Exercises for Section 12.1

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, find the domain of the vector-valued function.

1. $\mathbf{r}(t) = 5t\mathbf{i} - 4t\mathbf{j} - \frac{1}{t}\mathbf{k}$
2. $\mathbf{r}(t) = \sqrt{4 - t^2}\,\mathbf{i} + t^2\mathbf{j} - 6t\mathbf{k}$
3. $\mathbf{r}(t) = \ln t\mathbf{i} - e^t\mathbf{j} - t\mathbf{k}$
4. $\mathbf{r}(t) = \sin t\mathbf{i} + 4\cos t\mathbf{j} + t\mathbf{k}$
5. $\mathbf{r}(t) = \mathbf{F}(t) + \mathbf{G}(t)$ where
   $\mathbf{F}(t) = \cos t\mathbf{i} - \sin t\mathbf{j} + \sqrt{t}\,\mathbf{k}, \quad \mathbf{G}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$
6. $\mathbf{r}(t) = \mathbf{F}(t) - \mathbf{G}(t)$ where
   $\mathbf{F}(t) = \ln t\mathbf{i} + 5t\mathbf{j} - 3t^2\mathbf{k}, \quad \mathbf{G}(t) = \mathbf{i} + 4t\mathbf{j} - 3t^2\mathbf{k}$
7. $\mathbf{r}(t) = \mathbf{F}(t) \times \mathbf{G}(t)$ where
   $\mathbf{F}(t) = \sin t\mathbf{i} + \cos t\mathbf{j}, \quad \mathbf{G}(t) = \sin t\mathbf{j} + \cos t\mathbf{k}$
8. $\mathbf{r}(t) = \mathbf{F}(t) \times \mathbf{G}(t)$ where
   $\mathbf{F}(t) = t^3\mathbf{i} - t\mathbf{j} + t\mathbf{k}, \quad \mathbf{G}(t) = \sqrt[3]{t}\,\mathbf{i} + \frac{1}{t+1}\mathbf{j} + (t+2)\mathbf{k}$

In Exercises 9–12, evaluate (if possible) the vector-valued function at each given value of t.

9. $\mathbf{r}(t) = \frac{1}{2}t^2\mathbf{i} - (t - 1)\mathbf{j}$
   (a) $\mathbf{r}(1)$ (b) $\mathbf{r}(0)$ (c) $\mathbf{r}(s + 1)$
   (d) $\mathbf{r}(2 + \Delta t) - \mathbf{r}(2)$
10. $\mathbf{r}(t) = \cos t\mathbf{i} + 2\sin t\mathbf{j}$
   (a) $\mathbf{r}(0)$ (b) $\mathbf{r}(\pi/4)$ (c) $\mathbf{r}(\theta - \pi)$
   (d) $\mathbf{r}(\pi/6 + \Delta t) - \mathbf{r}(\pi/6)$
11. $\mathbf{r}(t) = \ln t\mathbf{i} + \frac{1}{t}\mathbf{j} + 3t\mathbf{k}$
   (a) $\mathbf{r}(2)$ (b) $\mathbf{r}(-3)$ (c) $\mathbf{r}(t - 4)$
   (d) $\mathbf{r}(1 + \Delta t) - \mathbf{r}(1)$
12. $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + t^{3/2}\mathbf{j} + e^{-t/4}\mathbf{k}$
   (a) $\mathbf{r}(0)$ (b) $\mathbf{r}(4)$ (c) $\mathbf{r}(c + 2)$
   (d) $\mathbf{r}(9 + \Delta t) - \mathbf{r}(9)$

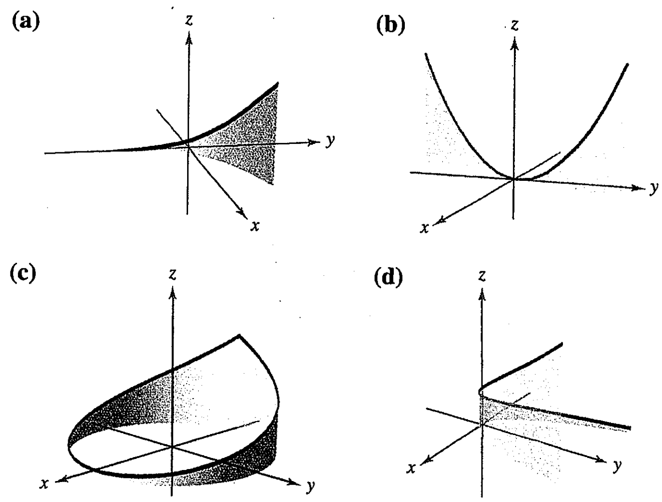
In Exercises 13 and 14, find $\|\mathbf{r}(t)\|$.

13. $\mathbf{r}(t) = \sin 3t\mathbf{i} + \cos 3t\mathbf{j} + t\mathbf{k}$
14. $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + 3t\mathbf{j} - 4t\mathbf{k}$

Think About It **In Exercises 15 and 16, find $\mathbf{r}(t) \cdot \mathbf{u}(t)$. Is the result a vector-valued function? Explain.**

15. $\mathbf{r}(t) = (3t - 1)\mathbf{i} + \frac{1}{4}t^3\mathbf{j} + 4\mathbf{k}$
    $\mathbf{u}(t) = t^2\mathbf{i} - 8\mathbf{j} + t^3\mathbf{k}$
16. $\mathbf{r}(t) = \langle 3\cos t, 2\sin t, t - 2\rangle$
    $\mathbf{u}(t) = \langle 4\sin t, -6\cos t, t^2\rangle$

In Exercises 17–20, match the equation with its graph. [The graphs are labeled (a), (b), (c), and (d).]

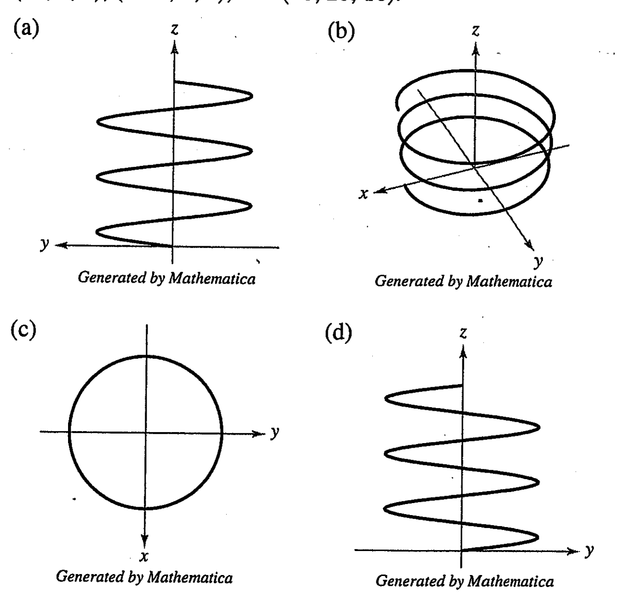


17. $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k}, \quad -2 \le t \le 2$
18. $\mathbf{r}(t) = \cos(\pi t)\mathbf{i} + \sin(\pi t)\mathbf{j} + t^2\mathbf{k}, \quad -1 \le t \le 1$
19. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + e^{0.75t}\mathbf{k}, \quad -2 \le t \le 2$
20. $\mathbf{r}(t) = t\mathbf{i} + \ln t\mathbf{j} + \frac{2t}{3}\mathbf{k}, \quad 0.1 \le t \le 5$

21. ***Think About It*** The four figures below are graphs of the vector-valued function

$$\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j} + \frac{t}{4}\mathbf{k}.$$

Match each of the four graphs with the point in space from which the helix is viewed. The four points are $(0, 0, 20)$, $(20, 0, 0)$, $(-20, 0, 0)$, and $(10, 20, 10)$.



22. Sketch three graphs of the vector-valued function

$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + 2\mathbf{k}$

as viewed from each point.

(a) $(0, 0, 20)$ (b) $(10, 0, 0)$ (c) $(5, 5, 5)$

In Exercises 23–38, sketch the curve represented by the vector-valued function and give the orientation of the curve.

23. $\mathbf{r}(t) = 3t\mathbf{i} + (t - 1)\mathbf{j}$

24. $\mathbf{r}(t) = (1 - t)\mathbf{i} + \sqrt{t}\mathbf{j}$

25. $\mathbf{r}(t) = t^3\mathbf{i} + t^2\mathbf{j}$

26. $\mathbf{r}(t) = (t^2 + t)\mathbf{i} + (t^2 - t)\mathbf{j}$

27. $\mathbf{r}(\theta) = \cos\theta\mathbf{i} + 3\sin\theta\mathbf{j}$

28. $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j}$

29. $\mathbf{r}(\theta) = 3\sec\theta\mathbf{i} + 2\tan\theta\mathbf{j}$

30. $\mathbf{r}(t) = 2\cos^3 t\mathbf{i} + 2\sin^3 t\mathbf{j}$

31. $\mathbf{r}(t) = (-t + 1)\mathbf{i} + (4t + 2)\mathbf{j} + (2t + 3)\mathbf{k}$

32. $\mathbf{r}(t) = t\mathbf{i} + (2t - 5)\mathbf{j} + 3t\mathbf{k}$

33. $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k}$

34. $\mathbf{r}(t) = 3\cos t\mathbf{i} + 4\sin t\mathbf{j} + \frac{t}{2}\mathbf{k}$

35. $\mathbf{r}(t) = 2\sin t\mathbf{i} + 2\cos t\mathbf{j} + e^{-t}\mathbf{k}$

36. $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j} + \frac{3}{2}t\mathbf{k}$

37. $\mathbf{r}(t) = \langle t, t^2, \frac{2}{3}t^3\rangle$

38. $\mathbf{r}(t) = \langle\cos t + t\sin t, \sin t - t\cos t, t\rangle$

In Exercises 39–42, use a computer algebra system to graph the vector-valued function and identify the common curve.

39. $\mathbf{r}(t) = -\frac{1}{2}t^2\mathbf{i} + t\mathbf{j} - \frac{\sqrt{3}}{2}t^2\mathbf{k}$

40. $\mathbf{r}(t) = t\mathbf{i} - \frac{\sqrt{3}}{2}t^2\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$

41. $\mathbf{r}(t) = \sin t\mathbf{i} + \left(\frac{\sqrt{3}}{2}\cos t - \frac{1}{2}t\right)\mathbf{j} + \left(\frac{1}{2}\cos t + \frac{\sqrt{3}}{2}\right)\mathbf{k}$

42. $\mathbf{r}(t) = -\sqrt{2}\sin t\mathbf{i} + 2\cos t\mathbf{j} + \sqrt{2}\sin t\mathbf{k}$

Think About It **In Exercises 43 and 44, use a computer algebra system to graph the vector-valued function $\mathbf{r}(t)$. For each $\mathbf{u}(t)$, make a conjecture about the transformation (if any) of the graph of $\mathbf{r}(t)$. Use a computer algebra system to verify your conjecture.**

43. $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + \frac{1}{2}t\mathbf{k}$

(a) $\mathbf{u}(t) = 2(\cos t - 1)\mathbf{i} + 2\sin t\mathbf{j} + \frac{1}{2}t\mathbf{k}$

(b) $\mathbf{u}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + 2t\mathbf{k}$

(c) $\mathbf{u}(t) = 2\cos(-t)\mathbf{i} + 2\sin(-t)\mathbf{j} + \frac{1}{2}(-t)\mathbf{k}$

(d) $\mathbf{u}(t) = \frac{1}{2}t\mathbf{i} + 2\sin t\mathbf{j} + 2\cos t\mathbf{k}$

(e) $\mathbf{u}(t) = 6\cos t\mathbf{i} + 6\sin t\mathbf{j} + \frac{1}{2}t\mathbf{k}$

44. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{2}t^3\mathbf{k}$

(a) $\mathbf{u}(t) = t\mathbf{i} + (t^2 - 2)\mathbf{j} + \frac{1}{2}t^3\mathbf{k}$

(b) $\mathbf{u}(t) = t^2\mathbf{i} + t\mathbf{j} + \frac{1}{2}t^3\mathbf{k}$

(c) $\mathbf{u}(t) = t\mathbf{i} + t^2\mathbf{j} + (\frac{1}{2}t^3 + 4)\mathbf{k}$

(d) $\mathbf{u}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{8}t^3\mathbf{k}$

(e) $\mathbf{u}(t) = (-t)\mathbf{i} + (-t)^2\mathbf{j} + \frac{1}{2}(-t)^3\mathbf{k}$

In Exercises 45–52, represent the plane curve by a vector-valued function. (There are many correct answers.)

45. $y = 4 - x$

46. $2x - 3y + 5 = 0$

47. $y = (x - 2)^2$

48. $y = 4 - x^2$

49. $x^2 + y^2 = 25$

50. $(x - 2)^2 + y^2 = 4$

51. $\frac{x^2}{16} - \frac{y^2}{4} = 1$

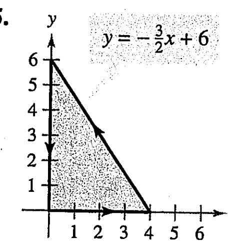
52. $\frac{x^2}{16} + \frac{y^2}{9} = 1$

53. A particle moves on a straight-line path that passes through the points $(2, 3, 0)$ and $(0, 8, 8)$. Find a vector-valued function for the path. Use a computer algebra system to graph your function. (There are many correct answers.)

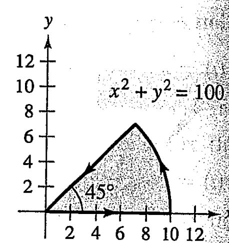
54. The outer edge of a playground slide is in the shape of a helix of radius 1.5 meters. The slide has a height of 2 meters and makes one complete revolution from top to bottom. Find a vector-valued function for the helix. Use a computer algebra system to graph your function. (There are many correct answers.)

In Exercises 55–58, find vector-valued functions forming the boundaries of the region in the figure. State the interval for the parameter of each function.

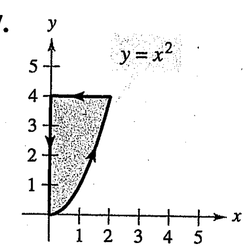
55.



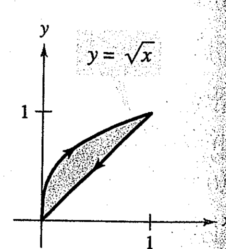
56.



57.



58.



In Exercises 59–66, sketch the space curve represented by the intersection of the surfaces. Then represent the curve by a vector-valued function using the given parameter.

	Surfaces	*Parameter*
59.	$z = x^2 + y^2,\ x + y = 0$	$x = t$
60.	$z = x^2 + y^2,\ z = 4$	$x = 2\cos t$
61.	$x^2 + y^2 = 4,\ z = x^2$	$x = 2\sin t$
62.	$4x^2 + 4y^2 + z^2 = 16,\ x = z^2$	$z = t$
63.	$x^2 + y^2 + z^2 = 4,\ x + z = 2$	$x = 1 + \sin t$
64.	$x^2 + y^2 + z^2 = 10,\ x + y = 4$	$x = 2 + \sin t$
65.	$x^2 + z^2 = 4,\ y^2 + z^2 = 4$	$x = t$ (first octant)
66.	$x^2 + y^2 + z^2 = 16,\ xy = 4$	$x = t$ (first octant)

67. Show that the vector-valued function

$\mathbf{r}(t) = t\mathbf{i} + 2t\cos t\mathbf{j} + 2t\sin t\mathbf{k}$

lies on the cone $4x^2 = y^2 + z^2$. Sketch the curve.

68. Show that the vector-valued function

$\mathbf{r}(t) = e^{-t}\cos t\mathbf{i} + e^{-t}\sin t\mathbf{j} + e^{-t}\mathbf{k}$

lies on the cone $z^2 = x^2 + y^2$. Sketch the curve.

In Exercises 69–74, evaluate the limit.

69. $\lim_{t\to 2}\left(t\mathbf{i} + \frac{t^2-4}{t^2-2t}\mathbf{j} + \frac{1}{t}\mathbf{k}\right)$

70. $\lim_{t\to 0}\left(e^t\mathbf{i} + \frac{\sin t}{t}\mathbf{j} + e^{-t}\mathbf{k}\right)$

71. $\lim_{t\to 0}\left(t^2\mathbf{i} + 3t\mathbf{j} + \frac{1-\cos t}{t}\mathbf{k}\right)$

72. $\lim_{t\to 1}\left(\sqrt{t}\,\mathbf{i} + \frac{\ln t}{t^2-1}\mathbf{j} + 2t^2\mathbf{k}\right)$

73. $\lim_{t\to 0}\left(\frac{1}{t}\mathbf{i} + \cos t\mathbf{j} + \sin t\mathbf{k}\right)$

74. $\lim_{t\to\infty}\left(e^{-t}\mathbf{i} + \frac{1}{t}\mathbf{j} + \frac{t}{t^2+1}\mathbf{k}\right)$

In Exercises 75–80, determine the interval(s) on which the vector-valued function is continuous.

75. $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{t}\mathbf{j}$

76. $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + \sqrt{t-1}\,\mathbf{j}$

77. $\mathbf{r}(t) = t\mathbf{i} + \arcsin t\mathbf{j} + (t-1)\mathbf{k}$

78. $\mathbf{r}(t) = 2e^{-t}\mathbf{i} + e^{-t}\mathbf{j} + \ln(t-1)\mathbf{k}$

79. $\mathbf{r}(t) = \langle e^{-t}, t^2, \tan t\rangle$

80. $\mathbf{r}(t) = \left\langle 8, \sqrt{t}, \sqrt[3]{t}\right\rangle$

Writing About Concepts

81. State the definition of a vector-valued function in the plane and in space.

82. If $\mathbf{r}(t)$ is a vector-valued function, is the graph of the vector-valued function $\mathbf{u}(t) = \mathbf{r}(t-2)$ a horizontal translation of the graph of $\mathbf{r}(t)$? Explain your reasoning.

83. Consider the vector-valued function

$\mathbf{r}(t) = t^2\mathbf{i} + (t-3)\mathbf{j} + t\mathbf{k}.$

Write a vector-valued function $\mathbf{s}(t)$ that is the specified transformation of $\mathbf{r}$.

(a) A vertical translation three units upward

(b) A horizontal translation two units in the direction of the negative x-axis

(c) A horizontal translation five units in the direction of the positive y-axis

84. State the definition of continuity of a vector-valued function. Give an example of a vector-valued function that is defined but not continuous at $t = 2$.

85. Let $\mathbf{r}(t)$ and $\mathbf{u}(t)$ be vector-valued functions whose limits exist as $t \to c$. Prove that

$$\lim_{t\to c}[\mathbf{r}(t) \times \mathbf{u}(t)] = \lim_{t\to c}\mathbf{r}(t) \times \lim_{t\to c}\mathbf{u}(t).$$

86. Let $\mathbf{r}(t)$ and $\mathbf{u}(t)$ be vector-valued functions whose limits exist as $t \to c$. Prove that

$$\lim_{t\to c}[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \lim_{t\to c}\mathbf{r}(t) \cdot \lim_{t\to c}\mathbf{u}(t).$$

87. Prove that if $\mathbf{r}$ is a vector-valued function that is continuous at c, then $\|\mathbf{r}\|$ is continuous at c.

88. Verify that the converse of Exercise 87 is not true by finding a vector-valued function $\mathbf{r}$ such that $\|\mathbf{r}\|$ is continuous at c but $\mathbf{r}$ is not continuous at c.

True or False? **In Exercises 89–92, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

89. If f, g, and h are first-degree polynomial functions, then the curve given by $x = f(t)$, $y = g(t)$, and $z = h(t)$ is a line.

90. If the curve given by $x = f(t)$, $y = g(t)$, and $z = h(t)$ is a line, then f, g, and h are first-degree polynomial functions of t.

91. Two particles traveling along the curves $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$ and $\mathbf{u}(t) = (2+t)\mathbf{i} + 8t\mathbf{j}$ will collide.

92. The vector-valued function $\mathbf{r}(t) = t^2\mathbf{i} + t\sin t\,\mathbf{j} + t\cos t\,\mathbf{k}$ lies on the paraboloid $x = y^2 + z^2$.

Section Project: Witch of Agnesi

In Section 3.5, you studied a famous curve called the **Witch of Agnesi.** In this project you will take a closer look at this function.

Consider a circle of radius a centered on the y-axis at $(0, a)$. Let A be a point on the horizontal line $y = 2a$, let O be the origin, and let B be the point where the segment OA intersects the circle. A point P is on the Witch of Agnesi if P lies on the horizontal line through B and on the vertical line through A.

(a) Show that the point A is traced out by the vector-valued function

$\mathbf{r}_A(\theta) = 2a\cot\theta\mathbf{i} + 2a\mathbf{j}, \quad 0 < \theta < \pi$

where θ is the angle that OA makes with the positive x-axis.

(b) Show that the point B is traced out by the vector-valued function

$\mathbf{r}_B(\theta) = a\sin 2\theta\mathbf{i} + a(1 - \cos 2\theta)\mathbf{j}, \quad 0 < \theta < \pi.$

(c) Combine the results in parts (a) and (b) to find the vector-valued function $\mathbf{r}(\theta)$ for the Witch of Agnesi. Use a graphing utility to graph this curve for $a = 1$.

(d) Describe the limits $\lim_{\theta\to 0^+}\mathbf{r}(\theta)$ and $\lim_{\theta\to \pi^-}\mathbf{r}(\theta)$.

(e) Eliminate the parameter θ and determine the rectangular equation of the Witch of Agnesi. Use a graphing utility to graph this function for $a = 1$ and compare your graph with that obtained in part (c).