Key

BC Calculus – 10.6 Notes – Ratio Test and Root Test

factorials example: $5! = 5\cdot4\cdot3\cdot2\cdot1$

Recall:

$\frac{(n+1)!}{n!} = \frac{(n+1)(n!)}{n!} \to \boxed{n+1}$ $\qquad$ $\frac{3^{n+1}}{3^n} = \frac{3^n \cdot 3^1}{3^n} \to \boxed{3}$

Ratio Test for Convergence

If $\sum_{n=1}^{\infty} a_n$ has positive terms and…

- $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, then the series converges
- $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$, then the series diverges
- $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 1$, then Ratio Test is inconclusive (use another Test)

Let's look at two series we already know.

1. $\sum_{n=1}^{\infty}\frac{1}{n}$ *harmonic series diverges

$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} \to \lim_{n\to\infty}\frac{\frac{1}{n+1}}{\frac{1}{n}}$

$\lim_{n\to\infty}\frac{1}{n+1}\cdot\frac{n}{1} = 1$ (inconclusive)

2. $\sum_{n=1}^{\infty}\frac{1}{n^2}$ → p-series converges

$\lim_{n\to\infty}\frac{1}{(n+1)^2}\cdot\frac{n^2}{1} = 1$ (inconclusive by Ratio Test)

Using the Ratio Test to find convergence or divergence.

3. $\sum_{n=1}^{\infty}\frac{n^2\cdot3^{n+1}}{5^n}$

$a_{n+1} = \frac{(n+1)^2\cdot3^{n+2}}{5^{n+1}}$

$\lim_{n\to\infty}\frac{(n+1)^2\cdot3^{n+2}}{5^{n+1}}\cdot\frac{5^n}{n^2\cdot3^{n+1}}$

$\lim_{n\to\infty}\frac{(n+1)^2}{n^2}\cdot\frac{\cancel{3^n}\cdot3^2\cdot\cancel{5^n}}{\cancel{5^n}\cdot5\cdot\cancel{3^n}\cdot3} = \frac{3}{5} < 1 \to$ converges by Ratio Test

4. $\sum_{n=1}^{\infty}\frac{4^n}{n!}$ $\qquad a_{n+1} = \frac{4^{n+1}}{(n+1)!}$

$\lim_{n\to\infty}\frac{4^{n+1}}{(n+1)!}\cdot\frac{n!}{4^n}$

$\lim_{n\to\infty}\frac{\cancel{4^n}\cdot4}{\cancel{4^n}}\cdot\frac{\cancel{n!}}{(n+1)(\cancel{n!})} = \lim_{n\to\infty}\frac{4}{n+1} < 0$

Series converges by Ratio Test

*Ratio Test usually good for fractional functions involving exponents and/or factorials

5. $\sum_{n=1}^{\infty}\frac{(2n)!}{n^5}$

$a_{n+1} = \frac{(2(n+1))!}{(n+1)^5} \to \frac{(2n+2)!}{(n+1)^5}$

$\lim_{n\to\infty}\frac{(2n+2)(2n+1)\cancel{(2n)!}}{(n+1)^5}\cdot\frac{n^5}{\cancel{(2n)!}} \to \lim_{n\to\infty}\frac{n^5}{(n+1)^5}\cdot(2n+2)(2n+1) = \boxed{\infty} > 1$

series diverge by Ratio Test

THEOREM
Root Test

Suppose $\sum_{k=1}^{\infty} a_k$ is a series of nonzero terms and $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$, a number.

- If $L < 1$, then $\sum_{k=1}^{\infty} a_k$ is absolutely convergent, so the series $\sum_{k=1}^{\infty} a_k$ converges.
- If $L > 1$, then $\sum_{k=1}^{\infty} a_k$ diverges.
- If $L = 1$, the test provides no information.

Write your questions and thoughts here!

5. $\sum_{n=1}^{\infty} \frac{(2n)!}{n^5}$

[redacted] Ratio Test

Calculus

Practice

Determine whether the following series converges or diverges.

1. $\sum_{n=1}^{\infty} \frac{(n+1)3^n}{n!}$

$a_{n+1} = \frac{(n+2)3^{n+1}}{(n+1)!}$

$\lim_{n\to\infty} \frac{(n+2)3^{n+1}}{(n+1)!} \cdot \frac{n!}{(n+1)3^n}$

$\lim_{n\to\infty} \frac{n+2}{(n+1)^2} \cdot \frac{3}{1} \to 0 < 1$

$\boxed{\text{Converges by Ratio Test}}$

2. $\sum_{n=1}^{\infty} \frac{n!}{5^n}$

$a_{n+1} = \frac{(n+1)!}{5^{n+1}}$

$\lim_{n\to\infty} \frac{(n+1)!}{5^{n+1}} \cdot \frac{5^n}{n!}$

$\lim_{n\to\infty} \frac{(n+1)\cdot\cancel{n!}\cdot\cancel{5^n}}{\cancel{5^n}\cdot 5\cdot\cancel{n!}} = \frac{n+1}{5} \to \infty > 1$

Diverges by Ratio Test

3. What are values of $x > 0$ for which the series $\sum_{n=1}^{\infty} \frac{n6^n}{x^n}$ converges?

$a_{n+1} = \frac{(n+1)6^{n+1}}{x^{n+1}}$

$\lim_{n\to\infty} \frac{(n+1)6^{n+1}}{x^{n+1}} \cdot \frac{x^n}{n6^n}$

$\lim_{n\to\infty} \frac{(n+1)}{n} \cdot \frac{\cancel{6^n}\cdot 6\cdot\cancel{x^n}}{\cancel{x^n}\cdot x\cdot\cancel{6^n}} < 1$

$\frac{6}{x} < 1$

$6 < x$ or $\boxed{x > 6}$

4. What are all positive values of p for which the series $\sum_{n=1}^{\infty} \frac{n^p}{7^n}$ will converge?

$a_{n+1} = \frac{(n+1)^p}{7^{n+1}}$

$\lim_{n\to\infty} \frac{(n+1)^p}{7^{n+1}} \cdot \frac{7^n}{n^p} \to \frac{1}{7} < 1$

$\boxed{\text{A. } p > 0}$ B. $0 < p < 7$

C. $p > 1$ D. There are no positive values where the series will converge.

5. Which of the following series converge?

I. $\sum_{n=1}^{\infty} \frac{7^n}{n!}$

$\lim_{n\to\infty} \frac{7^{n+1}}{(n+1)!} \cdot \frac{n!}{7^n} \to \frac{7}{n+1} = 0 < 1$

Converges ✓

II. $\sum_{n=1}^{\infty} \frac{n!}{n^{20}}$

$\lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{20}} \cdot \frac{n^{20}}{n!} \to n > 1$

Diverges

III. $\sum_{n=1}^{\infty} \frac{\pi^{-2n}}{n} \to \frac{1}{n \cdot \pi^{2n}}$

$\lim_{n\to\infty} \frac{1}{(n+1)\pi^{2(n+1)}} \cdot n\pi^{2n}$

$\lim_{n\to\infty} \frac{\pi^{2n}}{\pi^{2n} \cdot \pi^2} = \frac{1}{\pi^2} < 1$

Converges ✓

A. I only B. I and II only C. III only $\boxed{\text{D. I and III only}}$ E. I, II, and III

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6. If the Ratio Test is applied to the series $\sum_{n=1}^{\infty} \frac{n\pi^n}{15^n}$, which of the following inequalities results, implying that the series converges?

$a_{n+1} = \frac{(n+1)\pi^{n+1}}{15^{n+1}}$

$\lim_{n\to\infty} \frac{(n+1)\pi^{n+1}}{15^{n+1}} \cdot \frac{15^n}{n\pi^n} \to \frac{\pi}{15} < 1$

A. $\lim_{n\to\infty} \frac{n\pi^n}{15^n} < 1$ B. $\lim_{n\to\infty} \frac{15^n}{n\pi^n} < 1$ C. $\lim_{n\to\infty} \frac{(n+1)\pi^{n+1}}{15^{n+1}} < 1$ $\boxed{\text{D. } \lim_{n\to\infty} \frac{(n+1)\pi}{15n} < 1}$

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7. If $a_n > 0$ for all n and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 6$, which of the following series converges?

$\lim_{n\to\infty} \frac{a_{n+1}}{(n+1)^7} \cdot \frac{n^7}{a_n} \to 6 > 1$

Diverges

$\lim_{n\to\infty} \frac{a_{n+1}}{7^{n+1}} \cdot \frac{7^n}{a_n} = \frac{6}{7} < 1$

Converges

$\lim_{n\to\infty} \frac{(a_{n+1})^2}{7^{n+1}} \cdot \frac{7^n}{(a_n)^2} \quad \frac{6^2}{7} > 1$

Series diverge.

A. $\sum_{n=1}^{\infty} a_n$

Diverges

B. $\sum_{n=1}^{\infty} \frac{a_n}{n^7}$

$\boxed{\text{C. } \sum_{n=1}^{\infty} \frac{a_n}{7^n}}$

D. $\sum_{n=1}^{\infty} \frac{(a_n)^2}{7^n}$

8. Consider the series $\sum_{n=1}^{\infty} \frac{n!}{3^n}$. If the Ratio Test is applied to the series, which of the following inequalities results, implying the series diverges?

$a_{n+1} = \frac{(n+1)!}{3^{n+1}}$

$\lim_{n\to\infty} \frac{(n+1)!}{3^{n+1}} \cdot \frac{3^n}{n!} \to \lim_{n\to\infty} \frac{(n+1)\cancel{n!} \cdot \cancel{3^n}}{\cancel{3^n} \cdot 3 \cdot \cancel{n!}} \to \frac{n+1}{3} > 1$

A. $\lim_{n\to\infty} \frac{n!}{3^n} < 1$

B. $\lim_{n\to\infty} \frac{n!}{3^n} > 1$

C. $\lim_{n\to\infty} \frac{n+1}{3} < 1$

$\boxed{\text{D. } \lim_{n\to\infty} \frac{n+1}{3} > 1}$

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9. For which of the series is the Ratio Test inconclusive?

I. $\sum_{n=1}^{\infty} \frac{1}{3n}$

$\lim_{n\to\infty} \frac{1}{3(n+1)} \cdot \frac{3n}{1} \to 1$

(Inconclusive)

II. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}$

$\lim_{n\to\infty} \frac{\sqrt{n+1}}{n+2} \cdot \frac{n+1}{\sqrt{n}} = 1$

(Inconclusive)

III. $\sum_{n=1}^{\infty} \frac{e^n}{n!}$

$\lim_{n\to\infty} \frac{e^{n+1}}{(n+1)!} \cdot \frac{n!}{e^n}$

$\lim_{n\to\infty} \frac{e^n \cdot e}{(n+1)\cdot e^n} \to \frac{e}{n+1} < 1$

(Converges)

A. I only

B. II only

C. I and III only

$\boxed{\text{D. I and II only}}$

E. I, II, and III

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10. Apply any appropriate test to determine which of the following series diverges.

I. $\sum_{n=1}^{\infty} \frac{n}{2n^2+1}$

Compare with $\frac{1}{n}$ (harmonic series diverge)

$\lim_{n\to\infty} \frac{n}{2n^2+1} \cdot \frac{n}{1} \to \frac{n^2}{2n^2+1} = \frac{1}{2}$

Diverges by Limit Comparison Test

II. $\sum_{n=1}^{\infty} \frac{n!}{9^n}$

$\lim_{n\to\infty} \frac{(n+1)!}{9^{n+1}} \cdot \frac{9^n}{n!}$

$\lim_{n\to\infty} \frac{(n+1)}{9} > 1$

diverges by Ratio Test or Diverges by n^{th} term test

III. $\sum_{n=1}^{\infty} \frac{n+1}{4n+1}$

$\lim_{n\to\infty} a_n = \frac{1}{4} \neq 0$

(Diverges by n^{th} term test)

A. I only

B. II only

C. III only

D. I and II only

$\boxed{\text{E. I, II, and III}}$

Match the test for convergence of an infinite series with the conditions of convergence.

Convergence Test

11. F nth-Term Test
12. E Geometric Series
13. A p-series
14. G Alternating Series Test
15. H Integral Test
16. B Ratio Test
17. C Comparison Test
18. D Limit Comparison Test

Condition of convergence

A. $p > 1$

B. $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$

C. $0 < a_n \le b_n$ and $\sum\limits_{n=1}^{\infty} b_n$ converges

D. $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = L > 0$ and $\sum\limits_{n=1}^{\infty} b_n$ converges

E. $|r| < 1$

F. Inconclusive for convergence

G. $|a_{n+1}| \le |a_n|$ and $\lim\limits_{n\to\infty} a_n = 0$

H. $\int_1^{\infty} f(x)\,dx$ converges.

[redacted] Ratio Test

Test Prep

19. If the Ratio Test is applied to the series $\sum\limits_{n=1}^{\infty} \frac{7^n}{(n+1)!}$, which of the following limits results, implying that the series converges?

$$\lim_{n\to\infty} \frac{7^{n+1}}{(n+2)!} \cdot \frac{(n+1)!}{7^n} \rightarrow \lim_{n\to\infty} \frac{7}{(n+2)} = 0 < 1$$

A. $\lim\limits_{n\to\infty} \frac{7^n}{(n+1)!}$

B. $\boxed{\lim\limits_{n\to\infty} \frac{7}{n+2}}$

C. $\lim\limits_{n\to\infty} \frac{(n+1)!}{7^n}$

D. $\lim\limits_{n\to\infty} \frac{n+2}{7}$

20. Use the Ratio Test to determine the convergence or divergence of the series $\sum\limits_{n=1}^{\infty} \frac{n^n}{n!}$.

$$a_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$$

$$\lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n}$$

$$\lim_{n\to\infty} \frac{(n+1)^n \cdot \cancel{(n+1)} \cdot \cancel{n!}}{\cancel{(n+1)} \cdot \cancel{n!} \cdot n^n}$$

$$\lim_{n\to\infty} \frac{(n+1)^n}{n^n} \rightarrow \left(\frac{n+1}{n}\right)^n \rightarrow \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e > 1$$

$\boxed{\text{Diverges}}$