EXAMPLE 8 **Approximating the Derivative of a Function Defined by a Table**

The table below lists several values of a function $y = f(x)$ that is continuous on the interval $[-1, 5]$ and has a derivative at each number in the interval $(-1, 5)$. Approximate the derivative of f at 2.

x	0	1	2	3	4
$f(x)$	0	3	12	33	72

Solution

There are several ways to approximate the derivative of a function defined by a table. Each uses an average rate of change to approximate the rate of change of f at 2, which is the derivative of f at 2.

- Using the average rate of change from 2 to 3, we have

$$\frac{f(3) - f(2)}{3 - 2} = \frac{33 - 12}{1} = 21$$

With this choice, $f'(2)$ is approximately 21.

- Using the average rate of change from 1 to 2, we have

$$\frac{f(2) - f(1)}{2 - 1} = \frac{12 - 3}{1} = 9$$

With this choice, $f'(2)$ is approximately 9.

- A third approximation can be found by averaging the above two approximations. Then $f'(2)$ is approximately $\frac{21 + 9}{2} = 15$. ■

NOW WORK Problem 51 and AP® Practice Problem 8.

2.1 Assess Your Understanding

Concepts and Vocabulary

1. *True or False* The derivative is used to find instantaneous velocity.
2. *True or False* The derivative can be used to find the rate of change of a function.
3. The notation $f'(c)$ is read f ____ of c; $f'(c)$ represents the ____ of the tangent line to the graph of f at the point ____.
4. *True or False* If it exists, $\lim_{x \to 3} \frac{f(x) - f(3)}{x - 3}$ is the derivative of the function f at 3.
5. If $f(x) = 6x - 3$, then $f'(3) =$ ________.
6. The velocity of an object, the slope of a tangent line, and the rate of change of a function are three different interpretations of the mathematical concept called the ________.

Skill Building

In Problems 7–16,

(a) *Find an equation for the tangent line to the graph of each function at the indicated point.*

(b) *Find an equation of the normal line to each function at the indicated point.*

(c) *Graph the function, the tangent line, and the normal line at the indicated point on the same set of coordinate axes.*

7. $f(x) = 3x^2$ at $(-2, 12)$
8. $f(x) = x^2 + 2$ at $(-1, 3)$
9. $f(x) = x^3$ at $(-2, -8)$
10. $f(x) = x^3 + 1$ at $(1, 2)$
11. $f(x) = \frac{1}{x}$ at $(1, 1)$
12. $f(x) = \sqrt{x}$ at $(4, 2)$
13. $f(x) = \frac{1}{x+5}$ at $\left(1, \frac{1}{6}\right)$
14. $f(x) = \frac{2}{x+4}$ at $\left(1, \frac{2}{5}\right)$
15. $f(x) = \frac{1}{\sqrt{x}}$ at $(1, 1)$
16. $f(x) = \frac{1}{x^2}$ at $(1, 1)$

In Problems 17–20, find the rate of change of f at the indicated numbers.

17. $f(x) = 5x - 2$ at (a) $c = 0$, (b) $c = 2$
18. $f(x) = x^2 - 1$ at (a) $c = -1$, (b) $c = 1$
19. $f(x) = \frac{x^2}{x+3}$ at (a) $c = 0$, (b) $c = 1$
20. $f(x) = \frac{x}{x^2 - 1}$ at (a) $c = 0$, (b) $c = 2$

In Problems 21–30, find the derivative of each function at the given number.

21. $f(x) = 2x + 3$ at 1
22. $f(x) = 3x - 5$ at 2
23. $f(x) = x^2 - 2$ at 0
24. $f(x) = 2x^2 + 4$ at 1
25. $f(x) = 3x^2 + x + 5$ at -1
26. $f(x) = 2x^2 - x - 7$ at -1
27. $f(x) = \sqrt{x}$ at 4
28. $f(x) = \frac{1}{x^2}$ at 2
29. $f(x) = \frac{2 - 5x}{1 + x}$ at 0
30. $f(x) = \frac{2 + 3x}{2 + x}$ at 1

31. **Approximating Velocity** An object in rectilinear motion moves according to the position function $s(t) = 10t^2$ (s in centimeters and t in seconds). Approximate the velocity of the object at time $t_0 = 3$ s by letting Δt first equal 0.1 s, then 0.01 s, and finally 0.001 s. What limit does the velocity appear to be approaching? Organize the results in a table.

32. **Approximating Velocity** An object in rectilinear motion moves according to the position function $s(t) = 5 - t^2$ (s in centimeters and t in seconds). Approximate the velocity of the object at time $t_0 = 1$ by letting Δt first equal 0.1, then 0.01, and finally 0.001. What limit does the velocity appear to be approaching? Organize the results in a table.

33. **Rectilinear Motion** As an object in rectilinear motion moves, its signed distance s (in meters) from the origin after t seconds is given by the position function $s = f(t) = 3t^2 + 4t$. Find the velocity v at $t_0 = 0$. At $t_0 = 2$. At any time t_0.

34. **Rectilinear Motion** As an object in rectilinear motion moves, its signed distance s (in meters) from the origin after t seconds is given by the position function $s = f(t) = 2t^3 + 4$. Find the velocity v at $t_0 = 0$. At $t_0 = 3$. At any time t_0.

35. **Rectilinear Motion** As an object in rectilinear motion moves, its signed distance s from the origin at time t is given by the position function $s = s(t) = 3t^2 - \frac{1}{t}$, where s is in centimeters and t is in seconds. Find the velocity v of the object at $t_0 = 1$ and $t_0 = 4$.

36. **Rectilinear Motion** As an object in rectilinear motion moves, its signed distance s from the origin at time t is given by the position function $s = s(t) = 2\sqrt{t}$, where s is in centimeters and t is in seconds. Find the velocity v of the object at $t_0 = 1$ and $t_0 = 4$.

37. The Princeton Dinky is the shortest rail line in the country. It runs for 2.7 miles, connecting Princeton University to the Princeton Junction railroad station. The Dinky starts from the university and moves north toward Princeton Junction. Its distance from Princeton is shown in the graph (top, right), where the time t is in minutes and the distance s of the Dinky from Princeton University is in miles.

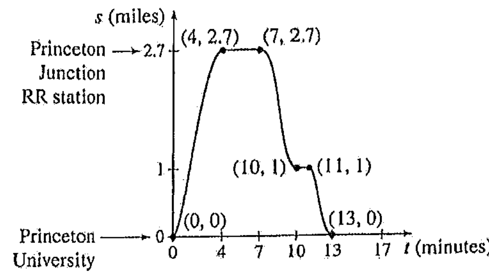
(a) When is the Dinky headed toward Princeton University?

(b) When is it headed toward Princeton Junction?

(c) When is the Dinky stopped?

(d) Find its average velocity on a trip from Princeton to Princeton Junction.

(e) Find its average velocity for the round-trip shown in the graph, that is, from $t = 0$ to $t = 13$.



38. Barbara walks to the deli, which is six blocks east of her house. After walking two blocks, she realizes she left her phone on her desk, so she runs home. After getting the phone, and closing and locking the door, Barbara starts on her way again. At the deli, she waits in line to buy a bottle of **vitaminwater**™, and then she jogs home. The graph below represents Barbara's journey. The time t is in minutes, and s is Barbara's distance, in blocks, from home.

(a) At what times is she headed toward the deli?

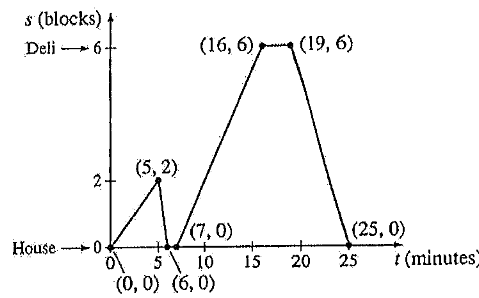
(b) At what times is she headed home?

(c) When is the graph horizontal? What does this indicate?

(d) Find Barbara's average velocity from home until she starts back to get her phone.

(e) Find Barbara's average velocity from home to the deli after getting her phone.

(f) Find her average velocity from the deli to home.



Applications and Extensions

39. **Slope of a Tangent Line** An equation of the tangent line to the graph of a function f at (2, 6) is $y = -3x + 12$. What is $f'(2)$?

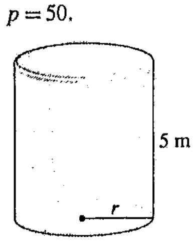
40. **Slope of a Tangent Line** An equation of the tangent line of a function f at (3, 2) is $y = \frac{1}{3}x + 1$. What is $f'(3)$?

41. **Tangent Line** Does the tangent line to the graph of $y = x^2$ at (1, 1) pass through the point (2, 5)?

42. **Tangent Line** Does the tangent line to the graph of $y = x^3$ at (1, 1) pass through the point (2, 5)?

43. **Respiration Rate** A human being's respiration rate R (in breaths per minute) is given by $R = R(p) = 10.35 + 0.59p$, where p is the partial pressure of carbon dioxide in the lungs. Find the rate of change in respiration when $p = 50$.

44. **Instantaneous Rate of Change** The volume V of the right circular cylinder of height 5 m and radius r m shown in the figure is $V = V(r) = 5\pi r^2$. Find the instantaneous rate of change of the volume with respect to the radius when $r = 3$ m.



45. **Market Share** During a month-long advertising campaign, the total sales S of a magazine is modeled by the function $S(x)=5x^2+100x+10{,}000$, where x, $0\le x\le 30$, represents the number of days since the campaign began.
    (a) What is the average rate of change of sales from $x=10$ to $x=20$ days?
    (b) What is the instantaneous rate of change of sales when $x=10$ days?

46. **Demand Equation** The demand equation for an item is $p=p(x)=90-0.02x$, where p is the price in dollars and x is the number of units (in thousands) made.
    (a) Assuming all units made can be sold, find the revenue function $R(x)=xp(x)$.
    (b) **Marginal Revenue** Marginal revenue is defined as the additional revenue earned by selling an additional unit. If we use $R'(x)$ to measure the marginal revenue, find the marginal revenue when 1 million units are sold.

47. **Gravity** If a ball is dropped from the top of the Empire State Building, 1002 ft above the ground, the distance s (in feet) it falls after t seconds is $s(t)=16t^2$.
    (a) What is the average velocity of the ball for the first 2 s?
    (b) How long does it take for the ball to hit the ground?
    (c) What is the average velocity of the ball during the time it is falling?
    (d) What is the velocity of the ball when it hits the ground?

48. **Velocity** A ball is thrown upward. Its height h in feet is given by $h(t)=100t-16t^2$, where t is the time elapsed in seconds.
    (a) What is the velocity v of the ball at $t=0$ s, $t=1$ s, and $t=4$ s?
    (b) At what time t does the ball strike the ground?
    (c) At what time t does the ball reach its highest point? *Hint:* At the time the ball reaches its maximum height, it is stationary. So, its velocity $v=0$.

49. **Gravity** A rock is dropped from a height of 88.2 m and falls toward Earth in a straight line. In t seconds the rock falls $4.9t^2$ m.
    (a) What is the average velocity of the rock for the first 2 s?
    (b) How long does it take for the rock to hit the ground?
    (c) What is the average velocity of the rock during its fall?
    (d) What is the velocity v of the rock when it hits the ground?

50. **Velocity** At a certain instant, the speedometer of an automobile reads V mi/h. During the next $\frac{1}{4}$ s the automobile travels 20 ft. Approximate V from this information.

PAGE 168 51. A tank is filled with 80 liters of water at 7 a.m. ($t=0$). Over the next 12 h the water is continuously used. The table below gives the amount of water $A(t)$ (in liters) remaining in the tank at selected times t, where t measures the number of hours after 7 a.m.

t	0	2	5	7	9	12
$A(t)$	80	71	66	60	54	50

    (a) Use the table to approximate $A'(5)$.
    (b) Using appropriate units, interpret $A'(5)$ in the context of the problem.

52. The table below lists the outside temperature T, in degrees Fahrenheit, in Naples, Florida, on a certain day in January, for selected times x, where x is the number of hours since 12 a.m.

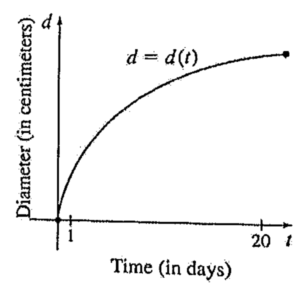
x	5	7	9	11	12	13	14	16	17
$T(x)$	62	71	74	78	81	83	84	85	78

    (a) Use the table to approximate $T'(11)$.
    (b) Using appropriate units, interpret $T'(11)$ in the context of the problem.

53. **Rate of Change** Show that the rate of change of a linear function $f(x)=mx+b$ is the slope m of the line $y=mx+b$.

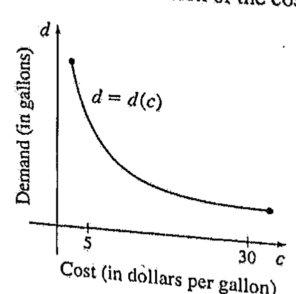
54. **Rate of Change** Show that the rate of change of a quadratic function $f(x)=ax^2+bx+c$ is a linear function of x.

55. **Agriculture** The graph represents the diameter d (in centimeters) of a maturing peach as a function of the time t (in days) it is on the tree.



    (a) Interpret the derivative $d'(t)$ as a rate of change.
    (b) Which is larger, $d'(1)$ or $d'(20)$?
    (c) Interpret both $d'(1)$ and $d'(20)$.

56. **Business** The graph represents the demand d (in gallons) for olive oil as a function of the cost c (in dollars per gallon) of the oil.



    (a) Interpret the derivative $d'(c)$.
    (b) Which is larger, $d'(5)$ or $d'(30)$? Give an interpretation to $d'(5)$ and $d'(30)$.

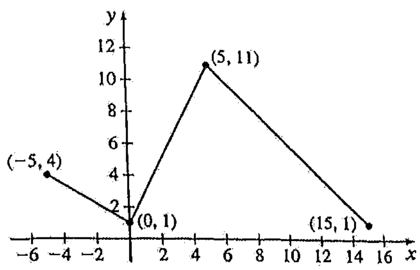
57. **Volume of a Cube** A metal cube with each edge of length x centimeters is expanding uniformly as a consequence of being heated.
    (a) Find the average rate of change of the volume of the cube with respect to an edge as x increases from 2.00 to 2.01 cm.
    (b) Find the instantaneous rate of change of the volume of the cube with respect to an edge at the instant when $x=2$ cm.

AP® Practice Problems

Preparing for the AP® Exam

1. The line $x + y = 5$ is tangent to the graph of $y = f(x)$ at the point where $x = 2$. The values $f(2)$ and $f'(2)$ are:
(A) $f(2) = 2$; $f'(2) = -1$ (B) $f(2) = 3$; $f'(2) = -1$
(C) $f(2) = 2$; $f'(2) = 1$ (D) $f(2) = 3$; $f'(2) = 2$

2. The graph of the function f, given below, consists of three line segments. Find $f'(3)$.

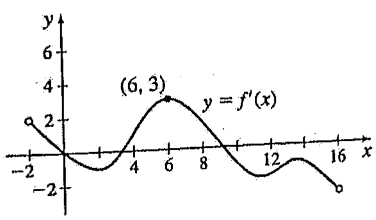


Graph of f

(A) 1 (B) 2 (C) 3 (D) $f'(3)$ does not exist

3. What is the instantaneous rate of change of the function $f(x) = 3x^2 + 5$ at $x = 2$?
(A) 5 (B) 7 (C) 12 (D) 17

4. The function f is defined on the closed interval $[-2, 16]$. The graph of the derivative of f, $y = f'(x)$, is given below.



The point $(6, -2)$ is on the graph of $y = f(x)$. An equation of the tangent line to the graph of f at $(6, -2)$ is
(A) $y = 3$ (B) $y + 2 = 6(x + 3)$
(C) $y + 2 = 6x$ (D) $y + 2 = 3(x - 6)$

5. If $x - 3y = 13$ is an equation of the normal line to the graph of f at the point $(2, 6)$, then $f'(2) =$
(A) $-\frac{1}{3}$ (B) $\frac{1}{3}$ (C) -3 (D) $-\frac{13}{3}$

6. If f is a function for which $\lim_{x\to -3} \frac{f(x) - f(-3)}{x+3} = 0$, then which of the following statements must be true?
(A) $x = -3$ is a vertical asymptote of the graph.
(B) The derivative of f at $x = -3$ exists.
(C) The function f is continuous at $x = 3$.
(D) f is not defined at $x = -3$.

7. If the position of an object on the x-axis at time t is $4t^2$, then the average velocity of the object over the interval $0 \le t \le 5$ is
(A) 5 (B) 20 (C) 40 (D) 100

8. A tank is filled with 80 liters of water at 7 a.m. ($t = 0$). Over the next 12 hours the water is continuously used and no water is added to replace it. The table below gives the amount of water $A(t)$ (in liters) remaining in the tank at selected times t, where t measures the number of hours after 7 a.m.

t	0	2	5	7	9	12
$A(t)$	80	71	66	60	54	50

Use the table to approximate $A'(5)$.

2.2 The Derivative as a Function; Differentiability

OBJECTIVES *When you finish this section, you should be able to:*

1 **Define the derivative function (p. 171)**
2 **Graph the derivative function (p. 173)**
3 **Identify where a function is not differentiable (p. 175)**

1 Define the Derivative Function

The derivative of f at a real number c has been defined as the real number

$$f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c} \qquad (1)$$

provided the limit exists. We refer to this representation of the derivative as Form (1).

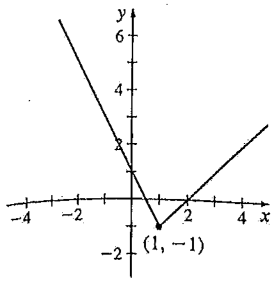


Figure 22 $g(x) = \begin{cases} 1-2x & \text{if } x \le 1 \\ x-2 & \text{if } x > 1 \end{cases}$

(b) See Figure 22. The function g is continuous at 1, which you should verify. To determine whether g is differentiable at 1, examine the one-sided limits at 1 using Form (1).

For $x < 1$,

$$\lim_{x\to 1^-} \frac{g(x)-g(1)}{x-1} = \lim_{x\to 1^-} \frac{(1-2x)-(-1)}{x-1} = \lim_{x\to 1^-} \frac{2-2x}{x-1}$$
$$= \lim_{x\to 1^-} \frac{-2(x-1)}{x-1} = \lim_{x\to 1^-}(-2) = -2$$

For $x > 1$,

$$\lim_{x\to 1^+} \frac{g(x)-g(1)}{x-1} = \lim_{x\to 1^+} \frac{(x-2)-(-1)}{x-1} = \lim_{x\to 1^+} \frac{x-1}{x-1} = 1$$

The one-sided limits are not equal, so $\lim_{x\to 1} \frac{g(x)-g(1)}{x-1}$ does not exist. That is, g is not differentiable at 1. ■

Notice in Figure 21 the tangent lines to the graph of f turn smoothly around the origin. On the other hand, notice in Figure 22 the tangent lines to the graph of g change abruptly at the point $(1, -1)$, where the graph of g has a corner.

NOW WORK Problem 41 and AP® Practice Problems 3, 4, 6, and 7.

2.2 Assess Your Understanding

Concepts and Vocabulary

1. *True or False* The domain of a function f and the domain of its derivative function f' are always equal.
2. *True or False* If a function is continuous at a number c, then it is differentiable at c.
3. *Multiple Choice* If f is continuous at a number c and if $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ is infinite, then the graph of f has [(a) a horizontal (b) a vertical (c) no] tangent line at c.
4. The instruction, "Differentiate f," means to find the ____ of f.

Skill Building

In Problems 5–10, find the derivative of each function f at any real number c. Use Form (1) on page 171.

5. $f(x) = 10$
6. $f(x) = -4$
7. $f(x) = 2x+3$
8. $f(x) = 3x-5$
9. $f(x) = 2-x^2$
10. $f(x) = 2x^2+4$

In Problems 11–16, differentiate each function f and determine the domain of f′. Use Form (2) on page 172.

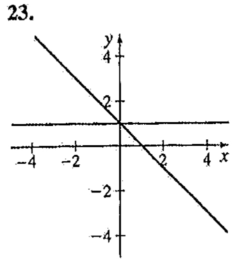
11. $f(x) = 5$
12. $f(x) = -2$
13. $f(x) = 3x^2+x+5$
14. $f(x) = 2x^2-x-7$
15. $f(x) = 5\sqrt{x-1}$
16. $f(x) = 4\sqrt{x+3}$

In Problems 17–22, differentiate each function f. Graph y = f(x) and y = f′(x) on the same set of coordinate axes.

17. $f(x) = \frac{1}{3}x+1$
18. $f(x) = -4x-5$
19. $f(x) = 2x^2-5x$
20. $f(x) = -3x^2+2$
21. $f(x) = x^3-8x$
22. $f(x) = -x^3-8$

In Problems 23–26, for each figure determine if the graphs represent a function f and its derivative f′. If they do, indicate which is the graph of f and which is the graph of f′.

23.



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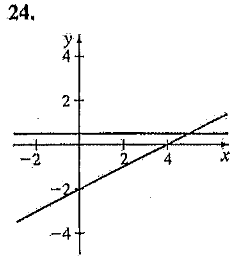

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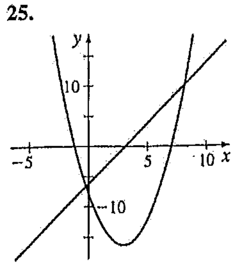

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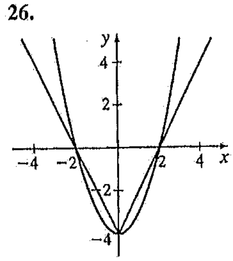
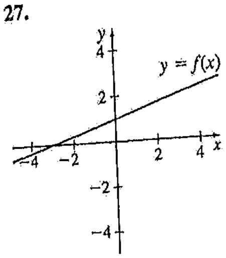


In Problems 27–30, use the graph of f to obtain the graph of f′.

27.



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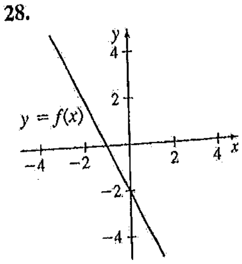

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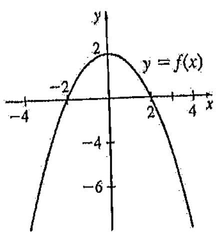

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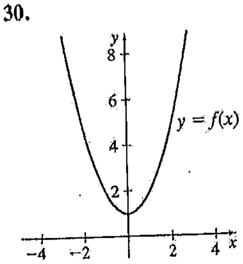
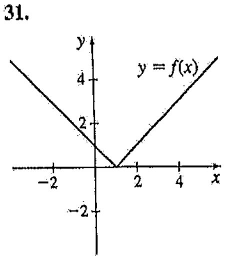


In Problems 31–34, the graph of a function f is given. Match each graph to the graph of its derivative f′ in A–D.

31.



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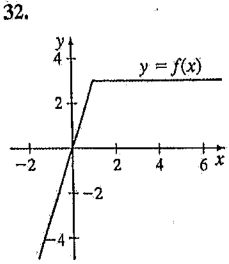

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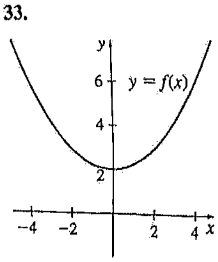

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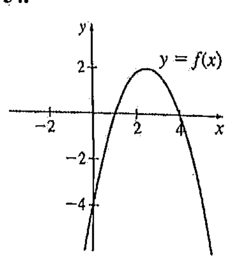
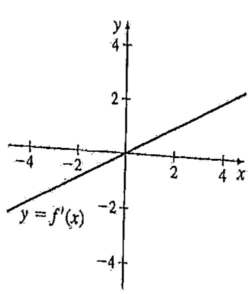
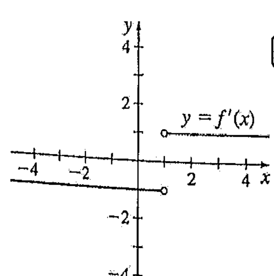




(A)



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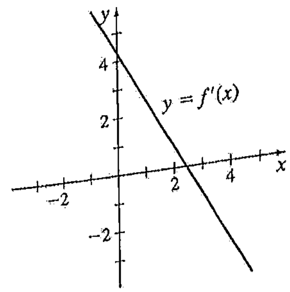

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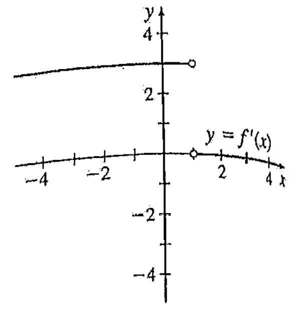


(D)

In Problems 35–44, determine whether each function f has a derivative at c. If it does, what is f′(c)? If it does not, give the reason why.

PAGE 176 **35.** $f(x)=x^{2/3}$ at $c=-8$

36. $f(x)=2x^{1/3}$ at $c=0$

37. $f(x)=|x^2-4|$ at $c=2$

38. $f(x)=|x^2-4|$ at $c=-$

PAGE 176 **39.** $f(x)=\begin{cases} 2x+3 & \text{if } x<1 \\ x^2+4 & \text{if } x\geq 1 \end{cases}$ at $c=1$

40. $f(x)=\begin{cases} 3-4x & \text{if } x<-1 \\ 2x+9 & \text{if } x\geq -1 \end{cases}$ at $c=-1$

PAGE 179 **41.** $f(x)=\begin{cases} -4+2x & \text{if } x\leq \frac{1}{2} \\ 4x^2-4 & \text{if } x>\frac{1}{2} \end{cases}$ at $c=\frac{1}{2}$

42. $f(x)=\begin{cases} 2x^2+1 & \text{if } x<-1 \\ -1-4x & \text{if } x\geq -1 \end{cases}$ at $c=-1$

PAGE 178 **43.** $f(x)=\begin{cases} 2x^2+1 & \text{if } x<-1 \\ 2+2x & \text{if } x\geq -1 \end{cases}$ at $c=-1$

44. $f(x)=\begin{cases} 5-2x & \text{if } x<2 \\ x^2 & \text{if } x\geq 2 \end{cases}$ at $c=2$

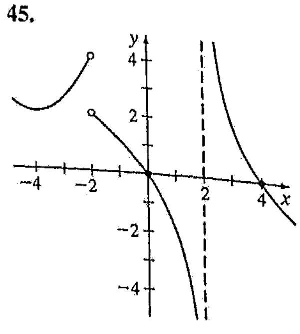
In Problems 45–48, each function f is continuous for all real numbers, and the graph of $y=f'(x)$ is given.

(a) *Does the graph of f have any horizontal tangent lines? If yes, explain why and identify where they occur.*

(b) *Does the graph of f have any vertical tangent lines? If yes, explain why, identify where they occur, and determine whether the point is a cusp of f.*

(c) *Does the graph of f have any corners? If yes, explain why and identify where they occur.*

PAGE 177 **45.**



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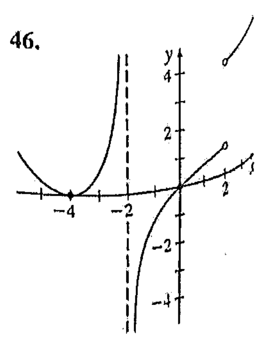

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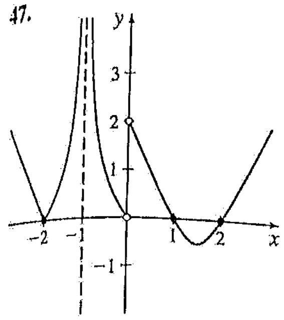

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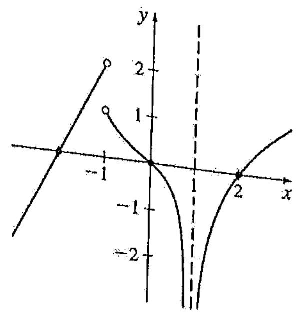
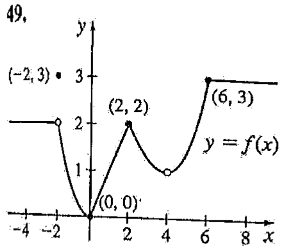


In Problems 49 and 50, use the given points $(c, f(c))$ on the graph of the function f.

(a) *For which numbers c does $\lim_{x \to c} f(x)$ exist but f is not continuous at c?*

(b) *For which numbers c is f continuous at c but not differentiable at c?*

49.



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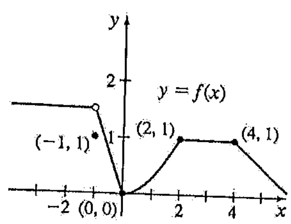


In Problems 51–54, find the derivative of each function.

51. $f(x) = mx + b$

52. $f(x) = ax^2 + bx + c$

53. $f(x) = \dfrac{1}{x^2}$

54. $f(x) = \dfrac{1}{\sqrt{x}}$

Applications and Extensions

In Problems 55–66, each limit represents the derivative of a function f at some number c. Determine f and c in each case.

55. $\lim_{h \to 0} \dfrac{(2+h)^2 - 4}{h}$

56. $\lim_{h \to 0} \dfrac{(2+h)^3 - 8}{h}$

57. $\lim_{x \to 1} \dfrac{x^2 - 1}{x - 1}$

58. $\lim_{x \to 1} \dfrac{x^4 - 1}{x - 1}$

59. $\lim_{h \to 0} \dfrac{\sqrt{9+h} - 3}{h}$

60. $\lim_{h \to 0} \dfrac{(8+h)^{1/3} - 2}{h}$

61. $\lim_{x \to \pi/6} \dfrac{\sin x - \frac{1}{2}}{x - \frac{\pi}{6}}$

62. $\lim_{x \to \pi/4} \dfrac{\cos x - \frac{\sqrt{2}}{2}}{x - \frac{\pi}{4}}$

63. $\lim_{x \to 0} \dfrac{2(x+2)^2 - (x+2) - 6}{x}$

64. $\lim_{x \to 0} \dfrac{3x^3 - 2x}{x}$

65. $\lim_{h \to 0} \dfrac{(3+h)^2 + 2(3+h) - 15}{h}$

66. $\lim_{h \to 0} \dfrac{3(h-1)^2 + h - 3}{h}$

67. **Units** The volume V (in cubic feet) of a balloon is expanding according to $V = V(t) = 4t$, where t is the time (in seconds). Find the rate of change of the volume of the balloon with respect to time. What are the units of $V'(t)$?

68. **Units** The area A (in square miles) of a circular patch of oil is expanding according to $A = A(t) = 2t$, where t is the time (in hours). At what rate is the area changing with respect to time? What are the units of $A'(t)$?

69. **Units** A manufacturer of precision digital switches has a daily cost C (in dollars) of $C(x) = 10{,}000 + 3x$, where x is the number of switches produced daily. What is the rate of change of cost with respect to x? What are the units of $C'(x)$?

70. **Units** A manufacturer of precision digital switches has daily revenue R (in dollars) of $R(x) = 5x - \dfrac{x^2}{2000}$, where x is the number of switches produced daily. What is the rate of change of revenue with respect to x? What are the units of $R'(x)$?

71. $f(x) = \begin{cases} x^3 & \text{if } x \le 0 \\ x^2 & \text{if } x > 0 \end{cases}$

(a) Determine whether f is continuous at 0.
(b) Determine whether $f'(0)$ exists.
(c) Graph the function f and its derivative f'.

72. For the function $f(x) = \begin{cases} 2x & \text{if } x \le 0 \\ x^2 & \text{if } x > 0 \end{cases}$

(a) Determine whether f is continuous at 0.
(b) Determine whether $f'(0)$ exists.
(c) Graph the function f and its derivative f'.

73. **Velocity** The distance s (in feet) of an automobile from the origin at time t (in seconds) is given by the position function

$$s = s(t) = \begin{cases} t^3 & \text{if } 0 \le t < 5 \\ 125 & \text{if } t \ge 5 \end{cases}$$

(This could represent a crash test in which a vehicle is accelerated until it hits a brick wall at $t = 5$ s.)

(a) Find the velocity just before impact (at $t = 4.99$ s) and just after impact (at $t = 5.01$ s).
(b) Is the velocity function $v = s'(t)$ continuous at $t = 5$?
(c) How do you interpret the answer to (b)?

74. **Population Growth** A simple model for population growth states that the rate of change of population size P with respect to time t is proportional to the population size. Express this statement as an equation involving a derivative.

75. **Atmospheric Pressure** Atmospheric pressure p decreases as the distance x from the surface of Earth increases, and the rate of change of pressure with respect to altitude is proportional to the pressure. Express this law as an equation involving a derivative.

76. **Electrical Current** Under certain conditions, an electric current I will die out at a rate (with respect to time t) that is proportional to the current remaining. Express this law as an equation involving a derivative.

77. **Tangent Line** Let $f(x) = x^2 + 2$. Find all points on the graph of f for which the tangent line passes through the origin.

78. **Tangent Line** Let $f(x) = x^2 - 2x + 1$. Find all points on the graph of f for which the tangent line passes through the point $(1, -1)$.

79. Area and Circumference of a Circle A circle of radius r has area $A=\pi r^2$ and circumference $C=2\pi r$. If the radius changes from r to $r+\Delta r$, find the:

(a) Change in area.
(b) Change in circumference.
(c) Average rate of change of area with respect to radius.
(d) Average rate of change of circumference with respect to radius.
(e) Rate of change of circumference with respect to radius.

80. Volume of a Sphere The volume V of a sphere of radius r is $V=\dfrac{4\pi r^3}{3}$. If the radius changes from r to $r+\Delta r$, find the:

(a) Change in volume.
(b) Average rate of change of volume with respect to radius.
(c) Rate of change of volume with respect to radius.

81. Use the definition of the derivative to show that $f(x)=|x|$ is not differentiable at 0.

82. Use the definition of the derivative to show that $f(x)=\sqrt[3]{x}$ is not differentiable at 0.

83. If f is an even function that is differentiable at c, show that its derivative function is odd. That is, show $f'(-c)=-f'(c)$.

84. If f is an odd function that is differentiable at c, show that its derivative function is even. That is, show $f'(-c)=f'(c)$.

85. Tangent Lines and Derivatives Let f and g be two functions each with derivatives at c. State the relationship between their tangent lines at c if:

(a) $f'(c)=g'(c)$ (b) $f'(c)=-\dfrac{1}{g'(c)}$ $g'(c)\neq 0$

Challenge Problems

86. Let f be a function defined for all real numbers x. Suppose f has the following properties:

$$f(u+v)=f(u)f(v) \qquad f(0)=1 \qquad f'(0) \text{ exists}$$

(a) Show that $f'(x)$ exists for all real numbers x.
(b) Show that $f'(x)=f'(0)f(x)$.

87. A function f is defined for all real numbers and has the following three properties:

$$f(1)=5 \qquad f(3)=21 \qquad f(a+b)-f(a)=kab+2b^2$$

for all real numbers a and b where k is a fixed real number independent of a and b.

(a) Use $a=1$ and $b=2$ to find k.
(b) Find $f'(3)$.
(c) Find $f'(x)$ for all real x.

88. A function f is **periodic** if there is a positive number p so that $f(x+p)=f(x)$ for all x. Suppose f is differentiable. Show that if f is periodic with period p, then f' is also periodic with period p.

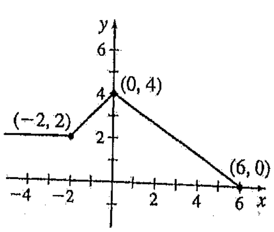
Preparing for the AP® Exam

AP® Practice Problems

PAGE 176 **1.** The function $f(x)=\begin{cases} x^2-ax & \text{if } x\le 1 \\ ax+b & \text{if } x>1\end{cases}$, where a and b are constants. If f is differentiable at $x=1$, then $a+b=$

(A) -3 (B) -2 (C) 0 (D) 2

PAGE 172 **2.** The graph of the function f, given below, consists of three line segments. Find $\lim\limits_{h\to 0}\dfrac{f(3+h)-f(3)}{h}$.



(A) -1 (B) $-\dfrac{2}{3}$ (C) $-\dfrac{3}{2}$ (D) does not exist

PAGE 179 **3.** If $f(x)=\begin{cases}\dfrac{x^2-25}{x-5} & \text{if } x\neq 5 \\ 5 & \text{if } x=5\end{cases}$

which of the following statements about f are true?

I. $\lim\limits_{x\to 5} f$ exists.
II. f is continuous at $x=5$.
III. f is differentiable at $x=5$.

(A) I only (B) I and II only
(C) I and III only (D) I, II, and III

PAGE 179 **4.** Suppose f is a function that is differentiable on the open interval $(-2, 8)$. If $f(0)=3$, $f(2)=-3$, and $f(7)=3$, which of the following must be true?

I. f has at least 2 zeros.
II. f is continuous on the closed interval $[-1, 7]$.
III. For some c, $0<c<7$, $f(c)=-2$.

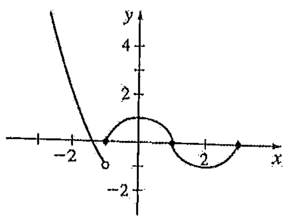
(A) I only (B) I and II only
(C) II and III only (D) I, II, and III

PAGE 176 **5.** If $f(x)=|x|$, which of the following statements about f are true?

I. f is continuous at 0.
II. f is differentiable at 0.
III. $f(0)=0$.

(A) I only (B) III only
(C) I and III only (D) I, II, and III

PAGE 179 **6.** The graph of the function f shown in the figure has horizontal tangent lines at the points $(0, 1)$ and $(2, -1)$ and a vertical tangent line at the point $(1, 0)$. For what numbers x in the open interval $(-2, 3)$ is f not differentiable?



(A) -1 only (B) -1 and 1 only

(C) $-1, 0,$ and 2 only (D) $-1, 0, 1,$ and 2

PAGE 179 **7.** Let f be a function for which $\lim_{h\to 0} \dfrac{f(1+h) - f(1)}{h} = -3$. Which of the following must be true?

I. f is continuous at 1.
II. f is differentiable at 1.
III. f' is continuous at 1.

(A) I only (B) II only

(C) I and II only (D) I, II, and III

8. At what point on the graph of $f(x) = x^2 - 4$ is the tangent line parallel to the line $6x - 3y = 2$?

(A) $(1, -3)$ (B) $(1, 2)$ (C) $(2, 0)$ (D) $(2, 4)$

PAGE 176 **9.** At $x = 2$, the function $f(x) = \begin{cases} 4x+1 & \text{if } x \le 2 \\ 3x^2 - 3 & \text{if } x > 2 \end{cases}$ is

(A) Both continuous and differentiable.
(B) Continuous but not differentiable.
(C) Differentiable but not continuous.
(D) Neither continuous nor differentiable.

PAGE 173 **10.** Oil is leaking from a tank. The amount of oil, in gallons, in the tank is given by $G(t) = 4000 - 3t^2$, where t, $0 \le t \le 24$ is the number of hours past midnight.

(a) Find $G'(5)$ using the definition of the derivative.
(b) Using appropriate units, interpret the meaning of $G'(5)$ in the context of the problem.

PAGE 173 **11.** A rod of length 12 cm is heated at one end. The table below gives the temperature $T(x)$ in degrees Celsius at selected numbers x cm from the heated end.

x	0	2	5	7	9	12
$T(x)$	80	71	66	60	54	50

(a) Use the table to approximate $T'(8)$.
(b) Using appropriate units, interpret $T'(8)$ in the context of the problem.

2.3 The Derivative of a Polynomial Function; The Derivative of $y = e^x$

OBJECTIVES ***When you finish this section, you should be able to:***

1. **Differentiate a constant function (p. 184)**
2. **Differentiate a power function (p. 184)**
3. **Differentiate the sum and the difference of two functions (p. 186)**
4. **Differentiate the exponential function $y = e^x$ (p. 189)**

Finding the derivative of a function from the definition can become tedious, especially if the function f is complicated. Just as we did for limits, we derive some basic derivative formulas and some properties of derivatives that make finding a derivative simpler.

Before getting started, we introduce other notations commonly used for the derivative $f'(x)$ of a function $y = f(x)$. The most common ones are

$$y' \qquad \frac{dy}{dx} \qquad Df(x)$$

Leibniz notation $\dfrac{dy}{dx}$ may be written in several equivalent ways as

$$\frac{dy}{dx} = \frac{d}{dx}y = \frac{d}{dx}f(x)$$

where $\dfrac{d}{dx}$ is an instruction to find the derivative (with respect to the independent variable x) of the function $y = f(x)$.

In **operator notation** $Df(x)$, D is said to *operate* on the function, and the result is the derivative of f. To emphasize that the operation is performed with respect to the independent variable x, it is sometimes written $Df(x) = D_x f(x)$.

We use prime notation or Leibniz notation, or sometimes a mixture of the two, depending on which is more convenient. We do not use the notation $Df(x)$ in this book.

Since $\frac{d}{dx}a^x = f'(0)\cdot a^x$, if $f(x)=e^x$, then $\frac{d}{dx}e^x = f'(0)\cdot e^x = 1\cdot e^x = e^x$.

THEOREM Derivative of the Exponential Function $y = e^x$

The derivative of the exponential function $y = e^x$ is

$$y' = \frac{d}{dx}e^x = e^x$$

CALC CLIP **EXAMPLE 7 Differentiating an Expression Involving $y = e^x$**

Find the derivative of $f(x) = 4e^x + x^3$.

Solution

The function f is the sum of $4e^x$ and x^3. Then

$$f'(x) = \frac{d}{dx}(4e^x + x^3) \underset{\text{Sum Rule}}{=} \frac{d}{dx}(4e^x) + \frac{d}{dx}x^3 \underset{\substack{\text{Constant Multiple Rule;}\\ \text{Simple Power Rule}}}{=} 4\frac{d}{dx}e^x + 3x^2 \underset{\text{Use (1).}}{=} 4e^x + 3x^2$$

NOTE We have not forgotten $y = \ln x$. Here is its derivative:

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

Use this result for now. We do not have the necessary mathematics to prove it until Chapter 3.

NOW WORK Problem 25 and AP® Practice Problems 4 and 9.

Now we know $\frac{d}{dx}e^x = e^x$. To find the derivative of $f(x) = a^x$, $a > 0$ and $a \neq 1$, we need more information. See Chapter 3.

2.3 Assess Your Understanding

Concepts and Vocabulary

[PAGE 185] **1.** $\frac{d}{dx}\pi^2 =$ ____; $\frac{d}{dx}x^3 =$ ____.

2. When n is a positive integer, the Simple Power Rule states that $\frac{d}{dx}x^n =$ ____.

3. *True or False* The derivative of a power function of degree greater than 1 is also a power function.

4. If k is a constant and f is a differentiable function, then $\frac{d}{dx}[kf(x)] =$ ____.

5. The derivative of $f(x) = e^x$ is ____.

6. *True or False* The derivative of an exponential function $f(x) = a^x$, where $a > 0$ and $a \neq 1$, is always a constant multiple of a^x.

Skill Building

In Problems 7–26, find the derivative of each function using the formulas of this section. (a, b, c, and d, when they appear, are constants.)

[PAGE 187] **7.** $f(x) = 3x + \sqrt{2}$

8. $f(x) = 5x - \pi$

9. $f(x) = x^2 + 3x + 4$

10. $f(x) = 4x^4 + 2x^2 - 2$

11. $f(u) = 8u^5 - 5u + 1$

12. $f(u) = 9u^3 - 2u^2 + 4u + 4$

13. $f(s) = as^3 + \frac{3}{2}s^2$

14. $f(s) = 4 - \pi s^2$

15. $f(t) = \frac{1}{3}(t^5 - 8)$

16. $f(x) = \frac{1}{5}(x^7 - 3x^2 + 2)$

17. $f(t) = \frac{t^3 + 2}{5}$

18. $f(x) = \frac{x^7 - 5x}{9}$

19. $f(x) = \frac{x^3 + 2x + 1}{7}$

20. $f(x) = \frac{1}{a}(ax^2 + bx + c), a \neq 0$

21. $f(x) = ax^2 + bx + c$

22. $f(x) = ax^3 + bx^2 + cx + d$

23. $f(x) = 4e^x$

24. $f(x) = -\frac{1}{2}e^x$

[PAGE 190] **25.** $f(u) = 5u^2 - 2e^u$

26. $f(u) = 3e^u + 10$

In Problems 27–32, find each derivative.

27. $\frac{d}{dt}\left(\sqrt{3}\,t + \frac{1}{2}\right)$

28. $\frac{d}{dt}\left(\frac{2t^4 - 5}{8}\right)$

29. $\frac{dA}{dR}$ if $A(R) = \pi R^2$

30. $\frac{dC}{dR}$ if $C = 2\pi R$

[PAGE 186] **31.** $\frac{dV}{dr}$ if $V = \frac{4}{3}\pi r^3$

32. $\frac{dP}{dT}$ if $P = 0.2T$

In Problems 33–36:

(a) Find the slope of the tangent line to the graph of each function f at the indicated point.

(b) Find an equation of the tangent line at the point.

(c) Find an equation of the normal line at the point.

(d) Graph f and the tangent line and normal line found in (b) and (c) on the same set of axes.

[PAGE 188] **33.** $f(x) = x^3 + 3x - 1$ at $(0, -1)$

34. $f(x) = x^4 + 2x - 1$ at $(1, 2)$

35. $f(x) = e^x + 5x$ at $(0, 1)$

36. $f(x) = 4 - e^x$ at $(0, 3)$

In Problems 37–42:

(a) Find the points, if any, at which the graph of each function f has a horizontal tangent line.

(b) Find an equation for each horizontal tangent line.

(c) Solve the inequality $f'(x) > 0$.

(d) Solve the inequality $f'(x) < 0$.

(e) Graph f and any horizontal lines found in (b) on the same set of axes.

(f) Describe the graph of f for the results obtained in parts (c) and (d).

37. $f(x) = 3x^2 - 12x + 4$ **38.** $f(x) = x^2 + 4x - 3$

39. $f(x) = x + e^x$ **40.** $f(x) = 2e^x - 1$

41. $f(x) = x^3 - 3x + 2$ **42.** $f(x) = x^4 - 4x^3$

43. Rectilinear Motion At t seconds, an object in rectilinear motion is s meters from the origin, where $s(t) = t^3 - t + 1$. Find the velocity of the object at $t = 0$ and at $t = 5$.

44. Rectilinear Motion At t seconds, an object in rectilinear motion is s meters from the origin, where $s(t) = t^4 - t^3 + 1$. Find the velocity of the object at $t = 0$ and at $t = 1$.

Rectilinear Motion *In Problems 45 and 46, each position function gives the signed distance s from the origin at time t of an object in rectilinear motion:*

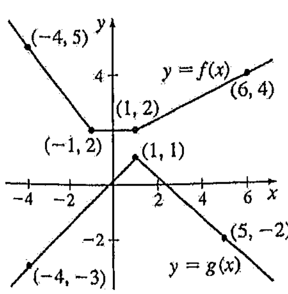
(a) Find the velocity v of the object at any time t.

(b) When is the velocity of the object 0?

45. $s(t) = 2 - 5t + t^2$ **46.** $s(t) = t^3 - \frac{9}{2}t^2 + 6t + 4$

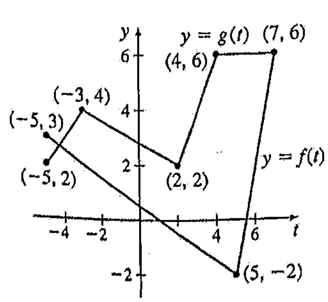
In Problems 47 and 48, use the graphs to find each derivative.

47. Let $u(x) = f(x) + g(x)$ and $v(x) = f(x) - g(x)$.



(a) $u'(0)$ (b) $u'(4)$
(c) $v'(-2)$ (d) $v'(6)$
(e) $3u'(5)$ (f) $-2v'(3)$

48. Let $F(t) = f(t) + g(t)$ and $G(t) = g(t) - f(t)$.



(a) $F'(0)$ (b) $F'(3)$
(c) $F'(-4)$ (d) $G'(-2)$
(e) $G'(-1)$ (f) $G'(6)$

In Problems 49 and 50, for each function f:

(a) Find $f'(x)$ by expanding $f(x)$ and differentiating the polynomial.

CAS *(b) Find $f'(x)$ using a CAS.*

(c) Show that the results found in parts (a) and (b) are equivalent.

49. $f(x) = (2x - 1)^3$ **50.** $f(x) = (x^2 + x)^4$

Applications and Extensions

In Problems 51–56, find each limit.

51. $\lim_{h \to 0} \dfrac{5\left(\frac{1}{2} + h\right)^8 - 5\left(\frac{1}{2}\right)^8}{h}$ **52.** $\lim_{h \to 0} \dfrac{6(2 + h)^5 - 6 \cdot 2^5}{h}$

53. $\lim_{h \to 0} \dfrac{\sqrt{3}(8 + h)^5 - \sqrt{3} \cdot 8^5}{h}$ **54.** $\lim_{h \to 0} \dfrac{\pi(1 + h)^{10} - \pi}{h}$

55. $\lim_{h \to 0} \dfrac{a(x + h)^3 - ax^3}{h}$ **56.** $\lim_{h \to 0} \dfrac{b(x + h)^n - bx^n}{h}$

In Problems 57–62, find an equation of the tangent line(s) to the graph of the function f that is (are) parallel to the line L.

57. $f(x) = 3x^2 - x$; $L: y = 5x$

58. $f(x) = 2x^3 + 1$; $L: y = 6x - 1$

59. $f(x) = e^x$; $L: y - x - 5 = 0$

60. $f(x) = -2e^x$; $L: y + 2x - 8 = 0$

61. $f(x) = \frac{1}{3}x^3 - x^2$; $L: y = 3x - 2$

62. $f(x) = x^3 - x$; $L: x + y = 0$

63. Tangent Lines Let $f(x) = 4x^3 - 3x - 1$.

(a) Find an equation of the tangent line to the graph of f at $x = 2$.

(b) Find the coordinates of any points on the graph of f where the tangent line is parallel to $y = x + 12$.

(c) Find an equation of the tangent line to the graph of f at any points found in (b).

(d) Graph f, the tangent line found in (a), the line $y = x + 12$, and any tangent lines found in (c) on the same screen.

64. Tangent Lines Let $f(x) = x^3 + 2x^2 + x - 1$.

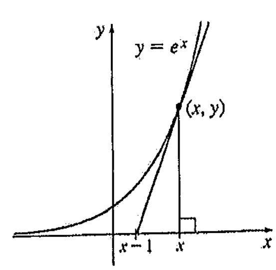
(a) Find an equation of the tangent line to the graph of f at $x = 0$.

(b) Find the coordinates of any points on the graph of f where the tangent line is parallel to $y = 3x - 2$.

(c) Find an equation of the tangent line to the graph of f at any points found in (b).

(d) Graph f, the tangent line found in (a), the line $y = 3x - 2$, and any tangent lines found in (c) on the same screen.

65. Tangent Line Show that the line perpendicular to the x-axis and containing the point (x, y) on the graph of $y = e^x$ and the tangent line to the graph of $y = e^x$ at the point (x, y) intersect the x-axis 1 unit apart. See the figure.



66. **Tangent Line** Show that the tangent line to the graph of $y=x^n$, $n \geq 2$ an integer, at $(1, 1)$ has y-intercept $1-n$.

67. **Tangent Lines** If n is an odd positive integer, show that the tangent lines to the graph of $y=x^n$ at $(1, 1)$ and at $(-1, -1)$ are parallel.

68. **Tangent Line** If the line $3x-4y=0$ is tangent to the graph of $y=x^3+k$ in the first quadrant, find k.

69. **Tangent Line** Find the constants a, b, and c so that the graph of $y=ax^2+bx+c$ contains the point $(-1, 1)$ and is tangent to the line $y=2x$ at $(0, 0)$.

70. **Tangent Line** Let T be the tangent line to the graph of $y=x^3$ at the point $\left(\frac{1}{2}, \frac{1}{8}\right)$. At what other point Q on the graph of $y=x^3$ does the line T intersect the graph? What is the slope of the tangent line at Q?

71. **Military Tactics** A dive bomber is flying from right to left along the graph of $y=x^2$. When a rocket bomb is released, it follows a path that is approximately along the tangent line. Where should the pilot release the bomb if the target is at $(1, 0)$?

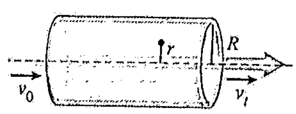
72. **Military Tactics** Answer the question in Problem 71 if the plane is flying from right to left along the graph of $y=x^3$.

73. **Fluid Dynamics** The velocity v of a liquid flowing through a cylindrical tube is given by the **Hagen–Poiseuille equation** $v=k(R^2-r^2)$, where R is the radius of the tube, k is a constant that depends on the length of the tube and the velocity of the liquid at its ends, and r is the variable distance of the liquid from the center of the tube. See the figure below.

(a) Find the rate of change of v with respect to r at the center of the tube.

(b) What is the rate of change halfway from the center to the wall of the tube?

(c) What is the rate of change at the wall of the tube?



74. **Rate of Change** Water is leaking out of a swimming pool that measures 20 ft by 40 ft by 6 ft. The amount of water in the pool at a time t is $W(t)=35{,}000-20t^2$ gallons, where t equals the number of hours since the pool was last filled. At what rate is the water leaking when $t=2$ h?

75. **Luminosity of the Sun** The luminosity L of a star is the rate at which it radiates energy. This rate depends on the temperature T and surface area A of the star's photosphere (the gaseous surface that emits the light). Luminosity is modeled by the equation $L=\sigma AT^4$, where σ is a constant known as the **Stefan–Boltzmann constant**, and T is expressed in the absolute (Kelvin) scale for which 0 K is absolute zero. As with most stars, the Sun's temperature has gradually increased over the 6 billion years of its existence, causing its luminosity to slowly increase.

(a) Find the rate at which the Sun's luminosity changes with respect to the temperature of its photosphere. Assume that the surface area A remains constant.

(b) Find the rate of change at the present time. The temperature of the photosphere is currently 5800 K (10,000 °F), the radius of the photosphere is $r=6.96 \times 10^8$ m, and $\sigma=5.67 \times 10^{-8} \frac{\text{W}}{\text{m}^2\,\text{K}^4}$.

(c) Assuming that the rate found in (b) remains constant, how much would the luminosity change if its photosphere temperature increased by 1 K (1 °C or 1.8 °F)? Compare this change to the present luminosity of the Sun.

76. **Medicine: Poiseuille's Equation** The French physician Poiseuille discovered that the volume V of blood (in cubic centimeters per unit time) flowing through an artery with inner radius R (in centimeters) can be modeled by

$$V(R)=kR^4$$

where $k=\frac{\pi}{8\nu l}$ is constant (here ν represents the viscosity of blood and l is the length of the artery).

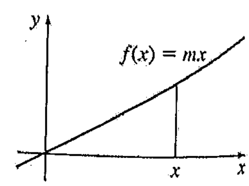
(a) Find the rate of change of the volume V of blood flowing through the artery with respect to the radius R.

(b) Find the rate of change when $R=0.03$ and when $R=0.04$.

(c) If the radius of a partially clogged artery is increased from 0.03 to 0.04 cm, estimate the effect on the rate of change of the volume V with respect to R of the blood flowing through the enlarged artery.

(d) How do you interpret the results found in (b) and (c)?

77. **Derivative of an Area** Let $f(x)=mx$, $m>0$. Let $F(x)$, $x>0$, be defined as the area of the shaded region in the figure. Find $F'(x)$.



78. **The Difference Rule** Prove that if f and g are differentiable functions and if $F(x)=f(x)-g(x)$, then

$$F'(x)=f'(x)-g'(x)$$

79. **Simple Power Rule** Let $f(x)=x^n$, where n is a positive integer. Use a factoring principle to show that

$$f'(c)=\lim_{x \to c} \frac{f(x)-f(c)}{x-c}=nc^{n-1}$$

80. **Normal Lines** For what nonnegative number b is the line given by $y=-\frac{1}{3}x+b$ normal to the graph of $y=x^3$?

81. **Normal Lines** Let N be the normal line to the graph of $y=x^2$ at the point $(-2, 4)$. At what other point Q does N meet the graph?

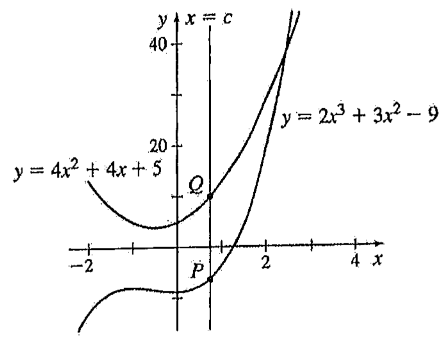
Challenge Problems

82. **Tangent Line** Find a, b, c, d so that the tangent line to the graph of the cubic $y = ax^3 + bx^2 + cx + d$ at the point $(1, 0)$ is $y = 3x - 3$ and at the point $(2, 9)$ is $y = 18x - 27$.

83. **Tangent Line** Find the fourth degree polynomial that contains the origin and to which the line $x + 2y = 14$ is tangent at both $x = 4$ and $x = -2$.

84. **Tangent Lines** Find equations for all the lines containing the point $(1, 4)$ that are tangent to the graph of $y = x^3 - 10x^2 + 6x - 2$. At what points do each of the tangent lines touch the graph?

85. The line $x = c$, where $c > 0$, intersects the cubic $y = 2x^3 + 3x^2 - 9$ at the point P and intersects the parabola $y = 4x^2 + 4x + 5$ at the point Q, as shown in the figure on the right.
    (a) If the line tangent to the cubic at the point P is parallel to the line tangent to the parabola at the point Q, find the number c.
    (b) Write an equation for each of the two tangent lines described in (a).



86. $f(x) = Ax^2 + B$, $A > 0$.
    (a) Find c, $c > 0$, in terms of A so that the tangent lines to the graph of f at $(c, f(c))$ and $(-c, f(-c))$ are perpendicular.
    (b) Find the slopes of the tangent lines in (a).
    (c) Find the coordinates, in terms of A and B, of the point of intersection of the tangent lines in (a).

Preparing for the AP® Exam

AP® Practice Problems

PAGE 185 1. If $g(x) = x$, then $g'(7) =$

(A) 0 (B) 1 (C) 7 (D) $\frac{49}{2}$

PAGE 188 2. The line $x + y = k$, where k is a constant, is a tangent line to the graph of the function $f(x) = x^2 - 5x + 2$. What is the value of k?

(A) -1 (B) 2 (C) -2 (D) -4

PAGE 189 3. An object moves along the x-axis so that its position at time t is $x(t) = 3t^2 - 9t + 7$. For what time t is the velocity of the object zero?

(A) -3 (B) 3 (C) $\frac{3}{2}$ (D) 7

PAGE 190 4. If $f(x) = e^x$, then $\ln(f'(3)) =$

(A) 3 (B) 0 (C) e^3 (D) $\ln 3$

PAGE 188 5. An equation of the normal line to the graph of $g(x) = x^3 + 2x^2 - 2x + 1$ at the point where $x = -2$ is

(A) $x + 2y = 12$ (B) $x - 2y = 8$
(C) $2x + y = -9$ (D) $x + 2y = 8$

PAGE 187 6. The line $9x - 16y = 0$ is tangent to the graph of $f(x) = 3x^3 + k$, where k is a constant, at a point in the first quadrant. Find k.

(A) $\frac{3}{32}$ (B) $\frac{3}{16}$ (C) $\frac{3}{64}$ (D) $\frac{9}{64}$

7. If $f(x) = 1 + |x - 4|$, find $f'(4)$.

(A) -1 (B) 0 (C) 1 (D) $f'(4)$ does not exist.

PAGE 187 8. The cost C (in dollars) of manufacturing x units of a product is $C(x) = 0.3x^2 + 4.02x + 3500$. What is the rate of change of C when $x = 1000$ units?

(A) 307.52 (B) 0.60402 (C) 604.02 (D) 1020

PAGE 190 9. $\frac{d}{dx}(5 \ln x) =$

(A) $\frac{1}{5x}$ (B) $5e^x$ (C) $-\frac{5}{\ln x}$ (D) $\frac{5}{x}$

PAGE 188 10. For the function $f(x) = x^2 + 4$
    (a) Find $f'(1)$.
    (b) Find an equation of the tangent line to the graph of f at $x = 1$.
    (c) Find $f'(-4)$.
    (d) Find an equation of the tangent line to the graph of f at $x = -4$.
    (e) Find the point of intersection of the two tangent lines found in (b) and (d).

PAGE 188 11. Which is an equation of the tangent line to the graph of $f(x) = x^4 + 3x^2 + 2$ at the point where $f'(x) = 2$?

(A) $y = 2x + 2$ (B) $y = 2x + 2.929$
(C) $y = 2x + 1.678$ (D) $y = 2x - 2.929$

The velocity v of the falling object is

$$v = \frac{ds}{dt} = \frac{d}{dt}(-ct^2) = -2ct$$

and its acceleration a is

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = -2c$$

which is a constant. Usually, we denote the constant $2c$ by g so $c = \frac{1}{2}g$. Then

$$a = -g \qquad v = -gt \qquad s = -\frac{1}{2}gt^2$$

The number g is called the **acceleration due to gravity**. For our planet, g is approximately 32 ft/s^2, or 9.8 m/s^2. On the planet Jupiter, $g \approx 26.0$ m/s^2, and on our moon, $g \approx 1.60$ m/s^2.

2.4 Assess Your Understanding

Concepts and Vocabulary

1. *True or False* The derivative of a product is the product of the derivatives.
2. If $F(x) = f(x)g(x)$, then $F'(x) =$ ______.
3. *True or False* $\frac{d}{dx}x^n = nx^{n+1}$, for any integer n.
4. If f and $g \neq 0$ are two differentiable functions, then $\frac{d}{dx}\frac{f(x)}{g(x)} =$ ______.
5. *True or False* $f(x) = \frac{e^x}{x^2}$ can be differentiated using the Quotient Rule or by writing $f(x) = \frac{e^x}{x^2} = x^{-2}e^x$ and using the Product Rule.
6. If $g \neq 0$ is a differentiable function, then $\frac{d}{dx}\frac{1}{g(x)} =$ ______.
7. If $f(x) = x$, then $f''(x) =$ ______.
8. When an object in rectilinear motion is modeled by the position function $s = s(t)$, then the acceleration a of the object at time t is given by $a = a(t) =$ ______.

Skill Building

In Problems 9–40, find the derivative of each function.

9. PAGE 195 $f(x) = xe^x$
10. $f(x) = x^2e^x$
11. $f(x) = x^2(x^3 - 1)$
12. $f(x) = x^4(x + 5)$
13. PAGE 196 $f(x) = (3x^2 - 5)(2x + 1)$
14. $f(x) = (3x - 2)(4x + 5)$
15. $s(t) = (2t^5 - t)(t^3 - 2t + 1)$
16. $F(u) = (u^4 - 3u^2 + 1)(u^2 - u + 2)$
17. $f(x) = (x^3 + 1)(e^x + 1)$
18. $f(x) = (x^2 + 1)(e^x + x)$
19. $g(s) = \frac{2s}{s + 1}$
20. $F(z) = \frac{z + 1}{2z}$
21. $G(u) = \frac{1 - 2u}{1 + 2u}$
22. $f(w) = \frac{1 - w^2}{1 + w^2}$
23. PAGE 197 $f(x) = \frac{4x^2 - 2}{3x + 4}$
24. $f(x) = \frac{-3x^3 - 1}{2x^2 + 1}$
25. PAGE 197 $f(w) = \frac{1}{w^3 - 1}$
26. $g(v) = \frac{1}{v^2 + 5v - 1}$
27. $s(t) = t^{-3}$
28. $G(u) = u^{-4}$
29. $f(x) = -\frac{4}{e^x}$
30. $f(x) = \frac{3}{4e^x}$
31. PAGE 198 $f(x) = \frac{10}{x^4} + \frac{3}{x^2}$
32. $f(x) = \frac{2}{x^5} - \frac{3}{x^3}$
33. $f(x) = 3x^3 - \frac{1}{3x^2}$
34. $f(x) = x^5 - \frac{5}{x^5}$
35. $s(t) = \frac{1}{t} - \frac{1}{t^2} + \frac{1}{t^3}$
36. $s(t) = \frac{1}{t} + \frac{1}{t^2} + \frac{1}{t^3}$
37. $f(x) = \frac{e^x}{x^2}$
38. $f(x) = \frac{x^2}{e^x}$
39. $f(x) = \frac{x^2 + 1}{xe^x}$
40. $f(x) = \frac{xe^x}{x^2 - x}$

In Problems 41–54, find f' and f'' for each function.

41. PAGE 199 $f(x) = 3x^2 + x - 2$
42. $f(x) = -5x^2 - 3x$
43. $f(x) = e^x - 3$
44. $f(x) = x - e^x$
45. PAGE 200 $f(x) = (x + 5)e^x$
46. $f(x) = 3x^4e^x$
47. $f(x) = (2x + 1)(x^3 + 5)$
48. $f(x) = (3x - 5)(x^2 - 2)$
49. $f(x) = x + \frac{1}{x}$
50. $f(x) = x - \frac{1}{x}$
51. $f(t) = \frac{t^2 - 1}{t}$
52. $f(u) = \frac{u + 1}{u}$
53. $f(x) = \frac{e^x + x}{x}$
54. $f(x) = \frac{e^x}{x}$
55. Find y' and y'' for (a) $y = \frac{1}{x}$ and (b) $y = \frac{2x - 5}{x}$.
56. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for (a) $y = \frac{5}{x^2}$ and (b) $y = \frac{2 - 3x}{x}$.

Rectilinear Motion *In Problems 57–60, find the velocity $v = v(t)$ and acceleration $a = a(t)$ of an object in rectilinear motion whose signed distance s from the origin at time t is modeled by the position function $s = s(t)$.*

57. $s(t) = 16t^2 + 20t$

58. $s(t) = 16t^2 + 10t + 1$

59. $s(t) = 4.9t^2 + 4t + 4$

60. $s(t) = 4.9t^2 + 5t$

In Problems 61–68, find the indicated derivative.

61. $f^{(4)}(x)$ if $f(x) = x^3 - 3x^2 + 2x - 5$

62. $f^{(5)}(x)$ if $f(x) = 4x^3 + x^2 - 1$

63. $\frac{d^8}{dt^8}\left(\frac{1}{8}t^8 - \frac{1}{7}t^7 + t^5 - t^3\right)$

64. $\frac{d^6}{dt^6}(t^6 + 5t^5 - 2t + 4)$

65. $\frac{d^7}{du^7}(e^u + u^2)$

66. $\frac{d^{10}}{du^{10}}(2e^u)$

67. $\frac{d^5}{dx^5}(-e^x)$

68. $\frac{d^8}{dx^8}(12x - e^x)$

In Problems 69–72:

(a) Find the slope of the tangent line for each function f at the given point.
(b) Find an equation of the tangent line to the graph of each function f at the given point.
(c) Find the points, if any, where the graph of the function has a horizontal tangent line.
(d) Graph each function, the tangent line found in (b), and any tangent lines found in (c) on the same set of axes.

69. $f(x) = \frac{x^2}{x-1}$ at $\left(-1, -\frac{1}{2}\right)$

70. $f(x) = \frac{x}{x+1}$ at $(0, 0)$

71. $f(x) = \frac{x^3}{x+1}$ at $\left(1, \frac{1}{2}\right)$

72. $f(x) = \frac{x^2+1}{x}$ at $\left(2, \frac{5}{2}\right)$

In Problems 73–80:

(a) Find the points, if any, at which the graph of each function f has a horizontal tangent line.
(b) Find an equation for each horizontal tangent line.
(c) Solve the inequality $f'(x) > 0$.
(d) Solve the inequality $f'(x) < 0$.
(e) Graph f and any horizontal lines found in (b) on the same set of axes.
(f) Describe the graph of f for the results obtained in (c) and (d).

73. $f(x) = (x+1)(x^2 - x - 11)$

74. $f(x) = (3x^2 - 2)(2x + 1)$

75. $f(x) = \frac{x^2}{x+1}$

76. $f(x) = \frac{x^2+1}{x}$

77. $f(x) = xe^x$

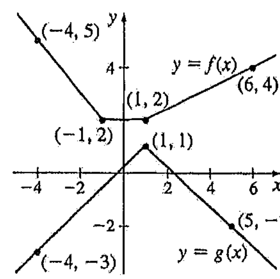
78. $f(x) = x^2e^x$

79. $f(x) = \frac{x^2-3}{e^x}$

80. $f(x) = \frac{e^x}{x^2+1}$

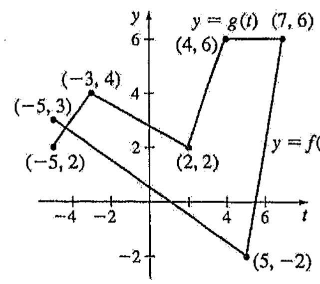
In Problems 81 and 82, use the graphs to determine each derivative.

81. Let $u(x) = f(x) \cdot g(x)$ and $v(x) = \frac{g(x)}{f(x)}$.



(a) $u'(0)$
(b) $u'(4)$
(c) $v'(-2)$
(d) $v'(6)$
(e) $\frac{d}{dx}\frac{1}{f(x)}$ at $x = -2$
(f) $\frac{d}{dx}\frac{1}{g(x)}$ at $x = 4$

82. Let $F(t) = f(t) \cdot g(t)$ and $G(t) = \frac{f(t)}{g(t)}$.



(a) $F'(0)$
(b) $F'(3)$
(c) $F'(-4)$
(d) $G'(-2)$
(e) $G'(-1)$
(f) $\frac{d}{dt}\frac{1}{f(t)}$ at $t = 3$

Applications and Extensions

83. **Vertical Motion** An object is propelled vertically upward from the ground with an initial velocity of 39.2 m/s. The distance s (in meters) of the object from the ground after t seconds is given by the position function $s = s(t) = -4.9t^2 + 39.2t$.

(a) What is the velocity of the object at time t?
(b) When will the object reach its maximum height?
(c) What is the maximum height?
(d) What is the acceleration of the object at any time t?
(e) How long is the object in the air?
(f) What is the velocity of the object upon impact with the ground? What is its speed?
(g) What is the total distance traveled by the object?

84. Vertical Motion A ball is thrown vertically upward from a height of 6 ft with an initial velocity of 80 ft/s. The distance s (in feet) of the ball from the ground after t seconds is given by the position function $s = s(t) = 6 + 80t - 16t^2$.

(a) What is the velocity of the ball after 2 s?
(b) When will the ball reach its maximum height?
(c) What is the maximum height the ball reaches?
(d) What is the acceleration of the ball at any time t?
(e) How long is the ball in the air?
(f) What is the velocity of the ball upon impact with the ground? What is its speed?
(g) What is the total distance traveled by the ball?

85. Environmental Cost The cost C, in thousands of dollars, for the removal of a pollutant from a certain lake is given by the function $C(x) = \dfrac{5x}{110 - x}$, where x is the percent of pollutant removed.

(a) What is the domain of C?
(b) Graph C.
(c) What is the cost to remove 80% of the pollutant?
(d) Find $C'(x)$, the rate of change of the cost C with respect to the amount of pollutant removed.
(e) Find the rate of change of the cost for removing 40%, 60%, 80%, and 90% of the pollutant.
(f) Interpret the answers found in (e).

86. Investing in Fine Art The value V of a painting t years after it is purchased is modeled by the function

$$V(t) = \frac{100t^2 + 50}{t} + 400 \quad 1 \le t \le 5$$

(a) Find the rate of change in the value V with respect to time.
(b) What is the rate of change in value after 2 years?
(c) What is the rate of change in value after 3 years?
(d) Interpret the answers in (b) and (c).

87. Drug Concentration The concentration of a drug in a patient's blood t hours after injection is given by the function $f(t) = \dfrac{0.4t}{2t^2 + 1}$ (in milligrams per liter).

(a) Find the rate of change of the concentration with respect to time.
(b) What is the rate of change of the concentration after 10 min? After 30 min? After 1 hour?
(c) Interpret the answers found in (b).
(d) Graph f for the first 5 hours after administering the drug.
(e) From the graph, approximate the time (in minutes) at which the concentration of the drug is highest. What is the highest concentration of the drug in the patient's blood?

88. Population Growth A population of 1000 bacteria is introduced into a culture and grows in number according to the formula $P(t) = 1000\left(1 + \dfrac{4t}{100 + t^2}\right)$, where t is measured in hours.

(a) Find the rate of change in population with respect to time.
(b) What is the rate of change in population at $t = 1$, $t = 2$, $t = 3$, and $t = 4$?
(c) Interpret the answers found in (b).
(d) Graph $P = P(t)$, $0 \le t \le 20$.
(e) From the graph, approximate the time (in hours) when the population is the greatest. What is the maximum population of the bacteria in the culture?

89. Economics The price-demand function for a popular e-book is given by $D(p) = \dfrac{100{,}000}{p^2 + 10p + 50}$, $4 \le p \le 20$, where $D = D(p)$ is the quantity demanded at the price p dollars.

(a) Find $D'(p)$, the rate of change of demand with respect to price.
(b) Find $D'(5)$, $D'(10)$, and $D'(15)$.
(c) Interpret the results found in (b).

90. Intensity of Light The intensity of illumination I on a surface is inversely proportional to the square of the distance r from the surface to the source of light. If the intensity is 1000 units when the distance is 1 m from the light, find the rate of change of the intensity with respect to the distance when the source is 10 meters from the surface.

PAGE 198 **91. Ideal Gas Law** The Ideal Gas Law, used in chemistry and thermodynamics, relates the pressure p, the volume V, and the absolute temperature T (in Kelvin) of a gas, using the equation $pV = nRT$, where n is the amount of gas (in moles) and $R = 8.31$ is the ideal gas constant. In an experiment, a spherical gas container of radius r meters is placed in a pressure chamber and is slowly compressed while keeping its temperature at 273 K.

(a) Find the rate of change of the pressure p with respect to the radius r of the chamber.
Hint: The volume V of a sphere is $V = \frac{4}{3}\pi r^3$.
(b) Interpret the sign of the answer found in (a).
(c) If the sphere contains 1.0 mol of gas, find the rate of change of the pressure when $r = \frac{1}{4}$ m.
Note: The metric unit of pressure is the pascal, Pa.

92. Body Density The density ρ of an object is its mass m divided by its volume V; that is, $\rho = \dfrac{m}{V}$. If a person dives below the surface of the ocean, the water pressure on the diver will steadily increase, compressing the diver and therefore increasing body density. Suppose the diver is modeled as a sphere of radius r.

(a) Find the rate of change of the diver's body density with respect to the radius r of the sphere.
Hint: The volume V of a sphere is $V = \frac{4}{3}\pi r^3$.
(b) Interpret the sign of the answer found in (a).
(c) Find the rate of change of the diver's body density when the radius is 45 cm and the mass is 80,000 g (80 kg).

Jerk and Snap *Problems 93–96 use the following discussion:* Suppose that an object is moving in rectilinear motion so that its signed distance s from the origin at time t is given by the position function $s = s(t)$. The velocity $v = v(t)$ of the object at time t is the rate of change of s with respect to time, namely, $v = v(t) = \dfrac{ds}{dt}$. The acceleration $a = a(t)$

of the object at time t is the rate of change of the velocity with respect to time,

$$a = a(t) = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}$$

There are also physical interpretations of the third derivative and the fourth derivative of $s = s(t)$. The **jerk** $J = J(t)$ of the object at time t is the rate of change of the acceleration a with respect to time; that is,

$$J = J(t) = \frac{da}{dt} = \frac{d}{dt}\left(\frac{dv}{dt}\right) = \frac{d^2v}{dt^2} = \frac{d^3s}{dt^3}$$

The **snap** $S = S(t)$ of the object at time t is the rate of change of the jerk J with respect to time; that is,

$$S = S(t) = \frac{dJ}{dt} = \frac{d^2a}{dt^2} = \frac{d^3v}{dt^3} = \frac{d^4s}{dt^4}$$

Engineers take jerk into consideration when designing elevators, aircraft, and cars. In these cases, they try to minimize jerk, making for a smooth ride. But when designing thrill rides, such as roller coasters, the jerk is increased, making for an exciting experience.

93. **Rectilinear Motion** As an object in rectilinear motion moves, its signed distance s from the origin at time t is given by the position function $s = s(t) = t^3 - t + 1$, where s is in meters and t is in seconds.

(a) Find the velocity v, acceleration a, jerk J, and snap S of the object at time t.

(b) When is the velocity of the object 0 m/s?

(c) Find the acceleration of the object at $t = 2$ and at $t = 5$.

(d) Does the jerk of the object ever equal 0 m/s^3?

(e) How would you interpret the snap for this object in rectilinear motion?

94. **Rectilinear Motion** As an object in rectilinear motion moves, its signed distance s from the origin at time t is given by the position function $s = s(t) = \frac{1}{6}t^4 - t^2 + \frac{1}{2}t + 4$, where s is in meters and t is in seconds.

(a) Find the velocity v, acceleration a, jerk J, and snap S of the object at any time t.

(b) Find the velocity of the object at $t = 0$ and at $t = 3$.

(c) Find the acceleration of the object at $t = 0$. Interpret your answer.

(d) Is the jerk of the object constant? In your own words, explain what the jerk says about the acceleration of the object.

(e) How would you interpret the snap for this object in rectilinear motion?

95. **Elevator Ride Quality** The ride quality of an elevator depends on several factors, two of which are acceleration and jerk. In a study of 367 persons riding in a 1600-kg elevator that moves at an average speed of 4 m/s, the majority of riders were comfortable in an elevator with vertical motion given by

$$s(t) = 4t + 0.8t^2 + 0.333t^3$$

(a) Find the acceleration that the riders found acceptable.

(b) Find the jerk that the riders found acceptable.

Source: Elevator Ride Quality, January 2007, http://www.lift-report.de/index.php/news/176/368/Elevator-Ride-Quality

96. **Elevator Ride Quality** In a hospital, the effects of high acceleration or jerk may be harmful to patients, so the acceleration and jerk need to be lower than in standard elevators. It has been determined that a 1600-kg elevator that is installed in a hospital and that moves at an average speed of 4 m/s should have vertical motion

$$s(t) = 4t + 0.55t^2 + 0.1167t^3$$

(a) Find the acceleration of a hospital elevator.

(b) Find the jerk of a hospital elevator.

Source: Elevator Ride Quality, January 2007, http://www.lift-report.de/index.php/news/176/368/Elevator-Ride-Quality

97. **Current Density in a Wire** The current density J in a wire is a measure of how much an electrical current is compressed as it flows through a wire and is modeled by the function $J(A) = \frac{I}{A}$, where I is the current (in amperes) and A is the cross-sectional area of the wire. In practice, current density, rather than merely current, is often important. For example, superconductors lose their superconductivity if the current density is too high.

(a) As current flows through a wire, it heats the wire, causing it to expand in area A. If a constant current is maintained in a cylindrical wire, find the rate of change of the current density J with respect to the radius r of the wire.

(b) Interpret the sign of the answer found in (a).

(c) Find the rate of change of current density with respect to the radius r when a current of 2.5 amps flows through a wire of radius $r = 0.50$ mm.

98. **Derivative of a Reciprocal Function** Prove that if a function g is differentiable, then $\frac{d}{dx}\frac{1}{g(x)} = -\frac{g'(x)}{[g(x)]^2}$, provided $g(x) \neq 0$.

99. **Extended Product Rule** Show that if f, g, and h are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)h(x)] = f(x)g(x)h'(x) + f(x)g'(x)h(x) + f'(x)g(x)h(x)$$

From this, deduce that

$$\frac{d}{dx}[f(x)]^3 = 3[f(x)]^2 f'(x)$$

In Problems 100–105, use the Extended Product Rule (Problem 99) to find y'.

100. $y = (x^2 + 1)(x - 1)(x + 5)$

101. $y = (x - 1)(x^2 + 5)(x^3 - 1)$

102. $y = (x^4 + 1)^3$

103. $y = (x^3 + 1)^3$

104. $y = (3x + 1)\left(1 + \frac{1}{x}\right)(x^{-5} + 1)$

105. $y = \left(1 - \frac{1}{x}\right)\left(1 - \frac{1}{x^2}\right)\left(1 - \frac{1}{x^3}\right)$

106. **(Further) Extended Product Rule** Write a formula for the derivative of the product of four differentiable functions. That is, find a formula for $\frac{d}{dx}[f_1(x)f_2(x)f_3(x)f_4(x)]$. Also find a formula for $\frac{d}{dx}[f(x)]^4$.

107. If f and g are differentiable functions, show that if $F(x)=\dfrac{1}{f(x)g(x)}$, then

$$F'(x)=-F(x)\left[\frac{f'(x)}{f(x)}+\frac{g'(x)}{g(x)}\right]$$

provided $f(x)\neq 0$, $g(x)\neq 0$.

108. Higher-Order Derivatives If $f(x)=\dfrac{1}{1-x}$, find a formula for the nth derivative of f. That is, find $f^{(n)}(x)$.

109. Let $f(x)=\dfrac{x^6-x^4+x^2}{x^4+1}$. Rewrite f in the form $(x^4+1)f(x)=x^6-x^4+x^2$. Now find $f'(x)$ without using the quotient rule.

110. If f and g are differentiable functions with $f\neq -g$, find the derivative of $\dfrac{fg}{f+g}$.

CAS **111.** $f(x)=\dfrac{2x}{x+1}$.

(a) Use technology to find $f'(x)$.
(b) Simplify f' to a single fraction using either algebra or a CAS.
(c) Use technology to find $f^{(5)}(x)$.
Hint: Your CAS may have a method for finding higher-order derivatives without finding other derivatives first.

Challenge Problems

112. Suppose f and g have derivatives up to the fourth order. Find the first four derivatives of the product fg and simplify the answers. In particular, show that the fourth derivative is

$$\frac{d^4}{dx^4}(fg)=f^{(4)}g+4f^{(3)}g^{(1)}+6f^{(2)}g^{(2)}+4f^{(1)}g^{(3)}+fg^{(4)}$$

Identify a pattern for the higher-order derivatives of fg.

113. Suppose $f_1(x),\ldots,f_n(x)$ are differentiable functions.

(a) Find $\dfrac{d}{dx}[f_1(x)\cdot\ldots\cdot f_n(x)]$.
(b) Find $\dfrac{d}{dx}\dfrac{1}{f_1(x)\cdot\ldots\cdot f_n(x)}$.

PAGE 201 6

114. Let a, b, c, and d be real numbers. Define

$$\begin{vmatrix} a & b\\ c & d\end{vmatrix}=ad-bc$$

This is called a 2×2 **determinant** and it arises in the study of linear equations. Let $f_1(x)$, $f_2(x)$, $f_3(x)$, and $f_4(x)$ be differentiable and let

$$D(x)=\begin{vmatrix} f_1(x) & f_2(x)\\ f_3(x) & f_4(x)\end{vmatrix}$$

PAGE 197

Show that

$$D'(x)=\begin{vmatrix} f_1'(x) & f_2'(x)\\ f_3(x) & f_4(x)\end{vmatrix}+\begin{vmatrix} f_1(x) & f_2(x)\\ f_3'(x) & f_4'(x)\end{vmatrix}$$

115. Let $f_0(x)=x-1$

$$f_1(x)=1+\frac{1}{x-1}$$

$$f_2(x)=1+\cfrac{1}{1+\cfrac{1}{x-1}}$$

$$f_3(x)=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{x-1}}}$$

(a) Write f_1, f_2, f_3, f_4, and f_5 in the form $\dfrac{ax+b}{cx+d}$.
(b) Using the results from (a), write the sequence of numbers representing the coefficients of x in the numerator, beginning with $f_0(x)=x-1$.
(c) Write the sequence in (b) as a recursive sequence.
Hint: Look at the sum of consecutive terms.
(d) Find f_0', f_1', f_2', f_3', f_4', and f_5'.

AP® Practice Problems

Preparing for the AP® Exam

PAGE 197 **1.** What is the instantaneous rate of change at $x=-2$ of the function $f(x)=\dfrac{x-1}{x^2+2}$?

(A) $-\frac{1}{6}$ (B) $\frac{1}{9}$ (C) $\frac{1}{2}$ (D) -1

PAGE 197 **2.** An equation of the tangent line to the graph of $f(x)=\dfrac{5x-3}{3x-6}$ at the point $(3,4)$ is

(A) $7x+3y=37$ (B) $7x+3y=33$
(C) $7x-3y=9$ (D) $13x+3y=51$

PAGE 197 **3.** If f, g, and h are nonzero differentiable functions of x, then $\dfrac{d}{dx}\left(\dfrac{gh}{f}\right)=$

(A) $\dfrac{fgh'+fg'h-f'gh}{f^2}$ (B) $\dfrac{g'h'-ghf'}{f^2}$
(C) $\dfrac{gh'+g'h}{f'}$ (D) $\dfrac{fgh'+fg'h+f'gh}{f^2}$

PAGE 195 **4.** If $y=x^3e^x$, then $\dfrac{dy}{dx}=$

(A) $3x^2e^x$ (B) $3x^2+e^x$
(C) $3x^2e^x(x+1)$ (D) $x^2e^x(x+3)$

PAGE 198 **5.** $\dfrac{d}{dt}\left(t^2-\dfrac{1}{t^2}+\dfrac{1}{t}\right)$ at $t=2$ is

(A) $\frac{7}{2}$ (B) $\frac{9}{2}$ (C) $\frac{9}{4}$ (D) 4

PAGE 201 6. The position of an object moving along a straight line at time t, in seconds, is given by $s(t) = 16t^2 - 5t + 20$ meters. What is the acceleration of the object when $t = 2$?

(A) 32 m/s (B) 0 m/s^2 (C) 32 m/s^2 (D) 64 m/s^2

PAGE 197 7. If $y = \dfrac{x-3}{x+3}$, $x \neq -3$, the instantaneous rate of change of y with respect to x at $x = 3$ is

(A) $-\dfrac{1}{6}$ (B) $\dfrac{1}{6}$ (C) $\dfrac{1}{36}$ (D) 1

PAGE 197 8. Find an equation of the normal line to the graph of the function $f(x) = \dfrac{x^2}{x+1}$ at $x = 1$.

(A) $8x + 6y = 11$ (B) $-8x + 6y = -5$
(C) $-3x + 4y = -1$ (D) $3x + 4y = 5$

PAGE 200 9. If $y = xe^x$, then the nth derivative of y is

(A) e^x (B) $(x+n)e^x$ (C) ne^x (D) $x^n e^x$

2.5 The Derivative of the Trigonometric Functions

OBJECTIVE ***When you finish this section, you should be able to:***

1 Differentiate trigonometric functions (p. 207)

1 Differentiate Trigonometric Functions

To find the derivatives of $y = \sin x$ and $y = \cos x$, we use the limits

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1 \qquad \text{and} \qquad \lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} = 0$$

that were established in Section 1.4.

THEOREM Derivative of $y = \sin x$

The derivative of $y = \sin x$ is $y' = \cos x$. That is,

$$\boxed{y' = \frac{d}{dx}\sin x = \cos x}$$

Proof

$$y' = \lim_{h\to 0}\frac{\sin(x+h) - \sin x}{h} \qquad \text{The definition of a derivative}$$

$$= \lim_{h\to 0}\frac{\sin x\cos h + \sin h\cos x - \sin x}{h} \qquad \sin(A+B) = \sin A\cos B + \sin B\cos A$$

$$= \lim_{h\to 0}\left[\frac{\sin x\cos h - \sin x}{h} + \frac{\sin h\cos x}{h}\right] \qquad \text{Rearrange terms.}$$

$$= \lim_{h\to 0}\left[\sin x\cdot\frac{\cos h - 1}{h} + \frac{\sin h}{h}\cdot\cos x\right] \qquad \text{Factor.}$$

$$= \left[\lim_{h\to 0}\sin x\right]\left[\lim_{h\to 0}\frac{\cos h - 1}{h}\right] + \left[\lim_{h\to 0}\cos x\right]\left[\lim_{h\to 0}\frac{\sin h}{h}\right] \qquad \text{Use properties of limits.}$$

$$= \sin x\cdot 0 + \cos x\cdot 1 = \cos x \qquad \lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} = 0;\ \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$ ■

NEED TO REVIEW? The trigonometric functions are discussed in Section P.6, pp. 52–58. Trigonometric identities are discussed in Appendix A.4, pp. A33 to A36.

The geometry of the derivative $\dfrac{d}{dx}\sin x = \cos x$ is shown in Figure 30. On the graph of $f(x) = \sin x$, the horizontal tangents are marked as well as the tangent lines that have slopes of 1 and -1. The derivative function is plotted on the second graph, and those points are connected with a smooth curve.

2.5 Assess Your Understanding

Concepts and Vocabulary

1. *True or False* $\frac{d}{dx}\cos x = \sin x$

2. *True or False* $\frac{d}{dx}\tan x = \cot x$

3. *True or False* $\frac{d^2}{dx^2}\sin x = -\sin x$

4. *True or False* $\frac{d}{dx}\sin\frac{\pi}{3} = \cos\frac{\pi}{3}$

Skill Building

In Problems 5–38, find y'.

5. $y = x - \sin x$
6. $y = \cos x - x^2$
7. $y = \tan x + \cos x$
8. $y = \sin x - \tan x$
9. $y = 3\sin\theta - 2\cos\theta$
10. $y = 4\tan\theta + \sin\theta$
11. $y = \sin x \cos x$
12. $y = \cot x \tan x$
13. $y = t\cos t$
14. $y = t^2 \tan t$
15. $y = e^x \tan x$
16. $y = e^x \sec x$
17. $y = \pi \sec u \tan u$
18. $y = \pi u \tan u$
19. $y = \frac{\cot x}{x}$
20. $y = \frac{\csc x}{x}$
21. $y = x^2 \sin x$
22. $y = t^2 \tan t$
23. $y = t\tan t - \sqrt{3}\sec t$
24. $y = x\sec x + \sqrt{2}\cot x$
25. $y = \frac{\sin\theta}{1-\cos\theta}$
26. $y = \frac{x}{\cos x}$
27. $y = \frac{\sin t}{1+t}$
28. $y = \frac{\tan u}{1+u}$
29. $y = \frac{\sin x}{e^x}$
30. $y = \frac{\cos x}{e^x}$
31. $y = \frac{\sin\theta + \cos\theta}{\sin\theta - \cos\theta}$
32. $y = \frac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta}$
33. $y = \frac{\sec t}{1 + t\sin t}$
34. $y = \frac{\csc t}{1 + t\cos t}$
35. $y = \csc\theta\cot\theta$
36. $y = \tan\theta\cos\theta$
37. $y = \frac{1+\tan x}{1-\tan x}$
38. $y = \frac{\csc x - \cot x}{\csc x + \cot x}$

In Problems 39–50, find y''.

39. $y = \sin x$
40. $y = \cos x$
41. $y = \tan\theta$
42. $y = \cot\theta$
43. $y = t\sin t$
44. $y = t\cos t$
45. $y = e^x \sin x$
46. $y = e^x \cos x$
47. $y = 2\sin u - 3\cos u$
48. $y = 3\sin u + 4\cos u$
49. $y = a\sin x + b\cos x$
50. $y = a\sec\theta + b\tan\theta$

In Problems 51–56:
(a) Find an equation of the tangent line to the graph of f at the indicated point.
(b) Graph the function and the tangent line.

51. $f(x) = \sin x$ at $(0, 0)$
52. $f(x) = \cos x$ at $\left(\frac{\pi}{3}, \frac{1}{2}\right)$
53. $f(x) = \tan x$ at $(0, 0)$
54. $f(x) = \tan x$ at $\left(\frac{\pi}{4}, 1\right)$
55. $f(x) = \sin x + \cos x$ at $\left(\frac{\pi}{4}, \sqrt{2}\right)$
56. $f(x) = \sin x - \cos x$ at $\left(\frac{\pi}{4}, 0\right)$

In Problems 57–60:
(a) Find all points on the graph of f where the tangent line is horizontal.
(b) Graph the function and the horizontal tangent lines on the interval $[-2\pi, 2\pi]$.

57. $f(x) = 2\sin x + \cos x$
58. $f(x) = \cos x - \sin x$
59. $f(x) = \sec x$
60. $f(x) = \csc x$

Applications and Extensions

In Problems 61 and 62, find the nth derivative of each function.

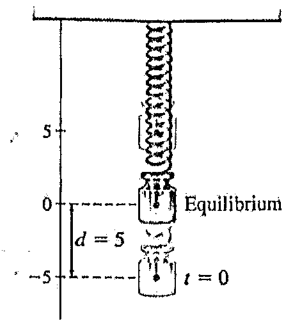
61. $f(x) = \sin x$
62. $f(\theta) = \cos\theta$

63. What is $\lim_{h\to 0}\frac{\cos\left(\frac{\pi}{2}+h\right) - \cos\frac{\pi}{2}}{h}$?

64. What is $\lim_{h\to 0}\frac{\sin(\pi + h) - \sin\pi}{h}$?

65. **Simple Harmonic Motion** The signed distance s (in meters) of an object from the origin at time t (in seconds) is modeled by the position function $s(t) = \frac{1}{8}\cos t$.
 (a) Find the velocity $v = v(t)$ of the object.
 (b) When is the speed of the object a maximum?
 (c) Find the acceleration $a = a(t)$ of the object.
 (d) When is the acceleration equal to 0?
 (e) Graph s, v, and a on the same screen.

66. **Simple Harmonic Motion** An object attached to a coiled spring is pulled down a distance $d = 5$ cm from its equilibrium position and then released as shown in the figure. The motion of the object at time t seconds is simple harmonic and is modeled by $d(t) = -5\cos t$.

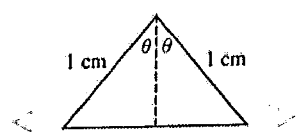


(a) As t varies from 0 to 2π, how does the length of the spring vary?
(b) Find the velocity $v = v(t)$ of the object.
(c) When is the speed of the object a maximum?
(d) Find the acceleration $a = a(t)$ of the object.
(e) When is the acceleration equal to 0?
(f) Graph d, v, and a on the same set of axes.

67. **Rate of Change** A large, 8-ft-high decorative mirror is placed on a wood floor and leaned against a wall. The weight of the mirror and the slickness of the floor cause the mirror to slip.

(a) If θ is the angle between the top of the mirror and the wall, and y is the distance from the floor to the top of the mirror, what is the rate of change of y with respect to θ?
(b) In feet/radian, how fast is the top of the mirror slipping down the wall when $\theta = \dfrac{\pi}{4}$?

68. **Rate of Change** The sides of an isosceles triangle are sliding outward. See the figure.



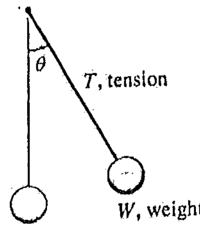
(a) Find the rate of change of the area of the triangle with respect to θ.
(b) How fast is the area changing when $\theta = \dfrac{\pi}{6}$?

69. **Sea Waves** Waves in deep water tend to have the symmetric form of the function $f(x) = \sin x$. As they approach shore, however, the sea floor creates drag, which changes the shape of the wave. The trough of the wave widens and the height of the wave increases, so the top of the wave is no longer symmetric with the trough. This type of wave can be represented by a function such as

$$w(x) = \frac{4}{2 + \cos x}$$

(a) Graph $w = w(x)$ for $0 \le x \le 4\pi$.
(b) What is the maximum and the minimum value of w?
(c) Find the values of x, $0 < x < 4\pi$, at which $w'(x) = 0$.
(d) Evaluate w' near the peak at π, using $x = \pi - 0.1$, and near the trough at 2π, using $x = 2\pi - 0.1$.
(e) Explain how these values confirm a nonsymmetric wave shape.

70. **Swinging Pendulum** A simple pendulum is a small-sized ball swinging from a light string. As it swings, the supporting string makes an angle θ with the vertical. See the figure. At an angle θ, the tension in the string is $T = \dfrac{W}{\cos\theta}$, where W is the weight of the swinging ball.



(a) Find the rate of change of the tension T with respect to θ when the pendulum is at its highest point ($\theta = \theta_{max}$).
(b) Find the rate of change of the tension T with respect to θ when the pendulum is at its lowest point.
(c) What is the tension at the lowest point?

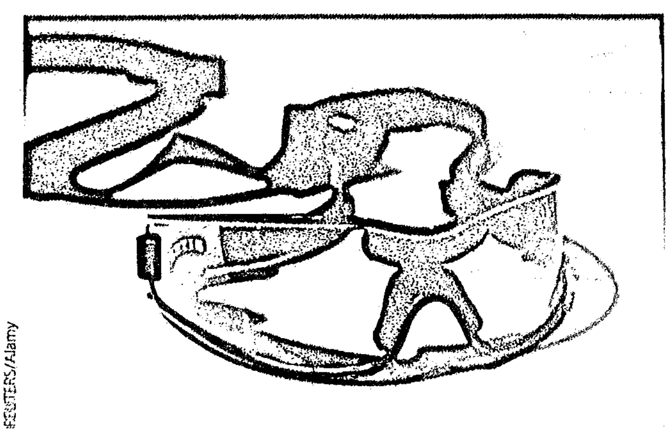
71. **Restaurant Sales** A restaurant in Naples, Florida, is very busy during the winter months and extremely slow over the summer. But every year the restaurant grows its sales. Suppose over the next two years, the revenue R, in units of \$10,000, is projected to follow the model

$$R = R(t) = \sin t + 0.3t + 1 \qquad 0 \le t \le 12$$

where $t = 0$ corresponds to November 1, 2018; $t = 1$ corresponds to January 1, 2019; $t = 2$ corresponds to March 1, 2019; and so on.

(a) What is the projected revenue for November 1, 2018; March 1, 2019; September 1, 2019; and January 1, 2020?
(b) What is the rate of change of revenue with respect to time?
(c) What is the rate of change of revenue with respect to time for January 1, 2020?
(d) Graph the revenue function and the derivative function $R' = R'(t)$.
(e) Does the graph of R support the facts that every year the restaurant grows its sales and that sales are higher during the winter and lower during the summer? Explain.

72. **Polarizing Sunglasses** Polarizing sunglasses are filters that transmit only light for which the electric field oscillations are in a specific direction. Light is polarized naturally by scattering off the molecules in the atmosphere and by reflecting off many (but not all) types of surfaces. If light of intensity I_0 is already polarized in a certain direction, and the transmission direction of the polarizing filter makes an angle with that direction, then the intensity I of the light after passing through the filter is given by **Malus's Law**, $I(\theta) = I_0 \cos^2\theta$.



REUTERS/Alamy

(a) As you rotate a polarizing filter, θ changes. Find the rate of change of the light intensity I with respect to θ.

(b) Find both the intensity $I(\theta)$ and the rate of change of the intensity with respect to θ, for the angles $\theta = 0°, 45°$, and $90°$. (Remember to use radians for θ.)

73. If $y = \sin x$ and $y^{(n)}$ is the nth derivative of y with respect to x, find the smallest positive integer n for which $y^{(n)} = y$.

74. Use the identity $\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$, with $A = x + h$ and $B = x$, to prove that

$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \cos x$$

75. Use the definition of a derivative to prove $\frac{d}{dx}\cos x = -\sin x$.

76. **Derivative of $y = \sec x$** Use a derivative rule to show that

$$\frac{d}{dx}\sec x = \sec x\tan x$$

77. **Derivative of $y = \csc x$** Use a derivative rule to show that

$$\frac{d}{dx}\csc x = -\csc x\cot x$$

78. **Derivative of $y = \cot x$** Use a derivative rule to show that

$$\frac{d}{dx}\cot x = -\csc^2 x$$

79. Let $f(x) = \cos x$. Show that finding $f'(0)$ is the same as finding $\lim_{x\to 0}\frac{\cos x - 1}{x}$.

80. Let $f(x) = \sin x$. Show that finding $f'(0)$ is the same as finding $\lim_{x\to 0}\frac{\sin x}{x}$.

81. If $y = A\sin t + B\cos t$, where A and B are constants, show that $y'' + y = 0$.

Challenge Problem

82. For a differentiable function f, let f^* be the function defined by

$$f^*(x) = \lim_{h\to 0}\frac{f(x+h)-f(x-h)}{h}$$

(a) Find $f^*(x)$ for $f(x) = x^2 + x$.

(b) Find $f^*(x)$ for $f(x) = \cos x$.

(c) Write an equation that expresses the relationship between the functions f^* and f', where f' denotes the derivative of f. Justify your answer.

AP® Practice Problems

Preparing for the AP® Exam

PAGE 208 1. If $y = x\sin x$, then $\frac{dy}{dx} =$

(A) $x\cos x + \sin x$ (B) $x\cos x - \sin x$
(C) $\cos x + \sin x$ (D) $(x+1)\cos x$

PAGE 209 2. What is $\lim_{h\to 0}\frac{\cos\left(\frac{\pi}{3}+h\right)-\cos\frac{\pi}{3}}{h}$?

(A) 0 (B) $\frac{1}{2}$ (C) $\frac{\sqrt{3}}{2}$ (D) $-\frac{\sqrt{3}}{2}$

PAGE 210 3. If $f(x) = \tan x$, then $f'\left(\frac{\pi}{3}\right)$ equals

(A) $2\sqrt{3}$ (B) 4 (C) 2 (D) $\frac{1}{4}$

PAGE 210 4. The position s (in meters) of an object moving along a horizontal line at time t, $0 \le t \le \frac{\pi}{2}$, (in seconds) is given by $s(t) = 6\sin t + \frac{3}{2}t^2 + 8$. What is the velocity of the object when its acceleration is zero?

(A) 6 m/s (B) $3 + \pi$ m/s
(C) $\frac{6\sqrt{3}+\pi}{2}$ m/s (D) $\left(3\sqrt{3}-\frac{\pi}{2}\right)$ m/s

PAGE 208 5. If $y = \sin x$, then $\frac{d^{50}}{dx^{50}}\sin x$ equals

(A) $\sin x$ (B) $-\sin x$ (C) $\cos x$ (D) $-\cos x$

PAGE 209 6. If $f(x) = \frac{x}{\cos x}$, find $f'\left(\frac{\pi}{3}\right)$.

(A) $2 - \frac{2\sqrt{3}}{3}\pi$ (B) $1 + \frac{\sqrt{3}}{3}\pi$
(C) $1 - \frac{\sqrt{3}}{3}\pi$ (D) $2 + \frac{2\sqrt{3}}{3}\pi$

PAGE 210 7. If $y = x - \tan x$, then $\frac{dy}{dx}$ equals

(A) $1 - \sec x\tan x$ (B) $-\tan^2 x$
(C) $\tan^2 x$ (D) $-\sec^2 x$

PAGE 209 8. If $g(x) = e^x\cos x + 2\pi$, then $g'(x) =$

(A) $e^x - \sin x$ (B) $e^x\cos x - e^x\sin x + 3\pi$
(C) $e^x\cos x - e^x\sin x$ (D) $e^x\cos x + e^x\sin x$

PAGE 209 9. At which of the following numbers x, $0 \le x \le 2\pi$, does the graph of $y = x + \cos x$ have a horizontal tangent line?

(A) 0 only (B) $\frac{\pi}{2}$ only
(C) $\frac{3\pi}{2}$ only (D) 0 and $\frac{\pi}{2}$ only

PAGE 208 10. An equation of the tangent line to the graph of $f(x) = \sin x$ at $x = \frac{2\pi}{3}$ is

(A) $3x + 6y = 4\pi - 3\sqrt{3}$ (B) $3x + 6y = 2\pi + 3\sqrt{3}$
(C) $6y - 3x = 2\pi - 3\sqrt{3}$ (D) $6y - 3x = 4\pi - 3\sqrt{3}$

Chapter Review

THINGS TO KNOW

2.1 Rates of Change and the Derivative

- **Definition** (Form 1) Derivative of a function f at a number c

$$f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

provided the limit exists. (p. 167)

Three Interpretations of the Derivative

- *Geometric* If $y=f(x)$, the derivative $f'(c)$ is the slope of the tangent line to the graph of f at the point $(c, f(c))$. (p. 167)
- *Rate of change of a function* If $y=f(x)$, the derivative $f'(c)$ is the rate of change of f with respect to x at c. (p. 167)
- *Physical* If the signed distance s from the origin at time t of an object in rectilinear motion is given by the position function $s=f(t)$, the derivative $f'(t_0)$ is the velocity of the object at time t_0. (p. 167)

2.2 The Derivative as a Function

- **Definition of a derivative function** (Form 2)

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

provided the limit exists. (p. 172)

- **Theorem** If a function f has a derivative at a number c, then f is continuous at c. (p. 177)
- **Corollary** If a function f is discontinuous at a number c, then f has no derivative at c. (p. 177)

2.3 The Derivative of a Polynomial Function; The Derivative of $y=e^x$

- **Leibniz notation** $\frac{dy}{dx}=\frac{d}{dx}y=\frac{d}{dx}f(x)$ (p. 183)
- **Basic derivatives**

$\frac{d}{dx}A=0$ A is a constant (p. 184) $\frac{d}{dx}x=1$ (p. 184)

$\frac{d}{dx}e^x=e^x$ (p. 190) $\frac{d}{dx}\ln x=\frac{1}{x}$ (p. 190)

- *Simple Power Rule* $\frac{d}{dx}x^n=nx^{n-1}$, $n\geq 1$, an integer (p. 185)

Properties of Derivatives

- *Sum Rule* (pp. 186, 187) $\frac{d}{dx}[f+g]=\frac{d}{dx}f+\frac{d}{dx}g$
  $(f+g)'=f'+g'$
- *Difference Rule* (p. 187) $\frac{d}{dx}[f-g]=\frac{d}{dx}f-\frac{d}{dx}g$
  $(f-g)'=f'-g'$
- *Constant Multiple Rule* (p. 186) If k is a constant,
  $\frac{d}{dx}[kf]=k\frac{d}{dx}f$
  $(kf)'=k\cdot f'$

2.4 Differentiating the Product and the Quotient of Two Functions; Higher-Order Derivatives

Properties of Derivatives

- *Product Rule* (p. 195) $\frac{d}{dx}(fg)=f\left(\frac{d}{dx}g\right)+\left(\frac{d}{dx}f\right)g$
  $(fg)'=fg'+f'g$
- *Quotient Rule* (p. 196) $\frac{d}{dx}\left(\frac{f}{g}\right)=\frac{\left(\frac{d}{dx}f\right)g-f\left(\frac{d}{dx}g\right)}{g^2}$
  $\left(\frac{f}{g}\right)'=\frac{f'g-fg'}{g^2}$
  provided $g(x)\neq 0$
- *Reciprocal Rule* (p. 197) $\frac{d}{dx}\left(\frac{1}{g}\right)=-\frac{\frac{d}{dx}g}{g^2}$
  $\left(\frac{1}{g}\right)'=-\frac{g'}{g^2}$
  provided $g(x)\neq 0$
- *Power Rule* $\frac{d}{dx}x^n=nx^{n-1}$, n an integer (p. 198)
- *Higher-order derivatives* See Table 3 (p. 199)
- *Position Function* $s=s(t)$ (p. 200)
- *Velocity* $v=v(t)=\frac{ds}{dt}$ (p. 200)
- *Acceleration* $a=a(t)=\frac{dv}{dt}=\frac{d^2s}{dt^2}$ (p. 200)

2.5 The Derivative of the Trigonometric Functions

Basic Derivatives

$\frac{d}{dx}\sin x=\cos x$ (p. 207) $\frac{d}{dx}\sec x=\sec x\tan x$ (p. 210)

$\frac{d}{dx}\cos x=-\sin x$ (p. 208) $\frac{d}{dx}\csc x=-\csc x\cot x$ (p. 210)

$\frac{d}{dx}\tan x=\sec^2 x$ (p. 210) $\frac{d}{dx}\cot x=-\csc^2 x$ (p. 210)

OBJECTIVES

Section	You should be able to ...	Examples	Review Exercises	Preparing for the AP® Exam: AP® Review Problems
2.1	1 Find equations for the tangent line and the normal line to the graph of a function (p. 162)	1	67–70	
	2 Find the rate of change of a function (p. 163)	2, 3	1, 2, 73 (a)	
	3 Find average velocity and instantaneous velocity (p. 164)	4, 5	71(a), (b); 72(a), (b)	
	4 Find the derivative of a function at a number (p. 166)	6–8	3–8, 75	5
2.2	1 Define the derivative function (p. 171)	1–3	9–12, 77	2
	2 Graph the derivative function (p. 173)	4, 5	9–12, 15–18	
	3 Identify where a function is not differentiable (p. 175)	6–10	13, 14, 75	4
2.3	1 Differentiate a constant function (p. 184)	1		
	2 Differentiate a power function (p. 184)	2, 3	19–22	
	3 Differentiate the sum and the difference of two functions (p. 186)	4–6	23–26, 33, 34, 40, 51, 52, 67	
	4 Differentiate the exponential function $y=e^x$ (p. 189)	7	44, 45, 53, 54, 56, 59, 69	7
2.4	1 Differentiate the product of two functions (p. 194)	1, 2	27, 28, 36, 46, 48–50, 53–56, 60	
	2 Differentiate the quotient of two functions (p. 196)	3–6	29–35, 37–43, 47, 57–59, 68, 73, 74	3, 10
	3 Find higher-order derivatives (p. 198)	7, 8	61–66, 71, 72, 76	
	4 Find the acceleration of an object in rectilinear motion (p. 200)	9	71, 72, 76	8
2.5	1 Differentiate trigonometric functions (p. 207)	1–6	49–60, 70	1, 6, 9

REVIEW EXERCISES

In Problems 1 and 2, use a definition of the derivative to find the rate of change of f at the indicated numbers.

1. $f(x)=\sqrt{x}$ at (a) $c=1$ (b) $c=4$ (c) c any positive real number

2. $f(x)=\dfrac{2}{x-1}$ at (a) $c=0$ (b) $c=2$ (c) c any real number, $c\neq 1$

In Problems 3–8, use a definition of the derivative to find the derivative of each function at the given number.

3. $F(x)=2x+5$ at 2
4. $f(x)=4x^2+1$ at -1
5. $f(x)=3x^2+5x$ at 0
6. $f(x)=\dfrac{3}{x}$ at 1
7. $f(x)=\sqrt{4x+1}$ at 0
8. $f(x)=\dfrac{x+1}{2x-3}$ at 1

In Problems 9–12, use a definition of the derivative to find the derivative of each function. Graph f and f′ on the same set of axes.

9. $f(x)=x-6$
10. $f(x)=7-3x^2$
11. $f(x)=\dfrac{1}{2x^3}$
12. $f(x)=\pi$

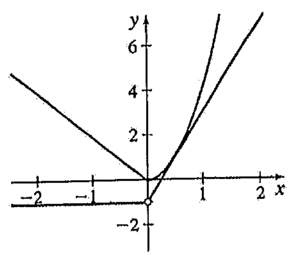
In Problems 13 and 14, determine whether the function f has a derivative at c. If it does, find the derivative. If it does not, explain why. Graph each function.

13. $f(x)=|x^3-1|$ at $c=1$

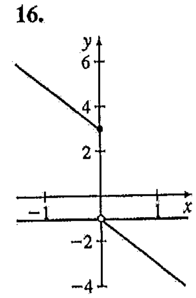
14. $f(x)=\begin{cases} 4-3x^2 & \text{if } x\le -1 \\ -x^3 & \text{if } x>-1 \end{cases}$ at $c=-1$

In Problems 15 and 16, determine whether the graphs represent a function f and its derivative f'. If they do, indicate which is the graph of f and which is the graph of f'.

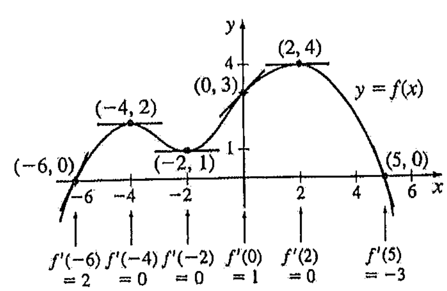
15.



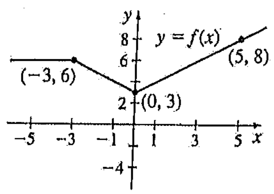
16.

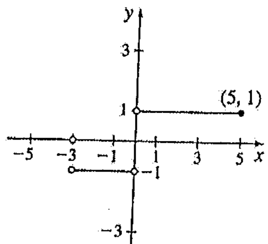


17. Use the information in the graph of $y=f(x)$ to sketch the graph of $y=f'(x)$.

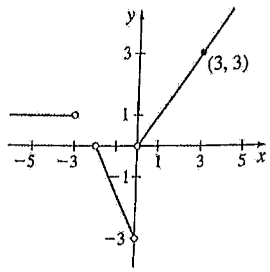


18. Match the graph of $y = f(x)$ with the graph of its derivative.

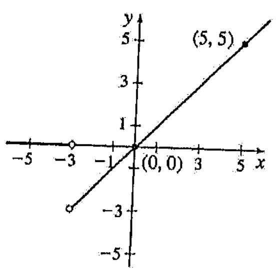




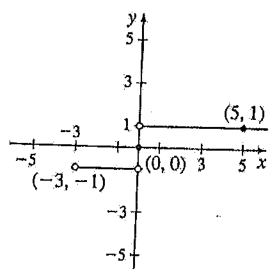
(A)



(B)



(C)



(D)

In Problems 19–60, find the derivative of each function. Treat a and b, if present, as constants.

19. $f(x) = x^5$

20. $f(x) = ax^3$

21. $f(x) = \dfrac{x^4}{4}$

22. $f(x) = -6x^2$

23. $f(x) = 2x^2 - 3x$

24. $f(x) = 3x^3 + \dfrac{2}{3}x^2 - 5x + 7$

25. $F(x) = 7(x^2 - 4)$

26. $F(x) = \dfrac{5(x+6)}{7}$

27. $f(x) = 5(x^2 - 3x)(x - 6)$

28. $f(x) = (2x^3 + x)(x^2 - 5)$

29. $f(x) = \dfrac{6x^4 - 9x^2}{3x^3}$

30. $f(x) = \dfrac{2x+2}{5x-3}$

31. $f(x) = \dfrac{7x}{x-5}$

32. $f(x) = 2x^{-12}$

33. $f(x) = 2x^2 - 5x^{-2}$

34. $f(x) = 2 + \dfrac{3}{x} + \dfrac{4}{x^2}$

35. $f(x) = \dfrac{a}{x} - \dfrac{b}{x^3}$

36. $f(x) = (x^3 - 1)^2$

37. $f(x) = \dfrac{3}{(x^2 - 3x)^2}$

38. $f(x) = \dfrac{x^2}{x+1}$

39. $s(t) = \dfrac{t^3}{t-2}$

40. $f(x) = 3x^{-2} + 2x^{-1} + 1$

41. $F(z) = \dfrac{1}{z^2+1}$

42. $f(v) = \dfrac{v-1}{v^2+1}$

43. $g(z) = \dfrac{1}{1 - z + z^2}$

44. $f(x) = 3e^x + x^2$

45. $s(t) = 1 - e^t$

46. $f(x) = ae^x(2x^2 + 7x)$

47. $f(x) = \dfrac{1+x}{e^x}$

48. $f(x) = (2xe^x)^2$

49. $f(x) = x \sin x$

50. $s(t) = \cos^2 t$

51. $G(u) = \tan u + \sec u$

52. $g(v) = \sin v - \dfrac{1}{3}\cos v$

53. $f(x) = e^x \sin x$

54. $f(x) = e^x \csc x$

55. $f(x) = 2 \sin x \cos x$

56. $f(x) = (e^x + b)\cos x$

57. $f(x) = \dfrac{\sin x}{\csc x}$

58. $f(x) = \dfrac{1 - \cot x}{1 + \cot x}$

59. $f(\theta) = \dfrac{\cos\theta}{2e^\theta}$

60. $f(\theta) = 4\theta \cot\theta \tan\theta$

In Problems 61–66, find the first derivative and the second derivative of each function.

61. $f(x) = (5x+3)^2$

62. $f(x) = xe^x$

63. $g(u) = \dfrac{u}{2u+1}$

64. $F(x) = e^x(\sin x + 2\cos x)$

65. $f(u) = \dfrac{\cos u}{e^u}$

66. $F(x) = \dfrac{\sin x}{x}$

In Problems 67–70, for each function:

(a) Find an equation of the tangent line to the graph of the function at the indicated point.

(b) Find an equation of the normal line to the function at the indicated point.

(c) Graph the function, the tangent line, and the normal line on the same screen.

67. $f(x) = 2x^2 - 3x + 7$ at $(-1, 12)$

68. $y = \dfrac{x^2+1}{2x-1}$ at $\left(2, \dfrac{5}{3}\right)$

69. $f(x) = x^2 - e^x$ at $(0, -1)$

70. $s(t) = 1 + 2\sin t$ at $(\pi, 1)$

71. Rectilinear Motion As an object in rectilinear motion moves, its signed distance s (in meters) from the origin at time t (in seconds) is given by the position function

$$s = f(t) = t^2 - 6t$$

(a) Find the average velocity of the object from 0 to 5 s.

(b) Find the velocity at $t = 0$, at $t = 5$, and at any time t.

(c) Find the acceleration at any time t.

72. Rectilinear Motion As an object in rectilinear motion moves, its signed distance s from the origin at time t is given by the position function $s(t) = t - t^2$, where s is in centimeters and t is in seconds.

(a) Find the average velocity of the object from 1 to 3 s.

(b) Find the velocity of the object at $t = 1$ s and $t = 3$ s.

(c) What is its acceleration at $t = 1$ and $t = 3$?

73. **Business** The price p in dollars per pound when x pounds of a commodity are demanded is modeled by the function

$$p(x) = \frac{10{,}000}{5x + 100} - 5$$

when between 0 and 90 lb are demanded (purchased).

(a) Find the rate of change of price with respect to demand.

(b) What is the revenue function R? (Recall, revenue R equals price times amount purchased.)

(c) What is the marginal revenue R' at $x = 10$ and at $x = 40$ lb?

74. If $f(x) = \dfrac{x-1}{x+1}$ for all $x \neq -1$, find $f'(1)$.

75. If $f(x) = 2 + |x - 3|$ for all x, determine whether the derivative f' exists at $x = 3$.

76. **Rectilinear Motion** An object in rectilinear motion moves according to the position function $s = 2t^3 - 15t^2 + 24t + 3$, where t is measured in minutes and s in meters.

(a) When is the object at rest?

(b) Find the object's acceleration when $t = 3$.

77. Find the value of the limit below and specify the function f for which this is the derivative.

$$\lim_{\Delta x \to 0} \frac{[4 - 2(x + \Delta x)]^2 - (4 - 2x)^2}{\Delta x}$$

AP® REVIEW PROBLEMS: CHAPTER 2

1. If $f(x) = \sec x$, then $f'\left(\frac{\pi}{4}\right) =$

(A) $\frac{\sqrt{2}}{2}$ (B) 2 (C) 1 (D) $\sqrt{2}$

2. If a function f is differentiable at c, then $f'(c)$ is given by

I. $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$

II. $\lim_{x \to c} \frac{f(x+h) - f(x)}{h}$

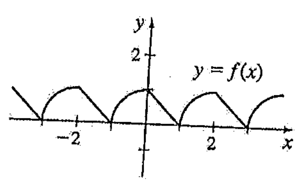
III. $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$

(A) I only (B) III only
(C) I and II only (D) I and III only

3. If $y = \frac{3}{x^2 - 5}$, then $\frac{dy}{dx} =$

(A) $\frac{6x}{(x^2-5)^2}$ (B) $-\frac{6x}{(x^2-5)^2}$
(C) $\frac{6x}{x^2-5}$ (D) $\frac{2x}{(x^2-5)^2}$

4. The graph of the function f is shown below. Which statement about the function is true?



(A) f is differentiable everywhere.
(B) $0 \le f'(x) \le 1$, for all real numbers.
(C) f is continuous everywhere.
(D) f is an even function.

5. The table displays select values of a differentiable function f. What is an approximate value of $f'(2)$?

x	1.996	1.998	2	2.002	2.004
$f(x)$	3.168	3.181	3.194	3.207	3.220

(A) 6.5 (B) 1.154 (C) 0.013 (D) 0.0016

6. If $y = \sin x + xe^x + 6$, what is the instantaneous rate of change of y with respect to x at $x = 5$?

(A) $\cos 5 + 6e^5$ (B) 2
(C) $\cos 5 + 5e^5$ (D) $6e^5 - \cos 5$

7. An equation of the normal line to the graph of $f(x) = 3xe^x + 5$ at $x = 0$ is

(A) $y = 3x + 5$ (B) $y = -\frac{1}{3}x + 5$
(C) $y = \frac{1}{3}x + 5$ (D) $y = -3x + 5$

8. An object moves along a horizontal line so that its position at time t is $s(t) = t^4 - 6t^3 - 2t - 1$. At what time t is the acceleration of the object zero?

(A) at 0 only (B) at 1 only
(C) at 3 only (D) at 0 and 3 only

9. If $f(x) = e^x(\sin x + \cos x)$, then $f'(x) =$

(A) $2e^x(\cos x + \sin x)$ (B) $e^x \cos x$
(C) $2e^x \cos x$ (D) $e^x(\cos^2 x - \sin^2 x)$

10. Find an equation of the tangent line to the graph of $f(x) = \frac{x+3}{x^2+2}$ at $x = 1$.

(A) $5x + 9y = 17$ (B) $9y - 5x = 7$
(C) $5x + 3y = 9$ (D) $5x + 9y = 7$

11. $\lim_{x \to \frac{\pi}{4}} \frac{\tan x - 1}{x - \frac{\pi}{4}} =$

(A) 0 (B) −1 (C) 2 (D) Does not exist.

Preparing for the AP® Exam

AP® CUMULATIVE REVIEW PROBLEMS: CHAPTERS 1–2

1. $\lim_{x \to 4} \frac{x-4}{4-x} =$

(A) -4 (B) -1 (C) 0 (D) does not exist

2. $\lim_{x \to 0} \frac{3x + \sin x}{2x} =$

(A) 0 (B) 1 (C) 2 (D) does not exist

3. Let h be defined by

$$h(x) = \begin{cases} f(x) \cdot g(x) & \text{if } x \le 1 \\ k + x & \text{if } x > 1 \end{cases}$$

where f and g are both continuous at all real numbers. If $\lim_{x \to 1} f(x) = 2$ and $\lim_{x \to 1} g(x) = -2$, then for what number k is h continuous?

(A) -5 (B) -4 (C) -2 (D) 2

4. Which function has the horizontal asymptotes $y = 1$ and $y = -1$?

(A) $f(x) = \frac{2}{\pi} \tan^{-1} x$ (B) $f(x) = e^{-x} + 1$

(C) $f(x) = \frac{1 - x^2}{1 + x^2}$ (D) $f(x) = \frac{2x^2 - 1}{2x^2 + x}$

5. Suppose the function f is continuous at all real numbers and $f(-2) = 1$ and $f(5) = -3$. Suppose the function g is also continuous at all real numbers and $g(x) = f^{-1}(x)$ for all x. The Intermediate Value Theorem guarantees that

(A) $g(c) = 2$ for at least one c between -3 and 1.

(B) $g(c) = 0$ for at least one c between -2 and 5.

(C) $f(c) = 0$ for at least one c between -3 and 1.

(D) $f(c) = 2$ for at least one c between -2 and 5.

6. The line $x = c$ is a vertical asymptote to the graph of the function f. Which of the following statements cannot be true?

(A) $\lim_{x \to c} f(x) = \infty$ (B) $\lim_{x \to \infty} f(x) = c$

(C) $f(c)$ is not defined. (D) f is continuous at $x = c$.

7. The position function of an object moving along a straight line is $s(t) = \frac{1}{15}t^3 - \frac{1}{2}t^2 + 5t^{-1}$. What is the object's acceleration at $t = 5$?

(A) $-\frac{27}{25}$ (B) $-\frac{1}{5}$ (C) $\frac{1}{5}$ (D) $\frac{27}{25}$

8. If the function $f(x) = \begin{cases} 2ax^2 + bx - 1 & \text{if } x \le 3 \\ bx^2 + bx - a & \text{if } x > 3 \end{cases}$ is continuous for all real numbers x, then

(A) $19a - 15b = 1$ (B) $18a - 9b = 1$

(C) $19a - 9b = 1$ (D) $19a + 15b = 1$

9. Find the slope of the tangent line to the graph of $f(x) = xe^x$ at the point $(1, e)$.

(A) 1 (B) e (C) $2e$ (D) e^2

10. An object in rectilinear motion is modeled by the position function

$$s(t) = 3t^4 - 8t^3 - 6t^2 + 24t \qquad t > 0$$

where s is in feet (ft) and t is in seconds (s). Find the acceleration of the object when its velocity is zero.

(A) -24 ft/s^2, 36 ft/s^2, and 72 ft/s^2 only

(B) 36 ft/s^2 only

(C) 36 ft/s^2 and 72 ft/s^2 only

(D) -24 ft/s^2 and 36 ft/s^2 only