left[\frac{\pi}{2} + \frac{3\sqrt{3}}{8}\right] - \left[\frac{\pi}{4} - \frac{3\sqrt{3}}{8}\right] = \frac{\pi}{4} + \frac{3\sqrt{3}}{4}.$$

25. $r = 1 + \cos\theta$

$r = 1 - \cos\theta$

Solving simultaneously,

$1 + \cos\theta = 1 - \cos\theta$

$2\cos\theta = 0$

$\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}.$

Replacing r by $-r$ and θ by $\theta + \pi$ in the first equation and solving, $-1 + \cos\theta = 1 - \cos\theta$, $\cos\theta = 1$, $\theta = 0$. Both curves pass through the pole, $(0, \pi)$, and $(0, 0)$, respectively.

Points of intersection: $\left(1, \dfrac{\pi}{2}\right), \left(1, \dfrac{3\pi}{2}\right), (0, 0)$

26. $r = 3(1 + \sin\theta)$

$r = 3(1 - \sin\theta)$

Solving simultaneously,

$3(1 + \sin\theta) = 3(1 - \sin\theta)$

$2\sin\theta = 0$

$\theta = 0, \pi.$

Replacing r by $-r$ and θ by $\theta + \pi$ in the first equation and solving, $-3(1 - \sin\theta) = 3(1 - \sin\theta)$, $\sin\theta = 1$, $\theta = \pi/2$. Both curves pass through the pole, $(0, 3\pi/2)$, and $(0, \pi/2)$, respectively.

Points of intersection: $(3, 0), (3, \pi), (0, 0)$

27. $r = 1 + \cos\theta$

$r = 1 - \sin\theta$

Solving simultaneously,

$1 + \cos\theta = 1 - \sin\theta$

$\cos\theta = -\sin\theta$

$\tan\theta = -1$

$\theta = \dfrac{3\pi}{4}, \dfrac{7\pi}{4}.$

Replacing r by $-r$ and θ by $\theta + \pi$ in the first equation and solving, $-1 + \cos\theta = 1 - \sin\theta$, $\sin\theta + \cos\theta = 2$, which has no solution. Both curves pass through the pole, $(0, \pi)$, and $(0, \pi/2)$, respectively.

Points of intersection:
$\left(\dfrac{2-\sqrt{2}}{2}, \dfrac{3\pi}{4}\right), \left(\dfrac{2+\sqrt{2}}{2}, \dfrac{7\pi}{4}\right), (0, 0)$

28. $r = 2 - 3\cos\theta$

$r = \cos\theta$

Solving simultaneously,

$2 - 3\cos\theta = \cos\theta$

$\cos\theta = \dfrac{1}{2}$

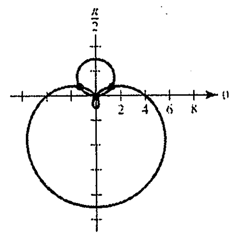
$\theta = \dfrac{\pi}{3}, \dfrac{5\pi}{3}.$

Both curves pass through the pole, $(0, \arccos 2/3)$, and $(0, \pi/2)$, respectively.

Points of intersection: $\left(\dfrac{1}{2}, \dfrac{\pi}{3}\right), \left(\dfrac{1}{2}, \dfrac{5\pi}{3}\right), (0, 0)$

29. $r = 4 - 5 \sin \theta$

$r = 3 \sin \theta$



Solving simultaneously,

$4 - 5 \sin \theta = 3 \sin \theta$

$$\sin \theta = \frac{1}{2}$$

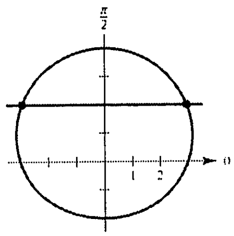
$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}.$$

Both curves pass through the pole, $(0, \arcsin 4/5)$, and $(0, 0)$, respectively.

Points of intersection: $\left(\frac{3}{2}, \frac{\pi}{6}\right), \left(\frac{3}{2}, \frac{5\pi}{6}\right), (0, 0)$

30. $r = 3 + \sin \theta$

$r = 2 \csc \theta$



The graph of $r = 3 + \sin \theta$ is a limaçon symmetric to $\theta = \pi/2$, and the graph of $r = 2 \csc \theta$ is the horizontal line $y = 2$. So, there are two points of intersection.

Solving simultaneously,

$$3 + \sin \theta = 2 \csc \theta$$

$$\sin^2 \theta + 3 \sin \theta - 2 = 0$$

$$\sin \theta = \frac{-3 \pm \sqrt{17}}{2}$$

$$\theta = \arcsin\left(\frac{\sqrt{17} - 3}{2}\right) \approx 0.596.$$

Points of intersection:

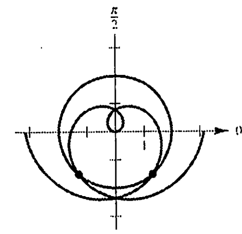
$$\left(\frac{\sqrt{17} + 3}{2}, \arcsin\left(\frac{\sqrt{17} - 3}{2}\right)\right),$$

$$\left(\frac{\sqrt{17} + 3}{2}, \pi - \arcsin\left(\frac{\sqrt{17} - 3}{2}\right)\right),$$

$(3.56, 0.596), (3.56, 2.545)$

31. $r = \dfrac{\theta}{2}$

$r = 2$



Solving simultaneously, you have

$\theta/2 = 2, \theta = 4.$

Points of intersection:

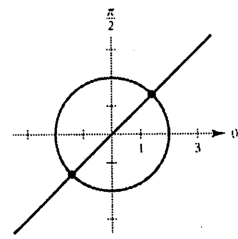
$(2, 4), (-2, -4)$

32. $\theta = \dfrac{\pi}{4}$

$r = 2$

Line of slope 1 passing through the pole and a circle of radius 2 centered at the pole.

Points of intersection: $\left(2, \frac{\pi}{4}\right), \left(-2, \frac{\pi}{4}\right)$



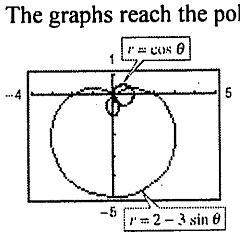
33. $r = \cos \theta$

$r = 2 - 3 \sin \theta$

Points of intersection:

$(0, 0), (0.935, 0.363), (0.535, -1.006)$

The graphs reach the pole at different times (θ values).

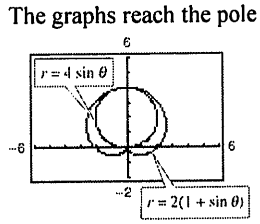


34. $r = 4 \sin \theta$

$r = 2(1 + \sin \theta)$

Points of intersection: $(0, 0), \left(4, \frac{\pi}{2}\right)$

The graphs reach the pole at different times (θ values).



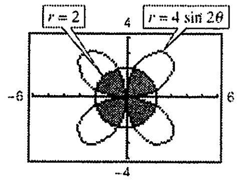
35. The points of intersection for one petal are $(2, \pi/12)$ and $(2, 5\pi/12)$. The area within one petal is

$$A = \frac{1}{2}\int_0^{\pi/12} (4\sin 2\theta)^2\,d\theta + \frac{1}{2}\int_{\pi/12}^{5\pi/12} (2)^2\,d\theta + \frac{1}{2}\int_{5\pi/12}^{\pi/2} (4\sin 2\theta)^2\,d\theta$$

$$= 16\int_0^{\pi/12} \sin^2(2\theta)\,d\theta + 2\int_{\pi/12}^{5\pi/12} d\theta \text{ (by symmetry of the petal)}$$

$$= 8\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/12} + \left[2\theta\right]_{\pi/12}^{5\pi/12} = \frac{4\pi}{3} - \sqrt{3}.$$

$$\text{Total area} = 4\left(\frac{4\pi}{3} - \sqrt{3}\right) = \frac{16\pi}{3} - 4\sqrt{3} = \frac{4}{3}\left(4\pi - 3\sqrt{3}\right)$$



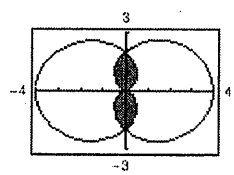
36. The common interior is 4 times the area in the first quadrant.

$$A = 4\frac{1}{2}\int_0^{\pi/2} \left[2(1 - \cos\theta)\right]^2\,d\theta$$

$$= 8\int_0^{\pi/2} \left(1 - 2\cos\theta + \frac{1 + \cos 2\theta}{2}\right)d\theta$$

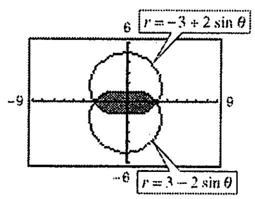
$$= 8\left[\frac{3\theta}{2} - 2\sin\theta + \frac{\sin 2\theta}{4}\right]_0^{\pi/2}$$

$$= 8\left[\frac{3}{2}\left(\frac{\pi}{2}\right) - 2\right] = 6\pi - 16$$



37. $A = 4\left[\frac{1}{2}\int_0^{\pi/2} (3 - 2\sin\theta)^2\,d\theta\right]$

$$= 2\left[11\theta + 12\cos\theta - \sin(2\theta)\right]_0^{\pi/2} = 11\pi - 24$$



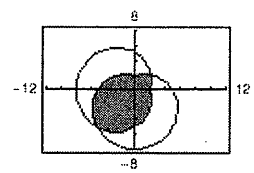
38. $r = 5 - 3\sin\theta$ and $r = 5 - 3\cos\theta$ intersect at $\theta = \pi/4$ and $\pi = 5\pi/4$.

$$A = 2\left[\frac{1}{2}\int_{\pi/4}^{5\pi/4} (5 - 3\sin\theta)^2\,d\theta\right]$$

$$= \left[\frac{59}{2}\theta + 30\cos\theta - \frac{9}{4}\sin 2\theta\right]_{\pi/4}^{5\pi/4}$$

$$= \left(\frac{59}{2}\left(\frac{5\pi}{4}\right) - 30\frac{\sqrt{2}}{2} - \frac{9}{4}\right) - \left(\frac{59}{2}\left(\frac{\pi}{4}\right) + 30\frac{\sqrt{2}}{2} - \frac{9}{4}\right)$$

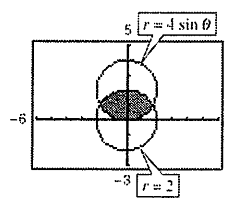
$$= \frac{59\pi}{2} - 30\sqrt{2} \approx 50.251$$



39. $A = 2\left[\frac{1}{2}\int_0^{\pi/6} (4\sin\theta)^2\,d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/2} (2)^2\,d\theta\right]$

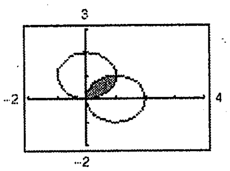
$$= 16\left[\frac{1}{2}\theta - \frac{1}{4}\sin(2\theta)\right]_0^{\pi/6} + \left[4\theta\right]_{\pi/6}^{\pi/2}$$

$$= \frac{8\pi}{3} - 2\sqrt{3} = \frac{2}{3}\left(4\pi - 3\sqrt{3}\right)$$



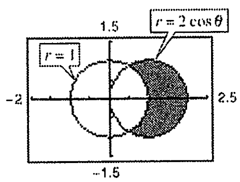
40. The common interior is given by

$$A = 2\frac{1}{2}\int_{\pi/4}^{\pi/2}[2\cos\theta]^2\,d\theta$$
$$= 4\int_{\pi/4}^{\pi/2}\frac{1+\cos 2\theta}{2}\,d\theta$$
$$= 2\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/4}^{\pi/2}$$
$$= 2\left[\frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{2}\right]$$
$$= \frac{\pi}{2} - 1$$



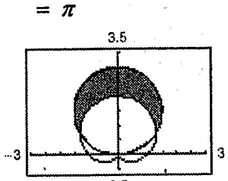
41. $r = 2\cos\theta = 1 \Rightarrow \theta = \pi/3$

$$A = 2\cdot\frac{1}{2}\int_0^{\pi/3}\left([2\cos\theta]^2 - 1\right)d\theta$$
$$= \int_0^{\pi/3}\left[2(1+\cos 2\theta) - 1\right]d\theta$$
$$= \left[\theta + \sin 2\theta\right]_0^{\pi/3}$$
$$= \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$

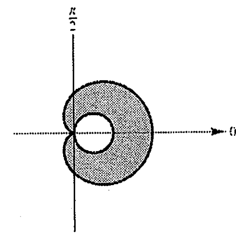


42. $3\sin\theta = 1 + \sin\theta \Rightarrow \sin\theta = 1/2 \Rightarrow \theta = \pi/6$

$$A = 2\cdot\frac{1}{2}\int_{\pi/6}^{\pi/2}\left([3\sin\theta]^2 - [1+\sin\theta]^2\right)d\theta$$
$$= \int_{\pi/6}^{\pi/2}\left[9\sin^2\theta - 1 - 2\sin\theta - \sin^2\theta\right]d\theta$$
$$= \int_{\pi/6}^{\pi/2}\left[4(1-\cos 2\theta) - 1 - 2\sin\theta\right]d\theta$$
$$= \left[3\theta - 2\sin 2\theta + 2\cos\theta\right]_{\pi/6}^{\pi/2}$$
$$= 3\frac{\pi}{2} - 3\frac{\pi}{6} + 2\frac{\sqrt{3}}{2} - 2\frac{\sqrt{3}}{2}$$
$$= \pi$$

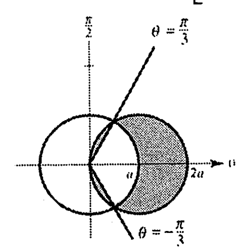


43. $$A = 2\left[\frac{1}{2}\int_0^{\pi}\left[a(1+\cos\theta)\right]^2 d\theta\right] - \frac{a^2\pi}{4}$$
$$= a^2\left[\frac{3}{2}\theta + 2\sin\theta + \frac{\sin 2\theta}{4}\right]_0^{\pi} - \frac{a^2\pi}{4}$$
$$= \frac{3a^2\pi}{2} - \frac{a^2\pi}{4} = \frac{5a^2\pi}{4}$$

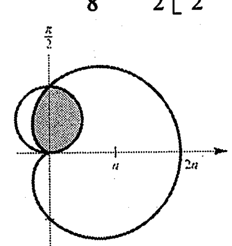


44. Area = Area of $r = 2a\cos\theta$ − Area of sector − twice area between $r = 2a\cos\theta$ and the lines

$$\theta = \frac{\pi}{3}, \theta = \frac{\pi}{2}.$$
$$A = \pi a^2 - \left(\frac{\pi}{3}\right)a^2 - 2\left[\frac{1}{2}\int_{\pi/3}^{\pi/2}(2a\cos\theta)^2\,d\theta\right]$$
$$= \frac{2\pi a^2}{3} - 2a^2\int_{\pi/3}^{\pi/2}(1+\cos 2\theta)\,d\theta$$
$$= \frac{2\pi a^2}{3} - 2a^2\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/3}^{\pi/2}$$
$$= \frac{2\pi a^2}{3} - 2a^2\left[\frac{\pi}{2} - \frac{\pi}{3} - \frac{\sqrt{3}}{4}\right] = \frac{2\pi a^2 + 3\sqrt{3}a^2}{6}$$



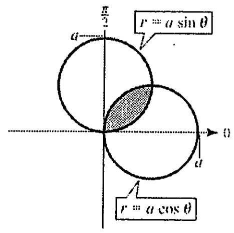
45. $$A = \frac{\pi a^2}{8} + \frac{1}{2}\int_{\pi/2}^{\pi}\left[a(1+\cos\theta)\right]^2 d\theta$$
$$= \frac{\pi a^2}{8} + \frac{a^2}{2}\int_{\pi/2}^{\pi}\left(\frac{3}{2} + 2\cos\theta + \frac{\cos 2\theta}{2}\right)d\theta$$
$$= \frac{\pi a^2}{8} + \frac{a^2}{2}\left[\frac{3}{2}\theta + 2\sin\theta + \frac{\sin 2\theta}{4}\right]_{\pi/2}^{\pi}$$
$$= \frac{\pi a^2}{8} + \frac{a^2}{2}\left[\frac{3\pi}{2} - \frac{3\pi}{4} - 2\right] = \frac{a^2}{2}[\pi - 2]$$



46. $r = a\cos\theta,\ r = a\sin\theta$

$\tan\theta = 1,\ \theta = \pi/4$

$$A = 2\left[\frac{1}{2}\int_0^{\pi/4}(a\sin\theta)^2\,d\theta\right]$$
$$= a^2\int_0^{\pi/4}\frac{1-\cos 2\theta}{2}\,d\theta$$
$$= \frac{1}{2}a^2\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/4}$$
$$= \frac{1}{2}a^2\left[\frac{\pi}{4} - \frac{1}{2}\right]$$
$$= \frac{1}{8}a^2\pi - \frac{1}{4}a^2$$

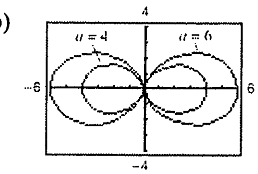


47. (a) $r = a\cos^2\theta$

$r^3 = ar^2\cos^2\theta$

$\left(x^2 + y^2\right)^{3/2} = ax^2$

(b)



(c) $$A = 4\left(\frac{1}{2}\right)\int_0^{\pi/2}\left[\left(6\cos^2\theta\right)^2 - \left(4\cos^2\theta\right)^2\right]d\theta$$
$$= 40\int_0^{\pi/2}\cos^4\theta\,d\theta$$
$$= 10\int_0^{\pi/2}(1+\cos 2\theta)^2\,d\theta$$
$$= 10\int_0^{\pi/2}\left(1 + 2\cos 2\theta + \frac{1-\cos 4\theta}{2}\right)d\theta$$
$$= 10\left[\frac{3}{2}\theta + \sin 2\theta + \frac{1}{8}\sin 4\theta\right]_0^{\pi/2} = \frac{15\pi}{2}$$

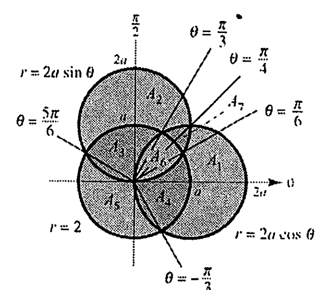
48. By symmetry, $A_1 = A_2$ and $A_3 = A_4$.

$$A_1 = A_2 = \frac{1}{2}\int_{-\pi/3}^{\pi/6}\left[(2a\cos\theta)^2 - (a)^2\right]d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/4}\left[(2a\cos\theta)^2 - (2a\sin\theta)^2\right]d\theta$$
$$= \frac{a^2}{2}\int_{-\pi/3}^{\pi/6}\left(4\cos^2\theta - 1\right)d\theta + 2a^2\int_{\pi/6}^{\pi/4}\cos 2\theta\,d\theta$$
$$= \frac{a^2}{2}[\theta + \sin 2\theta]_{-\pi/3}^{\pi/6} + a^2[\sin 2\theta]_{\pi/6}^{\pi/4} = \frac{a^2}{2}\left(\frac{\pi}{2} + \sqrt{3}\right) + a^2\left(1 - \frac{\sqrt{3}}{2}\right) = a^2\left(\frac{\pi}{4} + 1\right)$$

$$A_3 = A_4 = \frac{1}{2}\left(\frac{\pi}{2}\right)a^2 = \frac{\pi a^2}{4}$$

$$A_5 = \frac{1}{2}\left(\frac{5\pi}{6}\right)a^2 - 2\left(\frac{1}{2}\right)\int_{5\pi/6}^{\pi}(2a\sin\theta)^2\,d\theta$$
$$= \frac{5\pi a^2}{12} - 2a^2\int_{5\pi/6}^{\pi}(1-\cos 2\theta)\,d\theta$$
$$= \frac{5\pi a^2}{12} - a^2[2\theta - \sin 2\theta]_{5\pi/6}^{\pi} = \frac{5\pi a^2}{12} - a^2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) = a^2\left(\frac{\pi}{12} + \frac{\sqrt{3}}{2}\right)$$

$$A_6 = 2\left(\frac{1}{2}\right)\int_0^{\pi/6}(2a\sin\theta)^2\,d\theta + 2\left(\frac{1}{2}\right)\int_{\pi/6}^{\pi/4}a^2\,d\theta$$
$$= 2a^2\int_0^{\pi/6}(1-\cos 2\theta)\,d\theta + \left[a^2\theta\right]_{\pi/6}^{\pi/4}$$
$$= a^2[2\theta - \sin 2\theta]_0^{\pi/6} + \frac{\pi a^2}{12} = a^2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) + \frac{\pi a^2}{12} = a^2\left(\frac{5\pi}{12} - \frac{\sqrt{3}}{2}\right)$$



$$A_7 = 2\left(\frac{1}{2}\right)\int_{\pi/6}^{\pi/4}\left[(2a\sin\theta)^2 - (a)^2\right]d\theta$$
$$= a^2\int_{\pi/6}^{\pi/4}\left(4\sin^2\theta - 1\right)d\theta = a^2[\theta - \sin 2\theta]_{\pi/6}^{\pi/4} = a^2\left(\frac{\pi}{12} - 1 + \frac{\sqrt{3}}{2}\right)$$

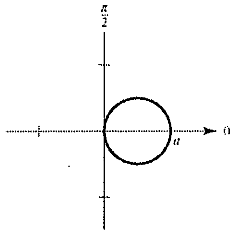
[**Note:** $A_1 + A_6 + A_7 + A_4 = \pi a^2$ = area of circle of radius a]

49. $r = a\cos(n\theta)$

For $n = 1$:

$r = a\cos\theta$

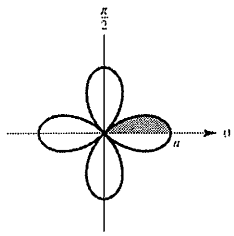
$$A = \pi\left(\frac{a}{2}\right)^2 = \frac{\pi a^2}{4}$$



For $n = 2$:

$r = a\cos 2\theta$

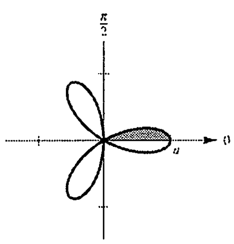
$$A = 8\left(\frac{1}{2}\right)\int_0^{\pi/4}(a\cos 2\theta)^2\,d\theta = \frac{\pi a^2}{2}$$



For $n = 3$:

$r = a\cos 3\theta$

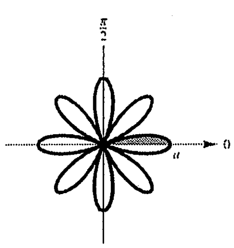
$$A = 6\left(\frac{1}{2}\right)\int_0^{\pi/6}(a\cos 3\theta)^2\,d\theta = \frac{\pi a^2}{4}$$



For $n = 4$:

$r = a\cos 4\theta$

$$A = 16\left(\frac{1}{2}\right)\int_0^{\pi/8}(a\cos 4\theta)^2\,d\theta = \frac{\pi a^2}{2}$$



In general, the area of the region enclosed by $r = a\cos(n\theta)$ for $n = 1, 2, 3, \ldots$ is $(\pi a^2)/4$ if n is odd and is $(\pi a^2)/2$ if n is even.

50. $$r = \sec\theta - 2\cos\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$r\cos\theta = 1 - 2\cos^2\theta$$

$$x = 1 - 2\left(\frac{r^2\cos^2\theta}{r^2}\right) = 1 - 2\left(\frac{x^2}{x^2 + y^2}\right)$$

$$(x^2 + y^2)x = x^2 + y^2 - 2x^2$$

$$y^2(x - 1) = -x^2 - x^3$$

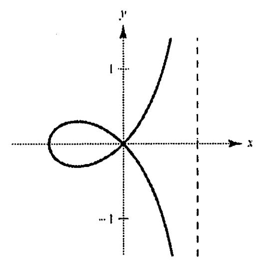
$$y^2 = \frac{x^2(1 + x)}{1 - x}$$

$$A = 2\left(\frac{1}{2}\right)\int_0^{\pi/4}(\sec\theta - 2\cos\theta)^2\,d\theta$$

$$= \int_0^{\pi/4}\left(\sec^2\theta - 4 + 4\cos^2\theta\right)d\theta$$

$$= \int_0^{\pi/4}\left(\sec^2\theta - 4 + 2(1 + \cos 2\theta)\right)d\theta$$

$$= \left[\tan\theta - 2\theta + \sin 2\theta\right]_0^{\pi/4} = 2 - \frac{\pi}{2}$$



51. $r = 8,\ r' = 0$

$$s = \int_0^{2\pi}\sqrt{8^2 + 0^2}\,d\theta = 8\theta\Big]_0^{2\pi} = 16\pi$$

(circumference of circle of radius 8)

52. $r = a$

$r' = 0$

$$s = \int_0^{2\pi}\sqrt{a^2 + 0^2}\,d\theta = \left[a\theta\right]_0^{2\pi} = 2\pi a$$

(circumference of circle of radius a)

53. $r = 4\sin\theta$

$r' = 4\cos\theta$

$$s = \int_0^{\pi}\sqrt{(4\sin\theta)^2 + (4\cos\theta)^2}\,d\theta$$

$$= \int_0^{\pi}4\,d\theta = \left[4\theta\right]_0^{\pi} = 4\pi$$

(circumference of circle of radius 2)

54. $r = 2a\cos\theta$

$r' = -2a\sin\theta$

$$s = \int_{-\pi/2}^{\pi/2}\sqrt{(2a\cos\theta)^2 + (-2a\sin\theta)^2}\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}2a\,d\theta = \left[2a\theta\right]_{-\pi/2}^{\pi/2} = 2\pi a$$

55. $r = 1 + \sin\theta$

$r' = \cos\theta$

$$s = 2\int_{\pi/2}^{3\pi/2}\sqrt{(1 + \sin\theta)^2 + (\cos\theta)^2}\,d\theta$$

$$= 2\sqrt{2}\int_{\pi/2}^{3\pi/2}\sqrt{1 + \sin\theta}\,d\theta$$

$$= 2\sqrt{2}\int_{\pi/2}^{3\pi/2}\frac{-\cos\theta}{\sqrt{1 - \sin\theta}}\,d\theta$$

$$= \left[4\sqrt{2}\sqrt{1 - \sin\theta}\right]_{\pi/2}^{3\pi/2}$$

$$= 4\sqrt{2}\left(\sqrt{2} - 0\right) = 8$$

56. $r = 8(1 + \cos\theta),\ 0 \le \theta \le 2\pi$

$r' = -8\sin\theta$

$$s = 2\int_0^{\pi}\sqrt{[8(1 + \cos\theta)]^2 + (-8\sin\theta]^2}\,d\theta$$

$$= 16\int_0^{\pi}\sqrt{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}\,d\theta$$

$$= 16\sqrt{2}\int_0^{\pi}\sqrt{1 + \cos\theta}\,d\theta$$

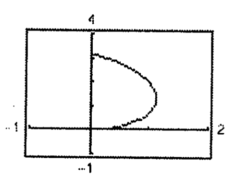
$$= 16\sqrt{2}\int_0^{\pi}\sqrt{1 + \cos\theta}\cdot\left(\frac{\sqrt{1 - \cos\theta}}{\sqrt{1 - \cos\theta}}\right)d\theta$$

$$= 16\sqrt{2}\int_0^{\pi}\frac{\sin\theta}{\sqrt{1 - \cos\theta}}\,d\theta$$

$$= \left[32\sqrt{2}\sqrt{1 - \cos\theta}\right]_0^{\pi}$$

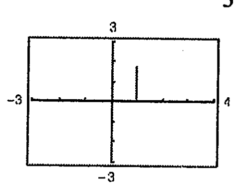
$$= 64$$

57. $r = 2\theta,\ 0 \le \theta \le \dfrac{\pi}{2}$



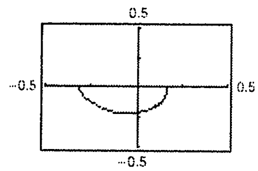
Length $\approx$ 4.16

58. $r = \sec\theta,\ 0 \le \theta \le \dfrac{\pi}{3}$



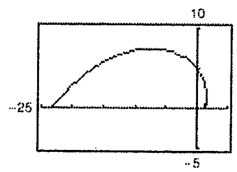
Length $\approx 1.73\ \left(\text{exact } \sqrt{3}\right)$

59. $r = \dfrac{1}{\theta},\ \pi \le \theta \le 2\pi$



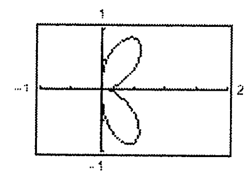
Length $\approx$ 0.71

60. $r = e^{\theta},\ 0 \le \theta \le \pi$



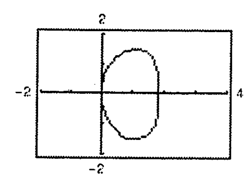
Length $\approx$ 31.31

61. $r = \sin(3\cos\theta),\ 0 \le \theta \le \pi$



Length $\approx$ 4.39

62. $r = 2\sin(2\cos\theta),\ 0 \le \theta \le \pi$



Length $\approx$ 7.78

63. $r = 6\cos\theta$

$r' = -6\sin\theta$

$$S = 2\pi\int_0^{\pi/2} 6\cos\theta\sin\theta\sqrt{36\cos^2\theta + 36\sin^2\theta}\,d\theta$$

$$= 72\pi\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta$$

$$= \left[36\pi\sin^2\theta\right]_0^{\pi/2}$$

$$= 36\pi$$

64. $r = a\cos\theta$

$r' = -a\sin\theta$

$$S = 2\pi\int_0^{\pi/2} a\cos\theta(\cos\theta)\sqrt{a^2\cos\theta + a^2\sin^2\theta}\,d\theta$$

$$= 2\pi a^2\int_0^{\pi/2}\cos^2\theta\,d\theta = \pi a^2\int_0^{\pi/2}(1 + \cos 2\theta)\,d\theta$$

$$= \left[\pi a^2\left(\theta + \frac{\sin 2\theta}{2}\right)\right]_0^{\pi/2} = \frac{\pi^2 a^2}{2}$$

65. $r = e^{a\theta}$

$r' = ae^{a\theta}$

$$S = 2\pi\int_0^{\pi/2} e^{a\theta}\cos\theta\sqrt{\left(e^{a\theta}\right)^2 + \left(ae^{a\theta}\right)^2}\,d\theta$$

$$= 2\pi\sqrt{1 + a^2}\int_0^{\pi/2} e^{2a\theta}\cos\theta\,d\theta$$

$$= 2\pi\sqrt{1 + a^2}\left[\frac{e^{2a\theta}}{4a^2 + 1}(2a\cos\theta + \sin\theta)\right]_0^{\pi/2}$$

$$= \frac{2\pi\sqrt{1 + a^2}}{4a^2 + 1}\left(e^{\pi a} - 2a\right)$$

66. $r = a(1 + \cos\theta)$

$r' = -a\sin\theta$

$$S = 2\pi\int_0^{\pi} a(1 + \cos\theta)\sin\theta\sqrt{a^2(1 + \cos\theta)^2 + a^2\sin^2\theta}\,d\theta = 2\pi a^2\int_0^{\pi}\sin\theta(1 + \cos\theta)\sqrt{2 + 2\cos\theta}\,d\theta$$

$$= -2\sqrt{2}\pi a^2\int_0^{\pi}(1 + \cos\theta)^{3/2}(-\sin\theta)\,d\theta = -\frac{4\sqrt{2}\pi a^2}{5}\left[(1 + \cos\theta)^{5/2}\right]_0^{\pi} = \frac{32\pi a^2}{5}$$

67. $r = 4\cos 2\theta$

$r' = -8\sin 2\theta$

$$S = 2\pi\int_0^{\pi/4} 4\cos 2\theta\sin\theta\sqrt{16\cos^2 2\theta + 64\sin^2\theta\, 2\theta}\,d\theta = 32\pi\int_0^{\pi/4}\cos 2\theta\sin\theta\sqrt{\cos^2 2\theta + 4\sin^2 2\theta}\,d\theta \approx 21.87$$

68. $r = \theta$

$r' = 1$

$$S = 2\pi\int_0^{\pi}\theta\sin\theta\sqrt{\theta^2 + 1}\,d\theta \approx 42.32$$

69. You will only find simultaneous points of intersection. There may be intersection points that do not occur with the same coordinates in the two graphs.

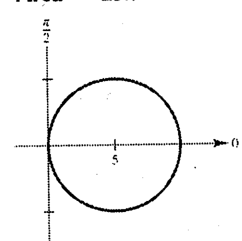
70. (a) $S = 2\pi\int_{\alpha}^{\beta} f(\theta)\sin\theta\sqrt{f(\theta)^2 + f'(\theta)^2}\,d\theta$

(b) $S = 2\pi\int_{\alpha}^{\beta} f(\theta)\cos\theta\sqrt{f(\theta)^2 + f'(\theta)^2}\,d\theta$

71. (a) $r = 10\cos\theta, 0 \le \theta < \pi$

Circle of radius 5

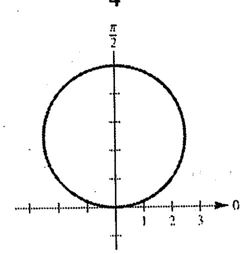
Area $= 25\pi$



(b) $r = 5\sin\theta, 0 \le \theta < \pi$

Circle radius $5/2$

Area $= \frac{25}{4}\pi$

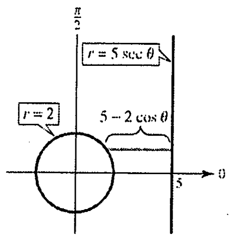


72. Graph (b) has a larger arc length because it has more leaves.

73. Revolve $r = 2$ about the line $r = 5 \sec \theta$.

$f(\theta) = 2, f'(\theta) = 0$

$$S = 2\pi \int_0^{2\pi} (5 - 2\cos\theta)\sqrt{2^2 + 0^2}\, d\theta$$
$$= 4\pi \int_0^{2\pi} (5 - 2\cos\theta)\, d\theta$$
$$= 4\pi\big[5\theta - 2\sin\theta\big]_0^{2\pi}$$
$$= 40\pi^2$$

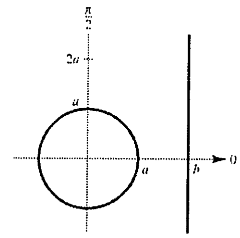


74. Revolve $r = a$ about the line $r = b \sec \theta$ where $b > a > 0$.

$f(\theta) = a$

$f'(\theta) = 0$

$$S = 2\pi \int_0^{2\pi} \big[b - a\cos\theta\big]\sqrt{a^2 + 0^2}\, d\theta$$
$$= 2\pi a\big[b\theta - a\sin\theta\big]_0^{2\pi}$$
$$= 2\pi a(2\pi b) = 4\pi^2 ab$$



75. $r = 8\cos\theta, 0 \le \theta \le \pi$

(a) $A = \frac{1}{2}\int_0^{\pi} r^2\, d\theta = \frac{1}{2}\int_0^{\pi} 64\cos^2\theta\, d\theta = 32\int_0^{\pi} \frac{1 + \cos 2\theta}{2}\, d\theta = 16\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi} = 16\pi$

$\left(\text{Area circle} = \pi r^2 = \pi 4^2 = 16\pi\right)$

(b)

θ	0.2	0.4	0.6	0.8	1.0	1.2	1.4
A	6.32	12.14	17.06	20.80	23.27	24.60	25.08

(c), (d) For $\frac{1}{4}$ of area $(4\pi \approx 12.57)$: 0.42

For $\frac{1}{2}$ of area $(8\pi \approx 25.13)$: $1.57\left(\frac{\pi}{2}\right)$

For $\frac{3}{4}$ of area $(12\pi \approx 37.70)$: 2.73

(e) No, it does not depend on the radius.

76. $r = 3\sin\theta, 0 \le \theta \le \pi$

(a) $A = \frac{1}{2}\int_0^{\pi} r^2\, d\theta = \frac{9}{2}\int_0^{\pi} \sin^2\theta\, d\theta = \frac{9}{4}\int_0^{\pi} (1 - \cos 2\theta)\, d\theta = \frac{9}{4}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi} = \frac{9}{4}\pi$

$\left[\textbf{Note: } \text{radius of circle is } \frac{3}{2} \Rightarrow A = \pi\left(\frac{3}{2}\right)^2 = \frac{9}{4}\pi\right]$

(b)

θ	0.2	0.4	0.6	0.8	1.0	1.2	1.4
A	0.0119	0.0930	0.3015	0.6755	1.2270	1.9401	2.7731

(c), (d) For $\frac{1}{8}$ of area $\left(\frac{1}{8}\frac{9}{4}\pi \approx 0.8836\right)$: $\theta \approx 0.88$

For $\frac{1}{4}$ of area $\left(\frac{1}{4}\frac{9}{4}\pi \approx 1.7671\right)$: $\theta \approx 1.15$

For $\frac{1}{2}$ of area $\left(\frac{1}{2}\frac{9}{4}\pi \approx 3.5343\right)$: $\theta = \frac{\pi}{2} \approx 1.57$

77.
$$r = a\sin\theta + b\cos\theta$$
$$r^2 = ar\sin\theta + br\cos\theta$$
$$x^2 + y^2 = ay + bx$$
$$x^2 + y^2 - bx - ay = 0 \text{ represents a circle.}$$

78. $r = \sin\theta + \cos\theta$, Circle

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta$$
$$= \frac{1}{2}\int_0^{\pi} (\sin\theta + \cos\theta)^2\,d\theta$$
$$= \frac{1}{2}\int_0^{\pi} (1 + 2\sin\theta\cos\theta)\,d\theta = \frac{1}{2}\Big[\theta + \sin^2\theta\Big]_0^{\pi} = \frac{\pi}{2}$$

Converting to rectangular form:

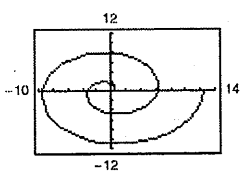
$$r^2 = r\sin\theta + r\cos\theta$$
$$x^2 + y^2 = y + x$$
$$\left(x^2 - x + \frac{1}{4}\right) + \left(y^2 - y + \frac{1}{4}\right) = \frac{1}{2}$$
$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}$$

Circle of radius $\frac{1}{\sqrt{2}}$ and center $\left(\frac{1}{2}, \frac{1}{2}\right)$

$$\text{Area} = \pi\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{\pi}{2}$$

79. (a) $r = \theta, \theta \ge 0$

As a increases, the spiral opens more rapidly. If $\theta < 0$, the spiral is reflected about the y-axis.



(b) $r = a\theta, \theta \ge 0$, crosses the polar axis for $\theta = n\pi$, n and integer. To see this

$r = a\theta \Rightarrow r\sin\theta = y = a\theta\sin\theta = 0$

for $\theta = n\pi$. The points are

$(r, \theta) = (an\pi, n\pi), n = 1, 2, 3, \ldots$

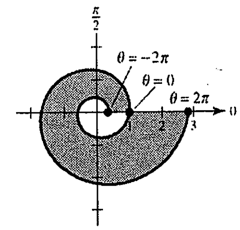
(c) $f(\theta) = \theta, f'(\theta) = 1$

$$s = \int_0^{2\pi} \sqrt{\theta^2 + 1}\,d\theta$$
$$= \frac{1}{2}\Big[\ln\left(\sqrt{x^2+1} + x\right) + x\sqrt{x^2+1}\Big]_0^{2\pi}$$
$$= \frac{1}{2}\ln\left(\sqrt{4\pi^2+1} + 2\pi\right) + \pi\sqrt{4\pi^2+1} \approx 21.2563$$

(d) $A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\,dr = \frac{1}{2}\int_0^{2\pi} \theta^2\,d\theta = \left[\frac{\theta^3}{6}\right]_0^{2\pi} = \frac{4}{3}\pi^3$

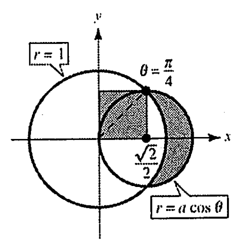
80. $r = e^{\theta/6}$

$$A = \frac{1}{2}\int_0^{2\pi} \left(e^{\theta/6}\right)^2 d\theta - \frac{1}{2}\int_{-2\pi}^{0} \left(e^{\theta/6}\right)^2 d\theta$$
$$= \frac{1}{2}\int_0^{2\pi} e^{\theta/3}\,d\theta - \frac{1}{2}\int_{-2\pi}^{0} e^{\theta/3}\,d\theta$$
$$= \left[\frac{3}{2}e^{\theta/3}\right]_0^{2\pi} - \left[\frac{3}{2}e^{\theta/3}\right]_{-2\pi}^{0}$$
$$= \frac{3}{2}e^{2\pi/3} - \frac{3}{2} - \frac{3}{2} + \frac{3}{2}e^{-2\pi/3} = \frac{3}{2}\left[e^{2\pi/3} + e^{-2\pi/3} - 2\right]$$
$$\approx 9.3655$$



81. The smaller circle has equation $r = a\cos\theta$. The area of the shaded lune is:

$$A = 2\left(\frac{1}{2}\right)\int_0^{\pi/4} \left[(a\cos\theta)^2 - 1\right] d\theta$$
$$= \int_0^{\pi/4} \left[\frac{a^2}{2}(1 + \cos 2\theta) - 1\right] d\theta$$
$$= \left[\frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) - \theta\right]_0^{\pi/4}$$
$$= \frac{a^2}{2}\left(\frac{\pi}{4} + \frac{1}{2}\right) - \frac{\pi}{4}$$



This equals the area of the square, $\left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}$.

$$\frac{a^2}{2}\left(\frac{\pi}{4} + \frac{1}{2}\right) - \frac{\pi}{4} = \frac{1}{2}$$
$$\pi a^2 + 2a^2 - 2\pi - 4 = 0$$
$$a^2 = \frac{4 + 2\pi}{2 + \pi} = 2$$
$$a = \sqrt{2}$$

Smaller circle: $r = \sqrt{2}\cos\theta$

82. $x = \dfrac{3t}{1+t^3}, y = \dfrac{3t^2}{1+t^3}$

(a) $x^3 + y^3 = \dfrac{27(t^3+t^6)}{(1+t^3)^3} = \dfrac{27t^3}{(1+t^3)^2}$

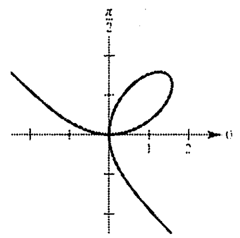
$3xy = \dfrac{27t^3}{(1+t^3)^2}$

So, $x^3 + y^3 = 3xy$.

$(r\cos\theta)^3 + (r\sin\theta)^3 = 3(r\cos\theta)(r\sin\theta)$

$r = \dfrac{3\cos\theta\sin\theta}{\cos^3\theta + \sin^3\theta}$

(b)



(c) $A = \dfrac{1}{2}\int_0^{\pi/2} r^2\, d\theta = \dfrac{3}{2}$

83. False. $f(\theta) = 1$ and $g(\theta) = -1$ have the same graphs.

84. False. $f(\theta) = 0$ and $g(\theta) = \sin 2\theta$ have only one point of intersection.

85. In parametric form,

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Using θ instead of t, you have
$x = r\cos\theta = f(\theta)\cos\theta$ and
$y = r\sin\theta = f(\theta)\sin\theta$. So,

$\dfrac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta$ and

$\dfrac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta.$

It follows that

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = [f(\theta)]^2 + [f'(\theta)]^2.$$

So, $s = \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta.$

Section 10.6 Polar Equations of Conics and Kepler's Laws

1. $r = \dfrac{2e}{1 + e\cos\theta}$

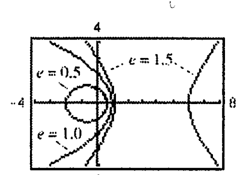
(a) $e = 1, r = \dfrac{2}{1+\cos\theta}$, parabola

(b) $e = 0.5$,

$r = \dfrac{1}{1 + 0.5\cos\theta} = \dfrac{2}{2+\cos\theta}$, ellipse

(c) $e = 1.5$,

$r = \dfrac{3}{1 + 1.5\cos\theta} = \dfrac{6}{2 + 3\cos\theta}$, hyperbola



2. $r = \dfrac{2e}{1 - e\cos\theta}$

(a) $e = 1, r = \dfrac{2}{1-\cos\theta}$, parabola

(b) $e = 0.5$,

$r = \dfrac{1}{1 - 0.5\cos\theta} = \dfrac{2}{2-\cos\theta}$, ellipse

(c) $e = 1.5$,

$r = \dfrac{3}{1 - 1.5\cos\theta} = \dfrac{6}{2 - 3\cos\theta}$, hyperbola