## Exercises for Section 12.2

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**In Exercises 1–6, sketch the plane curve represented by the vector-valued function, and sketch the vectors $\mathbf{r}(t_0)$ and $\mathbf{r}'(t_0)$ for the given value of $t_0$. Position the vectors such that the initial point of $\mathbf{r}(t_0)$ is at the origin and the initial point of $\mathbf{r}'(t_0)$ is at the terminal point of $\mathbf{r}(t_0)$. What is the relationship between $\mathbf{r}'(t_0)$ and the curve?**

**1.** $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j}, \quad t_0 = 2$

**2.** $\mathbf{r}(t) = t\mathbf{i} + t^3\mathbf{j}, \quad t_0 = 1$

**3.** $\mathbf{r}(t) = t^2\mathbf{i} + \dfrac{1}{t}\mathbf{j}, \quad t_0 = 2$

**4.** $\mathbf{r}(t) = (1 + t)\mathbf{i} + t^3\mathbf{j}, \quad t_0 = 1$

**5.** $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}, \quad t_0 = \dfrac{\pi}{2}$

**6.** $\mathbf{r}(t) = e^t\mathbf{i} + e^{2t}\mathbf{j}, \quad t_0 = 0$

**7.** ***Investigation*** Consider the vector-valued function

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}.$$

(a) Sketch the graph of $\mathbf{r}(t)$. Use a graphing utility to verify your graph.

(b) Sketch the vectors $\mathbf{r}(1/4)$, $\mathbf{r}(1/2)$, and $\mathbf{r}(1/2) - \mathbf{r}(1/4)$ on the graph in part (a).

(c) Compare the vector $\mathbf{r}'(1/4)$ with the vector

$$\frac{\mathbf{r}(1/2) - \mathbf{r}(1/4)}{1/2 - 1/4}.$$

**8.** ***Investigation*** Consider the vector-valued function

$$\mathbf{r}(t) = t\mathbf{i} + (4 - t^2)\mathbf{j}.$$

(a) Sketch the graph of $\mathbf{r}(t)$. Use a graphing utility to verify your graph.

(b) Sketch the vectors $\mathbf{r}(1)$, $\mathbf{r}(1.25)$, and $\mathbf{r}(1.25) - \mathbf{r}(1)$ on the graph in part (a).

(c) Compare the vector $\mathbf{r}'(1)$ with the vector $\dfrac{\mathbf{r}(1.25) - \mathbf{r}(1)}{1.25 - 1}$.

**In Exercises 9 and 10, (a) sketch the space curve represented by the vector-valued function, and (b) sketch the vectors $\mathbf{r}(t_0)$ and $\mathbf{r}'(t_0)$ for the given value of $t_0$.**

**9.** $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k}, \quad t_0 = \dfrac{3\pi}{2}$

**10.** $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{3}{2}\mathbf{k}, \quad t_0 = 2$

**In Exercises 11–18, find $\mathbf{r}'(t)$.**

**11.** $\mathbf{r}(t) = 6t\mathbf{i} - 7t^2\mathbf{j} + t^3\mathbf{k}$

**12.** $\mathbf{r}(t) = \dfrac{1}{t}\mathbf{i} + 16t\mathbf{j} + \dfrac{t^2}{2}\mathbf{k}$

**13.** $\mathbf{r}(t) = a\cos^3 t\mathbf{i} + a\sin^3 t\mathbf{j} + \mathbf{k}$

**14.** $\mathbf{r}(t) = 4\sqrt{t}\,\mathbf{i} + t^2\sqrt{t}\,\mathbf{j} + \ln t^2\mathbf{k}$

**15.** $\mathbf{r}(t) = e^{-t}\mathbf{i} + 4\mathbf{j}$

**16.** $\mathbf{r}(t) = \langle \sin t - t\cos t, \cos t + t\sin t, t^2\rangle$

**17.** $\mathbf{r}(t) = \langle t\sin t, t\cos t, t\rangle$

**18.** $\mathbf{r}(t) = \langle \arcsin t, \arccos t, 0\rangle$

**In Exercises 19–26, find (a) $\mathbf{r}''(t)$ and (b) $\mathbf{r}'(t) \cdot \mathbf{r}''(t)$.**

**19.** $\mathbf{r}(t) = t^3\mathbf{i} + \frac{1}{2}t^2\mathbf{j}$

**20.** $\mathbf{r}(t) = (t^2 + t)\mathbf{i} + (t^2 - t)\mathbf{j}$

**21.** $\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j}$

**22.** $\mathbf{r}(t) = 8\cos t\mathbf{i} + 3\sin t\mathbf{j}$

**23.** $\mathbf{r}(t) = \frac{1}{2}t^2\mathbf{i} - t\mathbf{j} + \frac{1}{6}t^3\mathbf{k}$

**24.** $\mathbf{r}(t) = t\mathbf{i} + (2t + 3)\mathbf{j} + (3t - 5)\mathbf{k}$

**25.** $\mathbf{r}(t) = \langle \cos t + t\sin t, \sin t - t\cos t, t\rangle$

**26.** $\mathbf{r}(t) = \langle e^{-t}, t^2, \tan t\rangle$

**In Exercises 27 and 28, a vector-valued function and its graph are given. The graph also shows the unit vectors $\mathbf{r}'(t_0)/\|\mathbf{r}'(t_0)\|$ and $\mathbf{r}''(t_0)/\|\mathbf{r}''(t_0)\|$. Find these two unit vectors and identify them on the graph.**

**27.** $\mathbf{r}(t) = \cos(\pi t)\mathbf{i} + \sin(\pi t)\mathbf{j} + t^2\mathbf{k}, \quad t_0 = -\frac{1}{4}$

**28.** $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + e^{0.75t}\mathbf{k}, \quad t_0 = \frac{1}{4}$

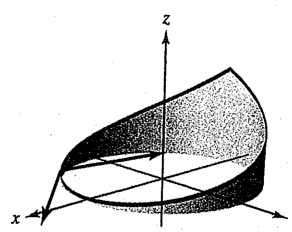


**Figure for 27**

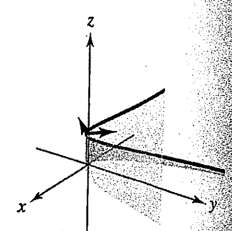


**Figure for 28**

**In Exercises 29–38, find the open interval(s) on which the curve given by the vector-valued function is smooth.**

**29.** $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$

**30.** $\mathbf{r}(t) = \dfrac{1}{t - 1}\mathbf{i} + 3t\mathbf{j}$

**31.** $\mathbf{r}(\theta) = 2\cos^3\theta\mathbf{i} + 3\sin^3\theta\mathbf{j}$

**32.** $\mathbf{r}(\theta) = (\theta + \sin\theta)\mathbf{i} + (1 - \cos\theta)\mathbf{j}$

**33.** $\mathbf{r}(\theta) = (\theta - 2\sin\theta)\mathbf{i} + (1 - 2\cos\theta)\mathbf{j}$

**34.** $\mathbf{r}(t) = \dfrac{2t}{8 + t^3}\mathbf{i} + \dfrac{2t^2}{8 + t^3}\mathbf{j}$

**35.** $\mathbf{r}(t) = (t - 1)\mathbf{i} + \dfrac{1}{t}\mathbf{j} - t^2\mathbf{k}$

**36.** $\mathbf{r}(t) = e^t\mathbf{i} - e^{-t}\mathbf{j} + 3t\mathbf{k}$

**37.** $\mathbf{r}(t) = t\mathbf{i} - 3t\mathbf{j} + \tan t\mathbf{k}$

**38.** $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + (t^2 - 1)\mathbf{j} + \frac{1}{4}t\mathbf{k}$

**In Exercises 39 and 40, use the properties of the derivative to find the following.**

**(a)** $\mathbf{r}'(t)$ **(b)** $\mathbf{r}''(t)$ **(c)** $D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)]$

**(d)** $D_t[3\mathbf{r}(t) - \mathbf{u}(t)]$ **(e)** $D_t[\mathbf{r}(t) \times \mathbf{u}(t)]$ **(f)** $D_t[\|\mathbf{r}(t)\|], \quad t > 0$

**39.** $\mathbf{r}(t) = t\mathbf{i} + 3t\mathbf{j} + t^2\mathbf{k}, \quad \mathbf{u}(t) = 4t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$

**40.** $\mathbf{r}(t) = t\mathbf{i} + 2\sin t\mathbf{j} + 2\cos t\mathbf{k},$

$$\mathbf{u}(t) = \frac{1}{t}\mathbf{i} + 2\sin t\mathbf{j} + 2\cos t\mathbf{k}$$

**In Exercises 41 and 42, find (a) $D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)]$ and (b) $D_t[\mathbf{r}(t) \times \mathbf{u}(t)]$ by differentiating the product, then applying the properties of Theorem 12.2.**

41. $\mathbf{r}(t) = t\mathbf{i} + 2t^2\mathbf{j} + t^3\mathbf{k}, \quad \mathbf{u}(t) = t^4\mathbf{k}$

42. $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}, \quad \mathbf{u}(t) = \mathbf{j} + t\mathbf{k}$

**In Exercises 43 and 44, find the angle $\theta$ between $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ as a function of $t$. Use a graphing utility to graph $\theta(t)$. Use the graph to find any extrema of the function. Find any values of $t$ at which the vectors are orthogonal.**

43. $\mathbf{r}(t) = 3\sin t\mathbf{i} + 4\cos t\mathbf{j}$ 44. $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j}$

**In Exercises 45–48, use the definition of the derivative to find $\mathbf{r}'(t)$.**

45. $\mathbf{r}(t) = (3t + 2)\mathbf{i} + (1 - t^2)\mathbf{j}$ 46. $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + \dfrac{3}{t}\mathbf{j} - 2t\mathbf{k}$

47. $\mathbf{r}(t) = \langle t^2, 0, 2t\rangle$ 48. $\mathbf{r}(t) = \langle 0, \sin t, 4t\rangle$

**In Exercises 49–56, find the indefinite integral.**

49. $\displaystyle\int (2t\mathbf{i} + \mathbf{j} + \mathbf{k})\,dt$ 50. $\displaystyle\int \left(4t^3\,\mathbf{i} + 6t\mathbf{j} - 4\sqrt{t}\,\mathbf{k}\right) dt$

51. $\displaystyle\int \left(\frac{1}{t}\mathbf{i} + \mathbf{j} - t^{3/2}\,\mathbf{k}\right) dt$ 52. $\displaystyle\int \left(\ln t\mathbf{i} + \frac{1}{t}\mathbf{j} + \mathbf{k}\right) dt$

53. $\displaystyle\int \left[(2t - 1)\mathbf{i} + 4t^3\mathbf{j} + 3\sqrt{t}\,\mathbf{k}\right] dt$

54. $\displaystyle\int (e^t\,\mathbf{i} + \sin t\mathbf{j} + \cos t\mathbf{k})\,dt$

55. $\displaystyle\int \left(\sec^2 t\mathbf{i} + \frac{1}{1 + t^2}\mathbf{j}\right) dt$

56. $\displaystyle\int (e^{-t}\sin t\mathbf{i} + e^{-t}\cos t\mathbf{j})\,dt$

**In Exercises 57–62, evaluate the definite integral.**

57. $\displaystyle\int_0^1 (8t\mathbf{i} + t\mathbf{j} - \mathbf{k})\,dt$ 58. $\displaystyle\int_{-1}^1 \left(t\mathbf{i} + t^3\mathbf{j} + \sqrt[3]{t}\,\mathbf{k}\right) dt$

59. $\displaystyle\int_0^{\pi/2} [(a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + \mathbf{k}]\,dt$

60. $\displaystyle\int_0^{\pi/4} [(\sec t\tan t)\mathbf{i} + (\tan t)\mathbf{j} + (2\sin t\cos t)\mathbf{k}]\,dt$

61. $\displaystyle\int_0^2 (t\mathbf{i} + e^t\mathbf{j} - te^t\mathbf{k})\,dt$ 62. $\displaystyle\int_0^3 \|t\mathbf{i} + t^2\mathbf{j}\|\,dt$

**In Exercises 63–68, find $\mathbf{r}(t)$ for the given conditions.**

63. $\mathbf{r}'(t) = 4e^{2t}\mathbf{i} + 3e^t\mathbf{j}, \quad \mathbf{r}(0) = 2\mathbf{i}$

64. $\mathbf{r}'(t) = 3t^2\mathbf{j} + 6\sqrt{t}\,\mathbf{k}, \quad \mathbf{r}(0) = \mathbf{i} + 2\mathbf{j}$

65. $\mathbf{r}''(t) = -32\mathbf{j}, \quad \mathbf{r}'(0) = 600\sqrt{3}\mathbf{i} + 600\mathbf{j}, \quad \mathbf{r}(0) = \mathbf{0}$

66. $\mathbf{r}''(t) = -4\cos t\mathbf{j} - 3\sin t\mathbf{k}, \quad \mathbf{r}'(0) = 3\mathbf{k}, \quad \mathbf{r}(0) = 4\mathbf{j}$

67. $\mathbf{r}'(t) = te^{-t^2}\mathbf{i} - e^{-t}\mathbf{j} + \mathbf{k}, \quad \mathbf{r}(0) = \frac{1}{2}\mathbf{i} - \mathbf{j} + \mathbf{k}$

68. $\mathbf{r}'(t) = \dfrac{1}{1 + t^2}\mathbf{i} + \dfrac{1}{t^2}\mathbf{j} + \dfrac{1}{t}\mathbf{k}, \quad \mathbf{r}(1) = 2\mathbf{i}$

## Writing About Concepts

69. State the definition of the derivative of a vector-valued function. Describe how to find the derivative of a vector-valued function and give its geometric interpretation.

70. How do you find the integral of a vector-valued function?

71. The three components of the derivative of the vector-valued function $\mathbf{u}$ are positive at $t = t_0$. Describe the behavior of $\mathbf{u}$ at $t = t_0$.

72. The $z$-component of the derivative of the vector-valued function $\mathbf{u}$ is 0 for $t$ in the domain of the function. What does this information imply about the graph of $\mathbf{u}$?

**In Exercises 73–80, prove the property. In each case, assume r, u, and v are differentiable vector-valued functions of $t$, $f$ is a differentiable real-valued function of $t$, and $c$ is a scalar.**

73. $D_t[c\mathbf{r}(t)] = c\mathbf{r}'(t)$

74. $D_t[\mathbf{r}(t) \pm \mathbf{u}(t)] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$

75. $D_t[f(t)\mathbf{r}(t)] = f(t)\mathbf{r}'(t) + f'(t)\mathbf{r}(t)$

76. $D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$

77. $D_t[\mathbf{r}(f(t))] = \mathbf{r}'(f(t))f'(t)$

78. $D_t[\mathbf{r}(t) \times \mathbf{r}'(t)] = \mathbf{r}(t) \times \mathbf{r}''(t)$

79. $D_t\{\mathbf{r}(t) \cdot [\mathbf{u}(t) \times \mathbf{v}(t)]\} = \mathbf{r}'(t) \cdot [\mathbf{u}(t) \times \mathbf{v}(t)] + \mathbf{r}(t) \cdot [\mathbf{u}'(t) \times \mathbf{v}(t)] + \mathbf{r}(t) \cdot [\mathbf{u}(t) \times \mathbf{v}'(t)]$

80. If $\mathbf{r}(t) \cdot \mathbf{r}(t)$ is a constant, then $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$.

81. ***Particle Motion*** A particle moves in the $xy$-plane along the curve represented by the vector-valued function $\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}$.
   (a) Use a graphing utility to graph $\mathbf{r}$. Describe the curve.
   (b) Find the minimum and maximum values of $\|\mathbf{r}'\|$ and $\|\mathbf{r}''\|$.

82. ***Particle Motion*** A particle moves in the $yz$-plane along the curve represented by the vector-valued function $\mathbf{r}(t) = (2\cos t)\mathbf{j} + (3\sin t)\mathbf{k}$.
   (a) Describe the curve.
   (b) Find the minimum and maximum values of $\|\mathbf{r}'\|$ and $\|\mathbf{r}''\|$.

***True or False?*** **In Exercises 83–86, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

83. If a particle moves along a sphere centered at the origin, then its derivative vector is always tangent to the sphere.

84. The definite integral of a vector-valued function is a real number.

85. $\dfrac{d}{dt}[\|\mathbf{r}(t)\|] = \|\mathbf{r}'(t)\|$

86. If $\mathbf{r}$ and $\mathbf{u}$ are differentiable vector-valued functions of $t$, then $D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}'(t) \cdot \mathbf{u}'(t)$.

87. Consider the vector-valued function

$$\mathbf{r}(t) = (e^t\sin t)\mathbf{i} + (e^t\cos t)\mathbf{j}.$$

Show that $\mathbf{r}(t)$ and $\mathbf{r}''(t)$ are always perpendicular to each other.