

Section 10.5 Area and Arc Length in Polar Coordinates

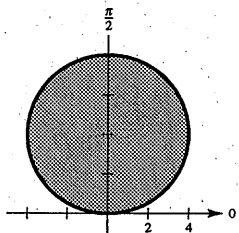
$$\begin{aligned} 1. A &= \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 d\theta \\ &= \frac{1}{2} \int_{\pi/2}^{\pi} (2 \sin \theta)^2 d\theta \\ &= 2 \int_{\pi/2}^{\pi} \sin^2 \theta d\theta \end{aligned}$$

$$\begin{aligned} 2. A &= \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 d\theta \\ &= \frac{1}{2} \int_{3\pi/4}^{5\pi/4} (\cos 2\theta)^2 d\theta \end{aligned}$$

$$\begin{aligned} 3. A &= \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 d\theta \\ &= \frac{1}{2} \int_{\pi/2}^{3\pi/2} (1 - \sin \theta)^2 d\theta \end{aligned}$$

$$\begin{aligned} 4. A &= \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 d\theta \\ &= \frac{1}{2} \int_0^{\pi} (1 - \cos 2\theta)^2 d\theta \end{aligned}$$

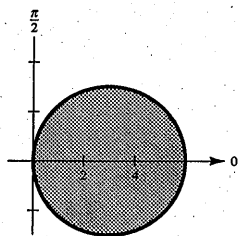
5. (a) $r = 8 \sin \theta$



$$A = \pi(4)^2 = 16\pi$$

$$\begin{aligned} (b) A &= 2 \left(\frac{1}{2} \right) \int_0^{\pi/2} [8 \sin \theta]^2 d\theta \\ &= 64 \int_0^{\pi/2} \sin^2 \theta d\theta \\ &= 32 \int_0^{\pi/2} (1 - \cos 2\theta) d\theta \\ &= 32 \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\pi/2} = 16\pi \end{aligned}$$

6. (a) $r = 3 \cos \theta$



$$A = \pi \left(\frac{3}{2} \right)^2 = \frac{9\pi}{4}$$

$$\begin{aligned} (b) A &= 2 \left(\frac{1}{2} \right) \int_0^{\pi/2} [3 \cos \theta]^2 d\theta \\ &= 9 \int_0^{\pi/2} \cos^2 \theta d\theta \\ &= \frac{9}{2} \int_0^{\pi/2} (1 + \cos 2\theta) d\theta \\ &= \frac{9}{2} \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/2} = \frac{9\pi}{4} \end{aligned}$$

$$7. A = 2 \left[\frac{1}{2} \int_0^{\pi/6} (2 \cos 3\theta)^2 d\theta \right] = 2 \left[\theta + \frac{1}{6} \sin 6\theta \right]_0^{\pi/6} = \frac{\pi}{3}$$

$$\begin{aligned} 8. A &= 2 \left[\frac{1}{2} \int_0^{\pi/4} (6 \sin 2\theta)^2 d\theta \right] = 36 \int_0^{\pi/4} \sin^2 2\theta d\theta \\ &= 36 \int_0^{\pi/4} \frac{1 - \cos 4\theta}{2} d\theta \\ &= 18 \left[\theta - \frac{\sin 4\theta}{4} \right]_0^{\pi/4} \\ &= 18 \left[\frac{\pi}{4} \right] = \frac{9\pi}{2} \end{aligned}$$

$$9. A = 2 \left[\frac{1}{2} \int_0^{\pi/4} (\cos 2\theta)^2 d\theta \right]$$

$$= \frac{1}{2} \left[\theta + \frac{1}{4} \sin 4\theta \right]_0^{\pi/4} = \frac{\pi}{8}$$

$$10. A = 2 \left[\frac{1}{2} \int_0^{\pi/10} (\cos 5\theta)^2 d\theta \right]$$

$$= \frac{1}{2} \left[\theta + \frac{1}{10} \sin(10\theta) \right]_0^{\pi/10} = \frac{\pi}{20}$$

$$11. A = 2 \left[\frac{1}{2} \int_{-\pi/2}^{\pi/2} (1 - \sin \theta)^2 d\theta \right]$$

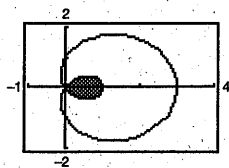
$$= \left[\frac{3}{2} \theta + 2 \cos \theta - \frac{1}{4} \sin 2\theta \right]_{-\pi/2}^{\pi/2} = \frac{3\pi}{2}$$

$$12. A = 2 \left[\frac{1}{2} \int_0^{\pi/2} (1 - \sin \theta)^2 d\theta \right]$$

$$= \left[\frac{3}{2} \theta + 2 \cos \theta - \frac{1}{4} \sin 2\theta \right]_0^{\pi/2} = \frac{3\pi - 8}{4}$$

$$13. A = 2 \frac{1}{2} \left[\int_{2\pi/3}^{\pi} (1 + 2 \cos \theta)^2 d\theta \right]$$

$$= \left[3\theta + 4 \sin \theta + \sin 2\theta \right]_{2\pi/3}^{\pi} = \frac{2\pi - 3\sqrt{3}}{2}$$

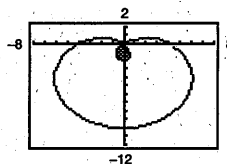


$$14. A = 2 \left[\frac{1}{2} \int_{\arcsin(2/3)}^{\pi/2} (4 - 6 \sin \theta)^2 d\theta \right]$$

$$= \int_{\arcsin(2/3)}^{\pi/2} [16 - 48 \sin \theta + 36 \sin^2 \theta] d\theta$$

$$= \int_{\arcsin(2/3)}^{\pi/2} \left[16 - 48 \sin \theta + 36 \left(\frac{1 - \cos 2\theta}{2} \right) \right] d\theta$$

$$= \left[34\theta + 48 \cos \theta - 9 \sin 2\theta \right]_{\arcsin(2/3)}^{\pi/2} \approx 1.7635$$

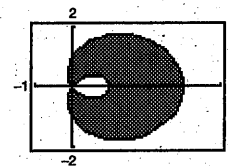


15. The area inside the outer loop is

$$2 \left[\frac{1}{2} \int_0^{2\pi/3} (1 + 2 \cos \theta)^2 d\theta \right] = \left[3\theta + 4 \sin \theta + \sin 2\theta \right]_0^{2\pi/3} = \frac{4\pi + 3\sqrt{3}}{2}$$

From the result of Exercise 13, the area between the loops is

$$A = \left(\frac{4\pi + 3\sqrt{3}}{2} \right) - \left(\frac{2\pi - 3\sqrt{3}}{2} \right) = \pi + 3\sqrt{3}$$



16. Four times the area in Exercise 15, $A = 4(\pi + 3\sqrt{3})$. More specifically, we see that the area inside the outer loop is

$$2 \left[\frac{1}{2} \int_{-\pi/6}^{\pi/2} (2(1 + 2 \sin \theta))^2 d\theta \right] = \int_{-\pi/6}^{\pi/2} (4 + 16 \sin \theta + 16 \sin^2 \theta) d\theta = 8\pi + 6\sqrt{3}$$

The area inside the inner loop is

$$2 \frac{1}{2} \left[\int_{7\pi/6}^{3\pi/2} (2(1 + 2 \sin \theta))^2 d\theta \right] = 4\pi - 6\sqrt{3}$$

Thus, the area between the loops is $(8\pi + 6\sqrt{3}) - (4\pi - 6\sqrt{3}) = 4\pi + 12\sqrt{3}$.

