Key

BC Calculus – 10.5 Notes – Alternating Series Test & Absolute Convergence

Alternating Series Test (AST)

If $a_n > 0$, then the alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ and $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converge if BOTH of the following conditions are met:

1. $\lim_{n\to\infty} a_n = 0$
2. $|a_{n+1}| \le |a_n|$ for all n ← each term is smaller than the previous term

Ways to check if a_n is decreasing.

- **Take the 1st derivative and see if it is negative.**
- **Usually, it is obvious.**
- **Could manipulate $a_{n+1} \le a_n$** → $\frac{a_{n+1}}{a_n} \le 1$ ← *see if the ratio of 2 terms is less than 1

Determine if the following series converge or diverge.

1. $\sum_{n=1}^{\infty}(-1)^n \underbrace{\frac{1}{n}}_{a_n}$

$\lim_{n\to\infty}\frac{1}{n} = 0$ $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots$

converges by Alternating Series test (AST)

2. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n+5}{(n+2)(n+3)}$

$\lim_{n\to\infty}\frac{n+5}{n^2+5n+6} = 0$

$a_{n+1} = \frac{n+6}{(n+3)(n+4)}$

$\boxed{\text{Converges by AST}}$

$\frac{n+6}{(\cancel{n+3})(n+4)} \cdot \frac{(n+2)(\cancel{n+3})}{n+5}$

$\frac{n^2+8n+12}{n^2+9n+20} \to \frac{n^2+8n+12}{n^2+8n+12+n+8}$

Ratio is less than 1 (proving this is a decreasing function)

3. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(n+7)}{n}$

$\lim_{n\to\infty}\frac{n+7}{n} \ne 0$

$\boxed{\text{Series diverge by } n^{th} \text{ term test}}$

4. $\sum_{n=1}^{\infty}\cos(n\pi)\frac{1}{n}$

$\cos(1\pi) = -1$
$\cos(2\pi) = 1$
$\cos(3\pi) = -1$

$\lim_{n\to\infty}\frac{1}{n} = 0$, $a_{n+1} < a_n$ ✓

$\boxed{\text{Series converges by AST}}$

5. The following is not an alternating series. Look carefully to see if you can tell why not.

$$\sum_{n=1}^{\infty}\frac{(-1)^n\cos(n\pi)\,n}{n^2+1}$$

$(-1)(-1) = 1$
$(+1)(1) = 1$

*Not an alternating series

$\frac{n}{n^2+1}$ → Compare with $\frac{1}{n}$ (harmonic series diverges)

$\lim_{n\to\infty}\frac{n}{n^2+1}\cdot\frac{n}{1} = \frac{n^2}{n^2+1} = 1$

$\boxed{\text{Diverges by LCT}}$

Absolute or Conditional Convergence Notes:

Three possibilities with regards to the series $\sum_{n=1}^{\infty} a_n$ dealing with convergence or divergence.

1. **Converges Absolutely.** If $\sum_{n=1}^{\infty} |a_n|$ converges, then the original series $\sum_{n=1}^{\infty} a_n$ also converges.
2. **Converges Conditionally.** If $\sum_{n=1}^{\infty} |a_n|$ diverges, but the original series $\sum_{n=1}^{\infty} a_n$ converges.
3. **Divergent.** Both $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} a_n$ diverge.

Find if the series converges absolutely, converges conditionally, or is divergent.

1. $\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$

$\left|\frac{(-3)^n}{n!}\right|$

$\sum_{n=1}^{\infty} \frac{3^n}{n!}$

$\lim_{n\to\infty} \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} = \frac{\cancel{3^n} \cdot 3 \cdot \cancel{n!}}{(n+1)\cancel{(n!)}\cancel{3^n}}$

$\lim_{n\to\infty} \frac{3}{n+1} = 0 < 1$

$\boxed{\text{Series converges Absolutely}}$ or Absolute convergence

2. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[3]{n}}$ $\to \left|\frac{1}{n^{1/3}}\right|$ series diverges by p-series $(p = 1/3)$

since $\lim_{n\to\infty} \frac{1}{\sqrt[3]{n}} = 0$ and $|a_{n+1}| < |a_n|$

$\boxed{\text{series converges conditionally}}$

3. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$

$\lim_{n\to\infty} \frac{n}{n+2} = \frac{1}{2}$

$\boxed{\text{Series diverges by } n^{th} \text{ term test}}$

Find the values of x that make the series converge absolutely.

4. $\sum_{n=1}^{\infty} \frac{(-1)^n n(x+4)^n}{6^n}$

$\lim_{n\to\infty} \left|\frac{(n+1)(x+4)^{n+1}}{6^{n+1}} \cdot \frac{6^n}{n(x+4)^n}\right| < 1$

$\lim_{n\to\infty} \left|\frac{n+1}{n} \cdot \frac{(x+4)}{6}\right| < 1$

$\frac{x+4}{6} < 1 \to |x+4| < 6$

$-6 < x+4 < 6 \to \boxed{-10 < x < 2}$

5. $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$

$\lim_{n\to\infty} \left|\frac{(x-1)^{n+1}}{n+1} \cdot \frac{n}{(x-1)^n}\right| < 1$

$\lim_{n\to\infty} \left|\frac{n}{n+1} \cdot \frac{x-1}{1}\right| < 1$

$|x-1| < 1$

$-1 < x-1 < 1$

$0 < x < 2$

for Absolute convergence

Conditional convergence at $x=0$

Write your questions and thoughts here!

5. The following is not an alternating series. Look carefully to see if you can tell why not.

$$\sum_{n=1}^{\infty} \frac{(-1)^n \cos(n\pi)\, n}{n^2+1}$$

Alternating Series Test

Calculus

Practice

1. Explain why the Alternating Series Test does not apply to the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+1)}{n}$.

In order to use AST, $\lim_{n\to\infty} a_n = 0$

But $\lim_{n\to\infty} \frac{n+1}{n} = 1 \neq 0$ so AST does not apply.

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2. The Alternating Series Test can be used to show convergence of which of the following alternating series?

✓ I. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

✓ $\lim_{n\to\infty} \frac{1}{n} = 0$

✓ $\frac{1}{n}$ is decreasing for $n > 1$

✓ II. $\sum_{n=2}^{\infty} (-1)^{n+1} \left(\frac{n}{n^2+4}\right)$

✓ $\lim_{n\to\infty} \frac{n}{n^2+4} = 0$

✓ $\frac{2}{2^2+4} > \frac{3}{3^2+4} \ldots$

X III. $\sum_{n=1}^{\infty} (-1)^n \left(\frac{4n}{5n+3}\right)$

$\lim_{n\to\infty} \frac{4n}{5n+3} = \frac{4}{5}$

A. I only

B. II only

C. III only

$\boxed{\text{D. I and II only}}$

E. I, II, and III

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3. Which of the following series converge?

A. $\sum_{n=1}^{\infty} (-1)^n \left(\frac{1-2n}{n}\right)$ $\lim_{n\to\infty} a_n = -2$

B. $\sum_{n=1}^{\infty} (-1)^n \left(\frac{n+1}{3n}\right)$ $\lim_{n\to\infty} a_n = \frac{1}{3}$

C. $\sum_{n=1}^{\infty} (-1)^n \left(\frac{n^3}{2\sqrt{n}}\right)$ $\lim_{n\to\infty} a_n = \infty$

$\boxed{\text{D. } \sum_{n=1}^{\infty} (-1)^n \left(\frac{2\sqrt{n}}{n^3}\right)}$ $\lim_{n\to\infty} a_n = 0$ and $a_{n+1} < a_n$

Use the Alternating Series Test to show the series are convergent.

4. $\sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{1}{n^2}\right)$

$\lim_{n\to\infty} \frac{1}{n^2} = 0$ ✓

$a_{n+1} < a_n$ for all n ✓

Series converge by AST.

5. $\sum_{n=1}^{\infty} (-1)^{n}\left(\frac{1}{3^n}\right)$

✓ $\lim_{n\to\infty} \frac{1}{3^n} = 0$

$a_{n+1} < a_n$

Series converge by AST.

6. **Calculator active.** Which of the following statements are true about the series $\sum_{n=2}^{\infty} a_n$, where $a_n = \frac{(-1)^n}{(-1)^n+\sqrt{n}}$

✓ I. The series is alternating.

✗ II. $|a_{n+1}| \leq |a_n|$ for $n \geq 2$.

✓ III. $\lim_{n\to\infty} a_n = 0$

n	a_n	sign
2	0.414	positive
3	-1.366	negative
4	0.333	positive
5	-0.809	negative

A. I only

B. I and II only

$\boxed{\text{C. I and III only}}$

D. I, II, and III

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7. **Calculator active.** Which of the following statements about the series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$, where $a_n = \frac{2+\cos n}{n^2}$ is true?

$\lim_{n\to\infty} a_n = 0$ ✓

$|a_1| = 0.396$

$|a_2| = 0.112$

$|a_3| = 0.0841$

$|a_4| = 0.0913$

not decreasing here

A. The series converges by the Alternating Series Test

B. The Alternating Series Test cannot be used because the series is not alternating.

C. The Alternating Series Test cannot be used because $\lim_{n\to\infty} a_n \neq 0$.

$\boxed{\text{D. The Alternating Series Test cannot be used because the terms of } a_n \text{ are not decreasing.}}$

8. The Alternating Series Test can be used to show convergence for which of the following series?

X A. $\frac{2}{1}-\frac{3}{2}+\frac{4}{3}-\frac{5}{4}+\frac{6}{5}-\cdots$, where $a_n = \frac{(-1)^{n+1}(n+1)}{n}$. $\lim_{n\to\infty} a_n = 1$

X B. $\frac{2}{1}-\frac{1}{1}+\frac{2}{2}-\frac{1}{2}+\frac{2}{3}-\frac{1}{3}+\frac{2}{4}-\frac{1}{4}+\cdots$ not decreasing

C. $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\frac{1}{36}+\cdots$, where $a_n = (-1)^{n+1}\frac{1}{n^2}$ $\lim_{n\to\infty} a_n = 0$ $|a_{n+1}| < |a_n|$

X D. $\frac{3}{2}-\frac{2}{2}+\frac{3}{3}-\frac{2}{3}+\frac{3}{4}-\frac{2}{4}+\cdots$ not decreasing

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9. For which of the following series can the Alternating Series Test not be used?

A. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^5}$

B. $\sum_{n=2}^{\infty} \frac{(-1)^n \ln(n^3)}{n}$

C. $\sum_{n=4}^{\infty} \frac{(-1)^n n}{n-3}$ $\lim_{n\to\infty} \frac{n}{n-3} = 1 \neq 0$

D. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

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10. Which of the following statements about the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$ is true? $\lim_{n\to\infty} \frac{1}{n!} = 0$ ✓

$|a_{n+1}| < |a_n|$ ✓

A. The series diverges by comparison to $\frac{1}{n}$.

B. The series converges by comparison to $\frac{1}{n}$.

C. The series diverges by the Alternating Series Test.

D. The series converges by the Alternating Series Test.

11. Which of the following statements are true about the series $\sum_{n=1}^{\infty} \frac{(-1)^n (n+1)!}{(n)!}$?

✓ I. The series is alternating.

✗ II. $|a_{n+1}| \leq |a_n|$ for $n \geq 1$.

✗ III. $\lim_{n \to \infty} a_n = 0$

$\lim_{n\to\infty} \frac{(n+1)!}{n!} \rightarrow \lim_{n\to\infty} (n+1) = \infty$

A. I only   B. I and II only   C. I and III only   D. I, II, and III

[redacted] Alternating Series Test

Test Prep

12. The Alternating Series Test can be used to show convergence for which of the following series?

✓ I. $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1}{n^2}\right)$

✗ II. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin n}{n^2}$ not always decreasing

✗ III. $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{2}+1} - \frac{1}{\sqrt{2}-1} + \frac{1}{\sqrt{3}+1} - \frac{1}{\sqrt{3}-1} + \frac{1}{\sqrt{4}+1} - \frac{1}{\sqrt{4}-1} + \cdots \right)$ not always decreasing

A. I only   B. I and II only   C. II and III only   D. I, II, and III

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13. If $\sum_{n=1}^{\infty} \frac{(-1)^n}{a_n}$ converges, which of the following must be true?

$0 < \frac{1}{a_{n+1}} < \frac{1}{a_n}$

A. $\lim_{n\to\infty} a_n = 0$ and $a_{n+1} \geq a_n > 0$ for $n \geq 1$.

B. $\lim_{n\to\infty} a_n = \infty$ and $a_{n+1} \leq a_n$ for $n \geq 1$.

C. $\lim_{n\to\infty} a_n = 0$ and $a_{n+1} \leq a_n$ for $n \geq 1$.

D. $\lim_{n\to\infty} a_n = \infty$ and $a_{n+1} \geq a_n > 0$ for $n \geq 1$.

since a_{n+1} and a_n are in the denominator

14. For what value of $k > 0$ will both $\underbrace{\sum_{n=1}^{\infty} \frac{(-1)^{kn}}{n}}$ and $\sum_{n=1}^{\infty} \left(\frac{6}{k}\right)^n$ diverge?

$k < 6$ if $\left(\frac{6}{k}\right)^n$ diverges

Diverges if k is even

(no longer an alternating series if k is even)

A. 3

$\boxed{\text{B. } 4}$

(even and less than 6)

C. 5

D. 7

Write your questions and thoughts here!

Find the values of x that make the series converge absolutely.

4. $\sum_{n=1}^{\infty} \frac{(-1)^n n(x+4)^n}{6^n}$

5. $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$

[redacted] Absolute or Conditional Convergence

Calculus

Practice

1. Which of the following series are conditionally convergent?

I. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$

$\sum \frac{1}{n^4}$ converges absolutely (p-series $p = 4 > 1$)

II. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

$\sum \frac{1}{n}$ diverges (p-series)

$\sum \frac{(-1)^n}{n}$ converges by AST (conditional convergence)

III. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$

$\sum \frac{1}{n^{1/3}}$ diverges by p-series ($p = 1/3 < 1$)

$\lim_{n \to \infty} a_n = 0$

converges conditionally by AST

A. I only

B. I and II only

C. I and III only

D. II and III only

Determine whether the series converges absolutely, converges conditionally, or diverges.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n^2+8)}{\pi^n}$

*Ratio Test

$$\lim_{n\to\infty} \frac{((n+1)^2+8)}{\pi^{n+1}} \cdot \frac{\pi^n}{(n^2+8)} \to \frac{(n^2+2n+9)}{(n^2+8)} \cdot \frac{\pi^n}{\pi^n \cdot \pi}$$

$$= \boxed{\frac{1}{\pi} < 1 \text{ converges absolutely}}$$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$ ← alternating series

$\sum \frac{1}{n}$ diverges (p-series)

$\lim_{n\to\infty} \frac{1}{n} = 0$ ✓

$\frac{1}{n}$ is decreasing ✓

$\boxed{\text{Converges by AST (conditionally)}}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{(n+1)^2}$

$$\lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1 \neq 0$$

$\boxed{\text{Diverges by } n^{th} \text{ term test}}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\frac{5}{2}}}$

$\sum \frac{1}{n^{5/2}}$ converges by p-series $p = 5/2 > 1$

$\boxed{\text{Converges Absolutely}}$

6. For which values x is the series $\displaystyle\sum_{n=1}^{\infty} \frac{nx^n}{4^n(n^2+1)}$ conditionally convergent?

*Ratio Test

$$\lim_{n\to\infty} \frac{(n+1)x^{n+1}}{4^{n+1}((n+1)^2+1)} \cdot \frac{4^n(n^2+1)}{nx^n}$$

$$\lim_{n\to\infty} \frac{(n+1)(n^2+1)}{n[(n+1)^2+1]} \cdot \frac{\cancel{4^n} \cdot \cancel{x^n} \cdot x}{\cancel{4^n} \cdot 4 \cdot \cancel{x^n}} < 1$$

$\left|\frac{x}{4}\right| < 1 \to -1 < \frac{x}{4} < 1$

$-4 \leq x < 4$ ← diverges at $x = 4$

conditionally convergent by AST at $x = -4$

$\sum \frac{(-4)^n}{4^n} \cdot \frac{n}{n^2+1}$

A. $x = 4$ B. $\boxed{x = -4}$ C. $x > 4$ D. $-4 < x < 4$

7. Which of the following statements is true about the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{1/3}}$.

$\sum \frac{1}{n^{1/3}}$ diverges by p-series $p = 1/3 < 1$

$\lim_{n \to \infty} \frac{1}{n^{1/3}} = 0$ ✓

$\frac{1}{n^{1/3}}$ decreases ✓

A. $\boxed{\text{The series converges conditionally.}}$

B. The series converges absolutely.

C. The series converges but neither conditionally nor absolutely.

D. The series diverges.

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8. Which of the following statements about the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^{3/2}}$ is true?

$\sum \frac{1}{n^{3/2}}$ converges by p-series $p = 3/2 < 1$

By comparison test,

$0 < \frac{1}{1+n^{3/2}} < \frac{1}{n^{3/2}}$

converges absolutely

A. The series converges conditionally.

B. $\boxed{\text{The series converges absolutely.}}$

C. The series converges but neither conditionally nor absolutely.

D. The series diverges.

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9. Which of the following statements about the series $\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$ is true?

I. Converges Absolutely

II. Diverges

$\boxed{\text{III. Converges Conditionally}}$

$\sum \frac{1}{n}$ diverges by harmonic series (p-series)

$\frac{1}{n} < \frac{\ln(n)}{n}$

Diverges

$\lim_{n \to \infty} \frac{\ln(n)}{n} = 0$ ✓

$\frac{\ln(n)}{n}$ decreases ✓

A. I only

B. II only

C. $\boxed{\text{III only}}$

D. I and III only

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10. For what values of x is the series $\sum_{n=1}^{\infty} \frac{n(x+5)^n}{7^n}$ absolutely convergent?

*Ratio Test

$$\lim_{n \to \infty} \frac{(n+1)(x+5)^{n+1}}{7^{n+1}} \cdot \frac{7^n}{n(x+5)^n} \rightarrow \lim_{n \to \infty} \frac{n+1}{n} \cdot \frac{\cancel{(x+5)^n} \cdot (x+5) \cdot \cancel{7^n}}{\cancel{(x+5)^n} \cdot \cancel{7^n} \cdot 7} < 1$$

$\left|\frac{x+5}{7}\right| < 1$

$|x+5| < 7$

$-7 < x+5 < 7$

$-12 < x < 2$

A. $x = -12$

B. $x = 2$

C. $x > 2$

D. $\boxed{-12 < x < 2}$