Key

BC Calculus – 10.8a Notes – Radius and Interval of Convergence of Power Series

Power Series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + \cdots + a_nx^n$$

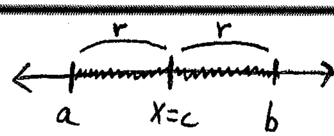
centered at $x=0$

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c)^1 + a_2(x-c)^2 + a_3(x-c)^3 + \cdots + a_n(x-c)^n$$

The domain of a power series is the set of all x-values for which the power series converges.

Note! The center is always part of the domain.

Three ways a power series may converge:



1. converges to an interval
   a. The radius is the distance from the center to the edge of interval
2. Converges to all real numbers
3. Converges to the center ($x=c$) only

The **Interval of Convergence** is the set of values for convergence. We use the Ratio Test to find the interval of convergence.

Ratio Test for Interval of Convergence

If you have a power series $\sum_{n=1}^{\infty} a_n$, find $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$.

- $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, then the series converges on an interval
- $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$, then the series converges for all values of x
- $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series converges only to center ($x=c$) (diverges everywhere else!)

Find the radius and interval of convergence.

1. $\sum_{n=1}^{\infty} \frac{n}{3^n}(x-5)^n$ $\lim_{n\to\infty}\left|\frac{n+1}{3^{n+1}}\cdot(x-5)^{n+1}\cdot\frac{3^n}{n(x-5)^n}\right|$

$\lim_{n\to\infty}\left|\frac{n+1}{n}\cdot\frac{(x-5)}{3}\right|<1 \rightarrow \left|\frac{x-5}{3}\right|<1$

$|x-c|<r$

$-1<\frac{x-5}{3}<1$

$-3<x-5<3$

$2<x<8$

test $x=8$ $|x-5|<3$

$\sum_{n=1}^{\infty}\frac{n}{3^n}(3)^n$ diverges (n^{th} term test)

* check convergence at endpoints!!

$\sum_{n=1}^{\infty}\frac{n}{3^n}(-3)^n=(-1)^n n$ diverges

at $x=2$

Interval of convergence

IOC: $2<x<8$

Radius: $r=3$

2. $\sum_{n=0}^{\infty} 3(x-2)^n$ $\lim_{n\to\infty}\left|3(x-2)^{n+1}\cdot\frac{1}{3(x-2)^n}\right|$

$\lim_{n\to\infty}|x-2|<1$

$|x-c|<r$

$-1<x-2<1$

$1<x<3$

*check endpts:

at $x=1$

$\sum 3(1-2)^n$

$\sum_{n=0}^{\infty}3(-1)^n$

diverges

at $x=3$

$\sum 3(3-2)^n$

$\sum 3(1)^n$

diverges

IOC: $1<x<3$

Radius: $r=1$

3. $\sum_{n=1}^{\infty}\frac{(x+2)^{n+1}}{n^3}$ $\lim_{n\to\infty}\left|\frac{(x+2)^{n+2}}{(n+1)^3}\cdot\frac{n^3}{(x+2)^{n+1}}\right|$

$\lim_{n\to\infty}\left|(x+2)\cdot\frac{n^3}{(n+1)^3}\right|<1$

$|x+2|<1$

$-1<x+2<1$

$-3<x<-1$

test $x=-3$

$\sum\frac{(-1)^{n+1}}{n^3}$

converges

test $x=-1$

$\sum\frac{(1)^{n+1}}{n^3}$

converges

IOC: $-3\leq x\leq -1$

Radius: $r=1$

4. $\sum_{n=0}^{\infty}\frac{(2n)!\,x^{2n}}{n!}$ $\lim_{n\to\infty}\left|\frac{(2n+2)!\,x^{2n+2}}{(n+1)!}\cdot\frac{n!}{(2n)!\,x^{2n}}\right|$

$\lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)\cancel{(2n)!}\cdot\cancel{x^{2n}}\cdot x^2\cdot\cancel{n!}}{(n+1)\cancel{n!}\cdot\cancel{(2n)!}\cdot\cancel{x^{2n}}}\right|=\infty$

Converges only at $x=0$

Radius $=0$

since there are no range of values

5. $\sum_{n=0}^{\infty}\frac{x^{3n}}{n!}$

$\lim_{n\to\infty}\left|\frac{x^{3n+3}}{(n+1)!}\cdot\frac{n!}{x^{3n}}\right|$

$\lim_{n\to\infty}\left|\frac{x^3\cdot n!}{(n+1)\cdot n!}\right|$

$\lim_{n\to\infty}\left|\frac{1}{n!}\cdot x^3\right|=0$

Converges for all values of x

IOC: $(-\infty,\infty)$

radius $=\infty$

[redacted] Radius and Interval of Convergence of Power Series

Calculus

Practice

Find the interval of convergence for each power series.

1. $\sum_{n=0}^{\infty} \frac{(x-1)^n}{4^n}$

$\lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{4^{n+1}} \cdot \frac{4^n}{(x-1)^n} \right|$

$\lim_{n\to\infty} \left| \frac{x-1}{4} \right| < 1 \rightarrow -1 < \frac{x-1}{4} < 1$

$-4 < x-1 < 4$

$-3 < x < 5$

*check endpoints to see if they converge or diverge

test $x=-3$

$\sum \frac{(-4)^n}{4^n} \rightarrow \sum (-1)^n$ diverges

test $x=5$

$\sum \frac{4^n}{4^n} \rightarrow (1)^n$ diverges

$\boxed{-3<x<5}$

2. $\sum_{n=0}^{\infty} \frac{(x+2)^n}{3^n}$

$\lim_{n\to\infty} \left| \frac{(x+2)^{n+1}}{3^{n+1}} \cdot \frac{3^n}{(x+2)^n} \right|$

$\lim_{n\to\infty} \left| \frac{(x+2)}{3} \right| < 1 \rightarrow -1 < \frac{x+2}{3} < 1$

$-3 < x+2 < 3$

$-5 < x < 1$

test $x=-5$

$\sum \frac{(-3)^n}{3^n} \rightarrow (-1)^n$ diverges

test $x=1$

$\sum \frac{(3)^n}{3^n} \rightarrow (1)^n$ diverges

$\boxed{-5<x<1}$

3. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n2^n}$

$\lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n \cdot 2^n}{(x-2)^n} \right|$

$\lim_{n\to\infty} \left| \frac{(x-2)}{2} \cdot \frac{n}{n+1} \right| < 1 \qquad -1 < \frac{x-2}{2} < 1$

$-2 < x-2 < 2$

$0 < x < 4$

test $x=0$

$\sum \frac{(-1)^{n+1}(-2)^n}{n(2^n)}$

$\sum \frac{(-1)^{n+1}(-1)^n}{n}$ diverges

test $x=4$

$\sum \frac{(-1)^{n+1}\cancel{(2)^n}}{n \cdot \cancel{2^n}}$ converges by AST

$\boxed{0<x\leq 4}$

4. $\sum_{n=0}^{\infty} (2n)! \left(\frac{x}{3}\right)^n$

$\lim_{n\to\infty} \left| \frac{[2(n+1)]! \left(\frac{x}{3}\right)^{n+1}}{1} \cdot \frac{\ }{(2n)! \left(\frac{x}{3}\right)^n} \right|$

$\lim_{n\to\infty} \left| \frac{(2n+2)! \left(\frac{x}{3}\right)}{(2n)!} \right| \rightarrow \lim_{n\to\infty} \left| \frac{(2n+2)(2n+1)\cancel{(2n)!}}{\cancel{(2n)!}} \left(\frac{x}{3}\right) \right|$

$= \boxed{\infty}$

$\boxed{\text{Converges only at center } x=0}$

Find the radius of convergence for each series.

5. $\sum_{n=1}^{\infty} \frac{(4x)^n}{n^2}$ $\lim_{n\to\infty}\left|\frac{(4x)^{n+1}}{(n+1)^2}\cdot\frac{n^2}{(4x)^n}\right|$

$\lim_{n\to\infty}\left|\frac{(4x)^n\cdot(4x)}{(4x)^n}\cdot\frac{n^2}{(n+1)^2}\right| = \lim_{n\to\infty}|4x|<1$

$|4x|<1 \rightarrow -1<4x<1$

$-\frac{1}{4}<x<\frac{1}{4} \rightarrow |x|<1/4$

$\boxed{\text{Radius} = 1/4}$

6. $\sum_{n=0}^{\infty} \frac{(x-4)^{n+1}}{2\cdot 3^{n+1}}$ $\lim_{n\to\infty}\left|\frac{(x-4)^{n+2}}{2\cdot 3^{n+2}}\cdot\frac{2\cdot 3^{n+1}}{(x-4)^{n+1}}\right|$

$\lim_{n\to\infty}\left|\frac{(x-4)}{3}\right|<1$

$|x-4|<3$

$\boxed{\text{radius} = 3}$

$* \; |x-c|<r$

7. $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ $\lim_{n\to\infty}\left|\frac{x^{2(n+1)}}{[2(n+1)]!}\cdot\frac{(2n)!}{x^{2n}}\right|$

$\lim_{n\to\infty}\left|\frac{x^2\cdot(2n)!}{(2n+2)!}\right| \rightarrow \lim_{n\to\infty}\left|\frac{x^2\cdot(2n)!}{(2n+2)(2n+1)(2n)!}\right|$

$\lim_{n\to\infty}\left|\frac{x^2}{(2n+2)(2n+1)}\right| = 0$ (always less than 1)

$\boxed{\text{Radius} = \infty}$

8. $\sum_{n=0}^{\infty} \frac{(2n)!\,x^{2n}}{n!}$ $\lim_{n\to\infty}\left|\frac{(2(n+1))!\,x^{2(n+1)}}{(n+1)!}\cdot\frac{n!}{(2n)!\,x^{2n}}\right|$

$\lim_{n\to\infty}\left|\frac{(2n+2)!\,x^{2n+2}\cdot n!}{(n+1)!\,(2n)!\,x^{2n}}\right|$

$\lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)\cancel{(2n)!}\,\cancel{x^{2n}}\cdot x^2\,\cancel{n!}}{(n+1)\cancel{n!}\,\cancel{(2n)!}\,\cancel{x^{2n}}}\right| = \infty$

$\boxed{\text{Radius} = 0}$

*converges only at center $(c=0)$

What are all values of x for which each series converges?

9. $\sum_{n=1}^{\infty} \left(\frac{4}{x^2+1}\right)^n$

$\lim_{n\to\infty}\left|\left(\frac{4}{x^2+1}\right)^{n+1}\cdot\left(\frac{x^2+1}{4}\right)^n\right|$

$\lim_{n\to\infty}\left|\frac{4}{x^2+1}\right|<1$

$4<(x^2+1)$

$x^2+1>4$

$x^2>3$

$x>\sqrt{3}$ or $x<-\sqrt{3}$

*test endpoints:

at $x=\sqrt{3}$:

$\sum\left(\frac{4}{\sqrt{3}^2+1}\right)^n$ diverges

at $x=-\sqrt{3}$

$\sum\left(\frac{4}{\sqrt{3}^2+1}\right)^n$ diverges

$\boxed{\text{converges when } x>\sqrt{3} \text{ or } x<-\sqrt{3}}$

10. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left(x+\frac{3}{2}\right)^n$ $\lim_{n\to\infty}\left|\frac{(x+3/2)^{n+1}}{n+1}\cdot\frac{n}{(x+3/2)^n}\right|$

$\lim_{n\to\infty}\left|\frac{(x+3/2)}{1}\right|<1$

$|x+3/2|<1$

$-1<x+3/2<1$ $\quad * \; |x-c|<r \quad r=1$

$-5/2<x<-1/2$

test $x=-5/2$:

$\sum\frac{(-1)^n}{n}\left(-\frac{5}{2}+\frac{3}{2}\right)^n$

$\sum\frac{(-1)^n}{n}(-1)^n$

diverges (p-series)

test $x=-1/2$:

$\sum\frac{(-1)^n}{n}\cdot(1)^n$

converges by AST

$\boxed{-\frac{5}{2}<x\le -1/2}$

11. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n \cdot 3^n}$ $\lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{(n+1)\cdot 3^{n+1}} \cdot \frac{n\cdot 3^n}{(x-2)^n} \right|$

$\lim_{n\to\infty} \left| \frac{(x-2)}{3} \cdot \frac{n}{n+1} \right| < 1$ $\left| \frac{x-2}{3} \right| < 1$

$|x-2| < 3$

$-3 < x-2 < 3$

$-1 < x < 5$

test $x=-1$:

$\sum \frac{(-1-2)^n}{n\cdot 3^n}$

$\sum \frac{(-3)^n}{n\cdot 3^n} \to \frac{(-1)^n}{n}$ converges by AST

test $x=5$:

$\sum \frac{(5-2)^n}{n\cdot 3^n} \to \sum \frac{(3)^n}{n\cdot 3^n}$ diverges (p-series)

$\boxed{-1 \le x < 5}$

12. $\sum_{n=0}^{\infty} \frac{x^{5n}}{n!}$ $\lim_{n\to\infty} \left| \frac{x^{5(n+1)}}{(n+1)!} \cdot \frac{n!}{x^{5n}} \right|$

$\lim_{n\to\infty} \left| \frac{x^5 \cdot n!}{(n+1)\cdot n!} \right| \to \lim_{n\to\infty} \left| \frac{x^5}{n+1} \right| = 0$ (always less than 1)

converges for all values of x:

$\boxed{(-\infty, \infty)}$

~~10.13~~ Radius and Interval of Convergence of Power Series

Test Prep

13. The radius of convergence for the power series $\sum_{n=1}^{\infty} \frac{(x-4)^{2n}}{n}$ is equal to 1. What is the interval of convergence?

$c=4$, $r=1$

$|x-c| < r$

$|x-4| < 1$

$-1 < x-4 < 1$

$3 < x < 5$

*check endpoints:

test $x=3$:

$\sum \frac{(3-4)^{2n}}{n} \to \frac{(-1)^{2n}}{n}$

diverges by p-series

test $x=5$:

$\sum \frac{(5-4)^{2n}}{n}$

$\sum \frac{(1)^{2n}}{n}$

diverges by p-series

$\boxed{3 < x < 5}$

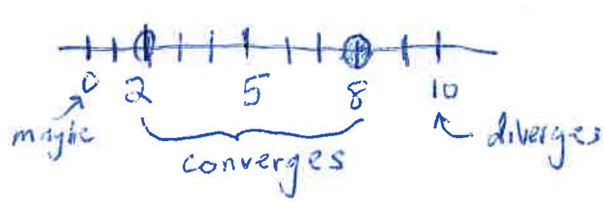
14. If the power series $\sum_{n=0}^{\infty} a_n (x-5)^n$ converges at $x=8$ and diverges at $x=10$, which of the following must be true?

*center at $x=5$

maybe I. The series converges at $x=2$. * Radius is between 3 and 5

yes II. The series converges at $x=3$.

maybe III. The series diverges at $x=0$.



(A) I only $\boxed{\text{(B) II only}}$ (C) I and II only (D) II and III only

15. The coefficients of the power series $\sum_{n=0}^{\infty} a_n(x-3)^n$ satisfy $a_0 = 6$ and $a_n = \left(\frac{2n+1}{3n+1}\right) a_{n-1}$ for all $n \geq 1$. What is the radius of convergence?

$\lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = ?$

$b_n = a_n(x-3)^n$

$b_{n+1} = a_{n+1}(x-3)^{n+1}$

$a_{n+1} = \left(\frac{2(n+1)+1}{3(n+1)+1}\right) \cdot a_n$

$a_{n+1} = \frac{2n+3}{3n+4} a_n$

$\lim_{n\to\infty} \left| \frac{a_{n+1}(x-3)^{n+1}}{a_n(x-3)^n} \right|$

$\frac{a_{n+1}}{a_n} = \frac{2n+3}{3n+4}$

$\lim_{n\to\infty} \left| \frac{2n+3}{3n+4} \cdot \frac{(x-3)^{n+1}}{(x-3)^n} \right|$

$\lim_{n\to\infty} \left| \frac{2}{3} \cdot (x-3) \right| < 1$

$-1 < \frac{2}{3}(x-3) < 1$

$-\frac{3}{2} < x-3 < \frac{3}{2}$

$\frac{3}{2} < x < \frac{9}{2}$

$\left|\frac{2}{3}(x-3)\right| < 1$

$|x-3| < 3/2$

$|x-c| < r$ *

$\boxed{\text{Radius} = \frac{3}{2}}$

16. The radius of convergence for the power series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n5^n}$ is 5, what is the interval of convergence?

Center is $x = 5$

radius $= 5$

$|x-5| < 5$

$-5 < x-5 < 5$

$0 < x < 10$

test endpoints:

$x = 0$:

$\sum \frac{(-1)^{n+1}(-5)^n}{n \cdot 5^n} \to \frac{(-1)(-5)^n}{5^n} \cdot \frac{(-1)}{n}$ diverges

test $x = 10$

$\sum \frac{(-1)^{n+1}(5)^n}{n \cdot 5^n}$ converges by AST

$\boxed{0 < x \leq 10}$

(A) $-5 < x < 5$ (B) $-5 < x \leq 5$ (C) $0 < x < 10$ $\boxed{\text{(D) } 0 < x \leq 10}$

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17. Let $a_n = \frac{1}{n \ln n}$ for $n \geq 3$ and let f be the function given by $f(x) = \frac{1}{x \ln x}$.

a. The function f is continuous, decreasing, and positive. Use the Integral Test to determine the convergence or divergence of the series $\sum_{n=3}^{\infty} a_n$.

$\int_3^{\infty} \frac{1}{x \ln x} dx$

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$dx = x\,du$

$\lim_{b\to\infty} \int_3^b \frac{1}{\cancel{x} \cdot u} \cdot \cancel{x}\,du$

$\ln|u| \to \ln|\ln x|$

$\ln|\ln x| \Big]_3^b \to \lim_{b\to\infty} \ln|\ln b| - \ln|\ln 3|$

$\infty - \ln|\ln 3| = \infty$

$\boxed{\text{Series Diverges by Integral Test}}$

b. Find the interval of convergence of the power series $\sum_{n=3}^{\infty} \frac{(x-2)^{n+1}}{n \ln n}$.

$$\lim_{n\to\infty} \left| \frac{(x-2)^{n+2}}{(n+1)\cdot \ln(n+1)} \cdot \frac{n \ln(n)}{(x-2)^{n+1}} \right|$$

$$\lim_{n\to\infty} \left| \frac{(x-2)}{1} \cdot \frac{n \ln(n)}{(n+1)\ln(n+1)} \right| < 1$$

$$\lim_{n\to\infty} |x-2| < 1$$

$$-1 < x-2 < 1$$

$$1 < x < 3$$

*test endpoints:

$x=1$:

$\sum \frac{(-1)^{n+1}}{n \cdot \ln(n)}$ converges by AST

$x=3$

$\sum \frac{(1)^{n+1}}{(n)\ln(n)}$ diverges by Integral Test

$$\boxed{1 \leq x < 3}$$