Key

BC Calculus – 10.9 Notes – Finding Taylor & Maclaurin Series for a Function

Taylor Series

If $f(x)$ has derivatives of all orders at $x = c$, then a Taylor Series may be formed that is equal to the function for many common functions.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \dots$$

If $c = 0$ it is a Maclaurin Series.

You need to know the following series: e^x, $\cos x$, $\sin x$, $\frac{1}{1+x}$

The Taylor series of these functions are exact when we go to ∞. They must be memorized!

Maclaurin Series for e^x.	Maclaurin Series for $\sin x$.
$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \dots$	$f(x) = \sin x \quad f(0) = 0$
e^x	$f'(x) = \cos x \quad f'(0) = 1$
$e^0 = 1$	$f''(x) = -\sin x \quad f''(0) = 0$
$e^x = 1 + 1x + \frac{1x^2}{2!} + \frac{1x^3}{3!} + \dots + \frac{x^n}{n!}$	$f'''(x) = -\cos x \quad f'''(0) = -1$
	$f^4(x) = \sin x \quad f^4(0) = 0$
$e^x = \sum_{n=0}^{\infty} \frac{x^2}{n!}$ | IOC: $(-\infty, \infty)$	$\sin x = 0 + 1x + \frac{0x^2}{2!} - \frac{x^3}{3!} + \frac{0x^4}{4!} - \frac{x^5}{5!} + \dots \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

Memorize the following!

Function	Series (expanded)	Series Notation	Int. of Conv.
$e^x =$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$-\infty < x < \infty$
$\sin x =$ (odd)	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$	$(-\infty, \infty)$
$\cos x =$ (even)	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$	$-\infty < x < \infty$
$\frac{1}{1+x} =$	$1 - x + x^2 - x^3 + \dots$	$\sum_{n=0}^{\infty} (-1)^n x^n$	$-1 < x < 1$

The function $f(x) = \frac{1}{1-x}$ is actually a geometric series.

Recall: $\sum_{n=k}^{\infty} ar^n = \frac{a_1}{1-r}, \ |r|<1$

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} 1(x)^n \text{ or } \boxed{\sum_{n=0}^{\infty} x^n}$$

$a_1 = 1$

$r = x$

Find the Maclaurin Series for each of the following functions.

3. $\sin x^2$

*look for parent function:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\sin(x^2) = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \dots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}$$

$$\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}}$$

4. $x^2 e^x$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

$$x^2 e^x = x^2\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}\right]$$

$$= x^2 + x^3 + \frac{x^4}{2!} + \frac{x^5}{3!} + \dots + \frac{x^{n+2}}{n!}$$

$$\boxed{\sum_{n=0}^{\infty} \frac{x^{n+2}}{n!}}$$

Practice Problems:

1. What is the coefficient of x^6 in the Taylor Series about $x = 0$ for the function $f(x) = \frac{e^{2x^2}}{2}$? $\frac{1}{2}e^{2x^2}$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \frac{x^6}{6!}$$

$$e^{2x^2} = 1 + (2x^2) + \frac{(2x^2)^2}{2} + \frac{(2x^2)^3}{3!}$$

$$\frac{1}{2}e^{2x^2} = \frac{1}{2}\left[1 + 2x^2 + \frac{4x^4}{2} + \frac{8x^6}{6}\right]$$

$$= \frac{1}{2} + x^2 + x^4 + \boxed{\frac{4}{6}x^6}$$

coefficient is $\frac{2}{3}$

2. If $f(x) = x \sin 3x$, what is the Taylor Series for f about $x = 0$? Write the first four non-zero terms.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$\sin(3x) = (3x) - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \frac{(3x)^7}{7!}$$

$$x\sin(3x) = x(3x) - \frac{x \cdot 27x^3}{6} + \frac{x \cdot 3^5 x^5}{5!} - \frac{x \cdot 3^7 x^7}{7!}$$

$$\boxed{3x^2 - \frac{27x^4}{3!} + \frac{243x^6}{5!} - \frac{2187x^8}{7!}}$$

3. What is the Maclaurin Series for $\frac{1}{(1-x)^2}$? Write the first four non-zero terms.

$$\frac{1}{1-x} \cdot \frac{1}{1-x}$$

$$[1 + x + x^2 + x^3 + x^4 \ldots] \cdot [1 + x + x^2 + x^3]$$

$$\begin{aligned} &1 + x + x^2 + x^3 \\ &\quad + x + x^2 + x^3 + x^4 \\ &\qquad + x^2 + x^3 + x^4 + x^5 + \ldots \\ &\qquad\quad + x^3 + x^4 + x^5 + \ldots \\ + &\qquad\qquad + x^4 + x^5 + \ldots \end{aligned}$$

$$\boxed{1 + 2x + 3x^2 + 4x^3}$$

4. What is the Maclaurin Series for the function $f(x) = \frac{1}{2}(e^x + e^{-x})$? Write the first four non-zero terms.

$$\frac{1}{2}e^x = \frac{1}{2}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!}\right)$$

$$\frac{1}{2}e^{-x} = \frac{1}{2}\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!}\right)$$

$+$

$$\frac{1}{2}\left[2 + 2\left(\frac{x^2}{2}\right) + 2\left(\frac{x^4}{4!}\right) + 2\left(\frac{x^6}{6!}\right)\right]$$

$$\boxed{1 + \frac{x^2}{2} + \frac{x^4}{4!} + \frac{x^6}{6!}}$$

5. Find the Maclaurin Series for the function $f(x) = \cos\sqrt{x}$. Write the first four non-zero terms.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

$$\cos(\sqrt{x}) = 1 - \frac{\sqrt{x}^2}{2!} + \frac{(\sqrt{x})^4}{4!} - \frac{(\sqrt{x})^6}{6!}$$

$$\boxed{= 1 - \frac{x}{2} + \frac{x^2}{4!} - \frac{x^3}{6!}}$$

6. Find the Maclaurin Series for the function $f(x) = \sin 5x$. Write the first four non-zero terms.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$\sin(5x) = 5x - \frac{(5x)^3}{3!} + \frac{(5x)^5}{5!} - \frac{(5x)^7}{7!}$$

$$\boxed{\sin(5x) = 5x - \frac{125}{3!}x^3 + \frac{5^5 x^5}{5!} - \frac{5^7 x^7}{7!}}$$

7. What is the Taylor series expansion about $x = 0$ for the function $f(x) = \frac{\sin x}{x}$? Write the first four non-zero terms.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$\frac{1}{x} \cdot \sin x = \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\right) = \boxed{1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!}}$$

8. The sum of the series $1 + \frac{3}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!} + \cdots + \frac{3^n}{n!}$ is

$$e^x = \sum \frac{x^n}{n!} \qquad e^3 = \sum_{n=0}^{\infty} \frac{3^n}{n!}$$

(A) $\ln 3$ (B) $\boxed{e^3}$ (C) $\cos 3$ (D) $\sin 3$

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9. What is the sum of the series $1 + \ln 3 + \frac{(\ln 3)^2}{2!} + \cdots + \frac{(\ln 3)^n}{n!}$?

$$e^x = \sum \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots \frac{x^n}{n!}$$

$$e^{(\ln 3)} = 1 + \ln 3 + \frac{(\ln 3)^2}{2!} + \cdots + \frac{(\ln 3)^n}{n!}$$

$$e^{\ln 3} = \boxed{3}$$

10. What is the Taylor Series about $x = 0$ for the function $f(x) = 1 + x^2 + \cos x$? Write the first four non-zero terms.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

$$f(x) = 1 + x^2 + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}\right)$$

$$f(x) = 1 + x^2 + 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

$$\boxed{f(x) = 2 + \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}}$$

11. What is the sum of the infinite series $1-\left(\frac{\pi}{2}\right)^2\left(\frac{1}{3!}\right)+\left(\frac{\pi}{2}\right)^4\left(\frac{1}{5!}\right)-\left(\frac{\pi}{2}\right)^6\left(\frac{1}{7!}\right)+\cdots+\frac{\left(\frac{\pi}{2}\right)^{2n}(-1)^n}{(2n+1)!}$?

$$\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\frac{\sin x}{x}=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}\cdot x^1}{x\cdot(2n+1)!}\rightarrow\frac{(-1)^n x^{2n}}{(2n+1)!}$$

$$\frac{\sin(\pi/2)}{\pi/2}=\frac{1}{\pi/2}=\boxed{\frac{2}{\pi}}$$

12. Find the Maclaurin Series for the function $f(x)=e^{-3x}$. Write the first four non-zero terms.

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}$$

$$e^{-3x}=1+(-3x)+\frac{(-3x)^2}{2!}+\frac{(-3x)^3}{3!}$$

$$e^{-3x}=1-3x+\frac{9x^2}{2}-\frac{27x^3}{6}$$

OR

$$\boxed{1-3x+\frac{9x^2}{2}-\frac{9}{2}x^3}$$

13. Find the Maclaurin Series for the function $f(x)=\frac{\sin x^2}{x}+\cos x$. Write the first four non-zero terms.

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}$$

$$\sin(x^2)=x^2-\frac{(x^2)^3}{3!}+\frac{(x^2)^5}{5!}-\frac{(x^2)^7}{7!}$$

$$\frac{1}{x}\left(\sin(x^2)\right)=\frac{1}{x}\left[x^2-\frac{x^6}{3!}+\frac{x^{10}}{5!}-\frac{x^{14}}{7!}\right]$$

$$=x-\frac{x^5}{3!}+\frac{x^9}{5!}-\frac{x^{13}}{7!}$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}$$

$$\boxed{\frac{\sin x^2}{x}+\cos x\approx 1+x-\frac{x^2}{2!}+\frac{x^4}{4!}}$$

14. Which of the following is the Maclaurin Series for the function $f(x)=x\cos 2x$?

$$\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

$$x\cos(2x)=\sum_{n=0}^{\infty}\frac{(-1)^n\cdot(2x)^{2n}\cdot x}{(2n)!}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n\cdot 2^{2n}\cdot x^{2n}\cdot x^1}{(2n)!}$$

(A) $\sum_{n=0}^{\infty}\frac{(-1)^n 2x^{2n}}{(2n)!}$ (B) $\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n}x^{2n}}{(2n)!}$ $\boxed{\text{(C) } \sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n}x^{2n+1}}{(2n)!}}$ (D) $\sum_{n=0}^{\infty}\frac{(-1)^n 2x^{2n+1}}{(2n)!}$

[redacted] Finding Taylor or Maclaurin Series

Test Prep

15. The Maclaurin series $x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots + \frac{x^{2n+1}}{(2n+1)!}$ represents which function $f(x)$

A) $\sin x = x - \frac{x^3}{3!} + \ldots$ (incorrect)

B) $-\sin x = -\left(x - \frac{x^3}{3!} + \ldots\right)$ (incorrect)

C) $\frac{1}{2}e^{x} - \frac{1}{2}e^{-x} \rightarrow \frac{1}{2}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}\right)$

$- \frac{1}{2}\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!}\right)$

$x + \frac{1}{2}\left(\frac{2x^3}{3!}\right) + \frac{1}{2}\left(\frac{2x^5}{5!}\right) \rightarrow x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$

(A) $\sin x$ (B) $-\sin x$ $\boxed{\text{(C) } \frac{1}{2}(e^x - e^{-x})}$ (D) $e^x - e^{-x}$

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16. The function f satisfies the equation $f'(x) = f(x) + x + 1$ and $f(0) = 2$. The Taylor Series for f about $x = 0$ converges to $f(x)$ for all x.

a. Write an equation for the line tangent to the curve of $y = f(x)$ at $x = 0$.

$f'(x) = f(x) + x + 1$

$f'(0) = f(0) + 0 + 1$

$f'(0) = 2 + 1 = 3$

point: $(0, 2)$

slope: $m = 3$

$\boxed{y - 2 = 3(x - 0) \quad \text{OR} \quad y = 3x + 2}$

b. Find $f''(0)$ and find the second-degree Taylor Polynomial for f about $x = 0$.

$f''(x) = f'(x) + 1$

$f''(0) = f'(0) + 1$

$= 3 + 1 = 4$

$P_2(x) = f(0) + f'(0)(x - 0) + \frac{f''(0)}{2!}(x - 0)^2$

$P_2(x) = 2 + 3(x - 0) + \frac{4}{2!}(x - 0)^2$

$\boxed{P_2(x) = 2 + 3x + 2x^2}$

c. Find the fourth-degree Taylor Polynomial for f about $x = 0$.

$$f'''(x) = f''(x) \rightarrow f'''(0) = f''(0) = 4$$

$$f^{4}(x) = f'''(x) \rightarrow f^{4}(0) = f'''(0) = 4$$

$$P_4(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \frac{f^4(0)}{4!}(x-0)^4$$

$$P_4(x) = 2 + 3(x-0) + \frac{4}{2!}(x-0)^2 + \frac{4}{3!}(x-0)^3 + \frac{4}{4!}(x-0)^4$$

$$\boxed{P_4(x) = 2 + 3x + 2x^2 + \frac{2}{3}x^3 + \frac{1}{6}x^4}$$

d. Find $f^{(n)}(0)$, the n^{th} derivative of f about $x = 0$, for $n \geq 2$. Use the Taylor Series for f about $x = 0$ and the Taylor Series for e^x about $x = 0$ to find $f(x) - 4e^x$.

$$f^{n}(0) = 4 \quad \text{for } n \geq 2$$

$$f(x) = 2 + 3x + \frac{4x^2}{2!} + \frac{4}{3!}x^3 + \frac{4}{4!}x^4 + \ldots$$

$$4e^x = 4\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right)$$

$$= 4 + 4x + \frac{4x^2}{2!} + \frac{4x^3}{3!} + \frac{4x^4}{4!}$$

$$f(x) - 4e^x = 2 - 4 + 3x - 4x + \cancel{\frac{4x^2}{2!}} - \cancel{\frac{4x^2}{2!}} + \cancel{\frac{4x^3}{3!}} - \cancel{\frac{4x^3}{3!}} + \ldots$$

$$\boxed{f(x) - 4e^x = -2 - x}$$