Key

BC Calculus – 10.8b Notes – Representing Functions as a Power Series

Recall:

Function	Series (expanded)	Series Notation	Int. of Conv.
$e^x =$	$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$	$\sum_{n=0}^{\infty}\frac{x^n}{n!}$	$-\infty<x<\infty$
$\sin x =$	$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$	$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$	$-\infty<x<\infty$
$\cos x =$	$1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$	$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$	$-\infty<x<\infty$
$\frac{1}{1+x}=$	$1-x+x^2-x^3+\cdots$	$\sum_{n=0}^{\infty}(-1)^n x^n$	$-1<x<1$

1. If $f(x)=\sum_{n=0}^{\infty}\frac{x^{5n}}{n!}$ then $f'(x)=$

$$f(x)=1+\frac{x^5}{1}+\frac{x^{10}}{2!}+\frac{x^{15}}{3!}+\cdots$$

$$f'(x)=0+5x^4+\frac{10x^9}{2}+\frac{15x^{14}}{3!}+\cdots$$

simplified

$$\boxed{f'(x)=5x^4+5x^9+\frac{5x^{14}}{2}+\cdots}$$

$\boxed{\text{OR}}$ take the derivative of the Rule

$$\sum_{n=1}^{\infty}\frac{5n\cdot x^{5n-1}}{n!} \rightarrow \frac{5\cancel{n}\cdot x^{5n-1}}{\cancel{n}(n-1)!}$$

$$\frac{5x^4}{1}+\frac{10x^9}{2!}+\cdots$$

simplified $\rightarrow \boxed{\sum_{n=1}^{\infty}\frac{5x^{5n-1}}{(n-1)!}}$

2. **Write the first 4 nonzero terms for the Maclaurin series that represents $\int_0^x \sin(t^7)\,dt$.**

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots+\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin(t^7)=t^7-\frac{t^{21}}{3!}+\frac{t^{35}}{5!}-\frac{t^{49}}{7!}+\cdots+\frac{(-1)^n(t^7)^{2n+1}}{(2n+1)!}$$

$$\int_0^x \sin(t^7)dt = \frac{t^8}{8}-\frac{t^{22}}{22\cdot 3!}+\frac{t^{36}}{36\cdot 5!}-\frac{t^{50}}{50\cdot 7!}\Bigg]_0^x = \boxed{\frac{x^8}{8}-\frac{x^{22}}{22\cdot 3!}+\frac{x^{36}}{36\cdot 5!}-\frac{x^{50}}{50\cdot 7!}}$$

Practice Problems:

1. What is the coefficient of x^2 in the Taylor Series for the function $f(x) = \sin^2 x$ about $x = 0$?

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$\sin^2 x = (\sin x)(\sin x) = \left[x - \frac{x^3}{3!}\cdots\right]\left[x - \frac{x^3}{3!} + \ldots\right] \rightarrow 1x^2 - \frac{x^4}{3!} - \frac{x^4}{3!} + \ldots$$

coefficient is $\boxed{1}$

2. If the function f is defined as $f(x) = \sum_{n=1}^{\infty} \frac{x^{2n}}{n!}$, then what is $f'(x)$? Write the first four nonzero terms and the general term.

$$f'(x) = \sum_{n=1}^{\infty} \frac{1}{n!}(2n)x^{2n-1} = 2x + \frac{4}{2!}x^3 + \frac{6}{3!}x^5 + \frac{8}{4!}x^7$$

$$= \boxed{2x + 2x^3 + x^5 + \frac{1}{3}x^7}$$

3. Use the power series expansion for $\cos x^6$ to evaluate the integral $\int_0^x \cos t^6\, dt$. Write the first four nonzero terms and the general term.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \rightarrow \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos(t^6) = 1 - \frac{(t^6)^2}{2!} + \frac{(t^6)^4}{4!} - \frac{(t^6)^6}{6!} \rightarrow \frac{(-1)^n (t^6)^{2n}}{(2n)!}$$

$$\int_0^x \cos(t^6) = x - \frac{t^{13}}{26} + \frac{t^{25}}{25\cdot 4!} - \frac{t^{37}}{37\cdot 6!} \rightarrow \boxed{\frac{(-1)^n t^{12n+1}}{(12n+1)\cdot(2n)!}}$$

4. For $x > 0$, the power series defined by $1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots + \frac{(-1)^n x^{2n}}{(2n+1)!}$ converges to which of the following?

(A) $\cos x$ (B) $\sin x$ $\boxed{\text{(C) } \frac{\sin x}{x}}$ (D) $e^x - e^{x^2}$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$\boxed{\frac{1}{x}\cdot \sin x} = \frac{1}{x}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right]$$

$$= \boxed{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots}$$

5. It is known that the Maclaurin series for $\frac{1}{1-x}$ is $\sum_{n=0}^{\infty} x^n$. Use this fact to assist in finding the first four nonzero terms and the general term for the power series expansion for the function $\frac{x^2}{1-x^2}$.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \dots$$

$$\frac{x^2}{1-x^2} = x^2\left[1 + x^2 + x^4 + x^6\right] \rightarrow \boxed{x^2 + x^4 + x^6 + x^8 + \dots \quad \sum_{n=0}^{\infty} x^{2n+2}}$$

6. Let f be the function with initial condition $f(0) = 0$ and derivative $f'(x) = \frac{1}{1+x^7}$. Write the first four nonzero terms of the Maclaurin series for the function f.

$c = 0$ since $f(0) = 0$

$$\frac{1}{1+x} \rightarrow \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + x^4 - \dots$$

$$\frac{1}{1+x^7} \rightarrow 1 - (x^7) + (x^7)^2 - (x^7)^3 + (x^7)^4 \dots$$

$$1 - x^7 + x^{14} - x^{21} + x^{28} + \dots$$

$$\int \frac{1}{1+x^7}dx = \boxed{x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22}} + C$$

7. Find the Maclaurin series for the function $f(x) = e^{3x}$. Write the first four nonzero terms and the general term.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \dots \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{3x} = 1 + (3x) + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} \dots \boxed{\sum_{n=0}^{\infty} \frac{(3x)^n}{n!}}$$

$$\boxed{e^{3x} \approx 1 + 3x + \frac{9x^2}{2!} + \frac{27x^3}{3!}}$$

8. If a function has the derivative $f'(x) = \sin(x^2)$ and initial conditions $f(0) = 0$, write the first four nonzero terms of the Maclaurin series for f.

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin(x^2) = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!}$$

$$\boxed{\int \sin(x^2)dx \approx \frac{x^3}{3!} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!}}$$

9. The function f has derivatives of all orders and the Maclaurin series for the function f is given by $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+3}$. Find the Maclaurin series for the derivative $f'(x)$. Write the first four nonzero terms and the general term.

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+3} = \frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} - \frac{x^7}{9} + \dots$$

$$\boxed{f'(x) = \frac{1}{3} - \frac{3}{5}x^2 + \frac{5}{7}x^4 - \frac{7}{9}x^6 \dots\dots + \frac{(-1)^n(2n+1)x^{2n}}{2n+3}}$$

10. Let the function f be defined by $f(x) = \frac{1}{1-x}$. Find the Maclaurin series for the derivative f'. Write the first four nonzero terms and the general term.

$$f = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$$

$$f'(x) = 1 + 2x + 3x^2 + 4x^3 + \dots\ nx^{n-1}$$

11. Find the second-degree Taylor Polynomial for the function $f(x) = \frac{\cos x}{1-x}$ about $x = 0$.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

$$\frac{1}{1-x} = 1 + x + x^2 + \dots$$

$$(\cos x)\left(\frac{1}{1-x}\right) = \left[1 - \frac{x^2}{2!}\dots\right]\left[1 + x + x^2 \dots\right]$$

$$= 1 + x + x^2 - \frac{x^2}{2!} - \frac{x^3}{2!} - \frac{x^4}{2!} + \dots$$

$$f(x) = 1 + x + x^2 - \frac{x^2}{2}$$

$$\boxed{f(x) = 1 + x + \frac{x^2}{2}}$$

12. What is the coefficient of x^2 in the Maclaurin series for the function $f(x) = \left(\frac{1}{1+x}\right)^2$?

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 \dots$$

$$\left(\frac{1}{1+x}\right)^2 = \left(\frac{1}{1+x}\right)\left(\frac{1}{1+x}\right) \rightarrow \left[1 - x + x^2 \dots\right]\left[1 - x + x^2 + \dots\right]$$

$$= 1 - x + \underline{x^2} - x + \underline{x^2} - x^3 + \underline{x^2} - x^3 + x^4$$

$$= 1 - 2x + \underline{\underline{3x^2}} - 2x^3 + x^4$$

Coefficient for x^2 term is $\boxed{3}$

13. Find the Maclaurin series for the function $f(x) = x\cos x^2$. Write the first four nonzero terms and the general term.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos(x^2) = 1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \frac{(x^2)^6}{6!} + \dots \frac{(-1)^n (x^2)^{2n}}{(2n)!}$$

$$x \cdot \cos(x^2) = x\left[1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} \dots\right] + \dots \frac{(-1)^n x^{4n} \cdot x}{(2n)!} \to \boxed{\frac{(-1)^n x^{4n+1}}{(2n)!}}$$

$$\boxed{x - \frac{x^5}{2!} + \frac{x^9}{4!} - \frac{x^{13}}{6!}}$$

14. Given that f is a function that has derivatives of all orders and $f(1) = 3, f'(1) = -2, f''(1) = 2$, and $f'''(1) = 4$. Write the second-degree Taylor Polynomial for the derivative f' about $x = 1$ and use it to find the approximate value of $f'(1.2)$.

$$P_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

$$P_3(x) = 3 + {}^-2(x-1) + \frac{2}{2!}(x-1)^2 + \frac{4}{3!}(x-1)^3$$

$$P_2'(x) = {}^-2 + 2(x-1) + \frac{4}{3!}(3)(x-1)^2 = -2 + 2(x-1) + 2(x-1)^2$$

$$P_2'(1.2) \approx -2 + 2(0.2) + 2(0.2)^2 = \boxed{-1.52}$$

15. Let the fourth-degree Taylor Polynomial be defined by $T = 7 - 3(x-4) + 5(x-4)^2 - 2(x-4)^3 + 6(x-4)^4$ for the function f about $x = 4$. Find the third-degree Taylor Polynomial for f' about $x = 4$ and then use it to approximate $f'(4.2)$.

$$T_3'(x) = -3 + 10(x-4) - 6(x-4)^2 + 24(x-4)^3$$

$$T_3'(4.2) = -3 + 10(0.2) - 6(0.2)^2 + 24(0.2)^3 \approx \boxed{-1.048}$$

[redacted] Representing Functions as Power Series

Test Prep

16. Given a function defined by $f(x) = \frac{\cos(2x)-1}{x^2}$ for $x \neq 0$ and is continuous for all real numbers x.

   a. What is the limit of the function $f(x)$ as x approaches 0?

$$\lim_{x \to 0} \frac{\cos(2x)-1}{x^2} \to \frac{1-1}{0} \to \frac{0}{0}$$

L'Hopital's $$\lim_{x \to 0} \frac{-\sin(2x) \cdot 2}{2x} \to \frac{0}{0}$$

L'Hopital's $$\lim_{x \to 0} \frac{-2\cos(2x) \cdot 2}{2} \to \frac{-4(1)}{2} = \boxed{-2}$$

b. Write the first four nonzero terms and the general term of the power series that represents the function $h(x) = \cos 2x$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos(2x) = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \dots \frac{(-1)^n (2x)^{2n}}{(2n)!}$$

$$\boxed{= 1 - \frac{4x^2}{2!} + \frac{2^4 x^4}{4!} - \frac{2^6 x^6}{6!} + \dots \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!}}$$

c. Use the results from part (b) to write the first three nonzero terms for $f(x) = \frac{\cos(2x)-1}{x^2}$.

$$\frac{1}{x^2}\left[\cos(2x) - 1\right]$$

$$\frac{1}{x^2}\left[1 - 2x^2 + \frac{16}{4!}x^4 - \frac{4}{45}x^6 - 1\right]$$

$$\boxed{-2 + \frac{2}{3}x^2 - \frac{4}{45}x^4}$$

d. Use the results from part (c) to determine if the function $f(x) = \frac{\cos(2x)-1}{x^2}$ has a relative maximum, a relative minimum or neither at $x = 0$. Justify your answer.

$$f(x) \approx -2 + \frac{2}{3}x^2 - \frac{4}{45}x^4$$

$$f'(x) = \frac{2}{3}\cdot 2x - \frac{16}{45}x^3$$

$$0 = x\left(\frac{4}{3} - \frac{16}{45}x^2\right)$$

$x=0$ is a critical point where slope $= 0$

$$f''(x) = \frac{4}{3} - \frac{48}{45}x^2$$

$$f''(0) = \frac{4}{3} - \frac{48}{45}(0)^2 = \frac{4}{3} > 0$$

so concave up at $x=0$

$\boxed{x=0 \text{ is a relative minimum because } f'(0)=0 \text{ and } f''(0)>0}$