Ch. 2.1 Definition of Derivative p.103-105

#1, 13, 17, 21, 23, 25a, 27a, 39-43 odd, 49, 51, 53, 55, 65, 67, 85-89 odd

Def. of Derivative: $f'(x) = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$ or $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

13) $f(x) = -10x \qquad f(x+\Delta x) = -10(x+\Delta x)$

$$f'(x) = \lim_{\Delta x \to 0} \frac{-10(x+\Delta x) - (-10x)}{\Delta x} = \frac{\cancel{-10x} - 10\Delta x \cancel{+10x}}{\Delta x} = \boxed{-10}$$

17) $f(x) = x^2 + x - 3 \qquad f(x+\Delta x) = (x+\Delta x)^2 + (x+\Delta x) - 3$

$$f'(x) = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 + x + \Delta x - 3 - (x^2+x-3)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 + x + \Delta x - 3 - x^2 - x + 3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}(2x + \Delta x + 1)}{\cancel{\Delta x}} = 2x + 0 + 1 = \boxed{2x+1}$$

21) $f(x) = \dfrac{1}{x-1} \qquad f(x+\Delta x) = \dfrac{1}{x+\Delta x - 1}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\frac{1}{x+\Delta x-1} - \frac{1}{x-1}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{x-1-(x+\Delta x-1)}{(x+\Delta x-1)(x-1)}}{\cancel{\Delta x}} \cdot \frac{\frac{1}{\Delta x}}{\cancel{\frac{1}{\Delta x}}}$$

$$= \lim_{\Delta x \to 0} \frac{x - 1 - x - \Delta x + 1}{(x+\Delta x-1)(x-1)\cdot\Delta x} = \frac{-\cancel{\Delta x}}{(x+\Delta x-1)(x-1)\cdot\cancel{\Delta x}}$$

$$= \lim_{\Delta x \to 0} \frac{-1}{(x+\Delta x-1)(x-1)} = \frac{-1}{(x+0-1)(x-1)} = \boxed{\frac{-1}{(x-1)^2}}$$

23) $f(x) = \sqrt{x+4}$ $\quad f(x+\Delta x) = \sqrt{(x+\Delta x)+4} = \sqrt{x+\Delta x+4}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x+4} - \sqrt{x+4}}{\Delta x}$$

* multiply numerator and denominator by conjugate of numerator.

$$f'(x) = \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x+4} - \sqrt{x+4}}{\Delta x} \cdot \frac{(\sqrt{x+\Delta x+4} + \sqrt{x+4})}{(\sqrt{x+\Delta x+4} + \sqrt{x+4})}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{x+\Delta x+4-(x+4)}{\Delta x(\sqrt{x+\Delta x+4} + \sqrt{x+4})} = \frac{x+\Delta x+4-x-4}{\Delta x(\sqrt{x+\Delta x+4} + \sqrt{x+4})}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}}{\cancel{\Delta x}(\sqrt{x+\Delta x+4} + \sqrt{x+4})} = \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x+4} + \sqrt{x+4})}$$

$$= \frac{1}{\sqrt{x+0+4} + \sqrt{x+4}} = \boxed{\frac{1}{2\sqrt{x+4}}}$$

25a) $f(x) = x^2+3$ $\quad (-1,4)$ $\quad$ Find equation of tangent line at point:

$$f(x) = x^2+3 \qquad f(x+\Delta x) = (x+\Delta x)^2+3$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2+3-(x^2+3)}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^2+2x\Delta x+\Delta x^2+3-x^2-3}{\Delta x} = \frac{\Delta x(2x+\Delta x)}{\cancel{\Delta x}}$$

$$f'(x) = \lim_{\Delta x \to 0} 2x+\Delta x = 2x+0 = 2x$$

$f'(x) = 2x$

$f'(-1) = 2(-1) = -2$

point: $(-1,4)$

slope: $m=-2$

$y - y_1 = m(x-x_1)$

$y-4 = -2(x-(-1))$

$\boxed{y-4 = -2(x+1)}$

27) $f(x) = x^3 \quad (2,8)$

$f(x) = x^3 \qquad f(x+\Delta x) = (x+\Delta x)^3$

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1

$(x+\Delta x)^3 = x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3$

$$f'(x) = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^3 - x^3}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 - x^3}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}(3x^2 + 3x\Delta x + \Delta x^2)}{\cancel{\Delta x}} = 3x^2 + 3x(0) + (0)^2 = 3x^2$$

$f'(x) = 3x^2$

$f'(2) = 3(2)^2 = 12$

point: $(2,8)$

slope: $m = 12$

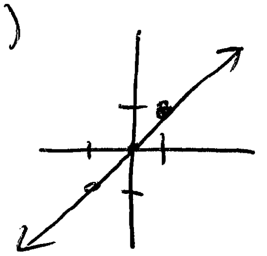
$y - y_1 = m(x - x_1)$

$\boxed{y - 8 = 12(x-2)}$

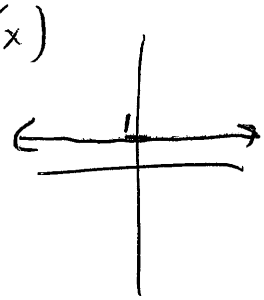
39) Sketch $f'(x)$ graph

$f(x)$

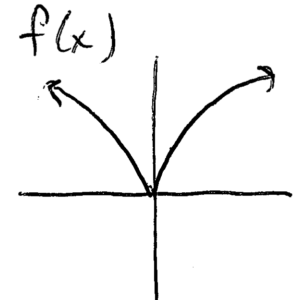
slope $= 1$



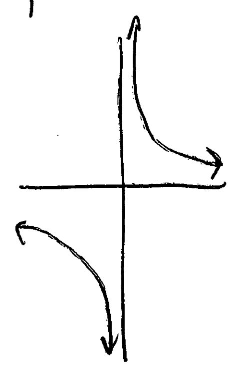
$f'(x)$



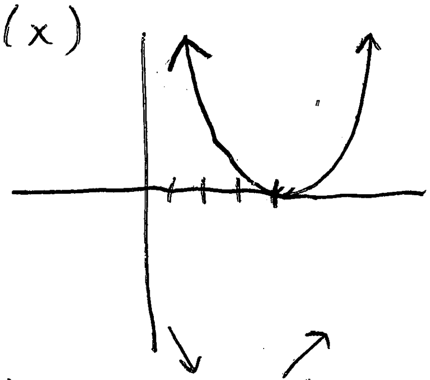
43) $f(x)$



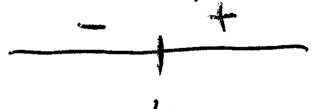
$f'(x)$



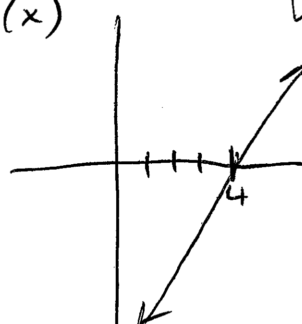
41) $f(x)$



$f'(x)$



$f'(x)$



Working backwards

Limit Def. of Derivative

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

49) $\lim_{\Delta x \to 0} \frac{[5-3(1+\Delta x)] - 2}{\Delta x}$

$\boxed{f(x) = 5 - 3x \qquad c = 1}$

Alt. Limit Definition of Derivative

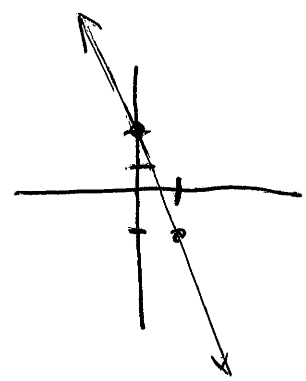
$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

51) $\lim_{x \to 6} \frac{-x^2+36}{x-6}$

$\boxed{f(x) = -x^2 \qquad c = 6}$

53) Sketch graph with given characteristic:

$f(0) = 2 \quad f'(x) = -3$ for $-\infty < x < \infty$



55) Find equation of tangent lines:

$f(x) = 4x - x^2$

point: $(1,3)$ and $(2,5)$

slope: $m = \frac{5-3}{2-1} = 2$

$y - y_1 = m(x - x_1)$

$\boxed{\begin{array}{c} y - 3 = 2(x-1) \\ y = 2x + 1 \end{array}}$

point $(3,3)$ and $(2,5)$

slope $m = \frac{5-3}{2-3} = -2$

$\boxed{\begin{array}{c} y - 3 = -2(x-3) \\ y = -2x + 9 \end{array}}$

Use Alt. Form of Derivative: $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$

65) $f(x) = x^2 - 5 \quad c = 3 \quad f(3) = 3^2 - 5 = 4$

$$f'(3) = \lim_{x \to 3} \frac{x^2 - 5 - f(3)}{x - 3} = \lim_{x \to 3} \frac{x^2 - 5 - 4}{x - 3} = \lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} \frac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}} = 6$$

$\boxed{f'(3) = 6}$

67) $f(x) = x^3 + 2x^2 + 1 \quad c = -2 \quad f(-2) = (-2)^3 + 2(-2)^2 + 1$
$= -8 + 8 + 1 = 1$

$$f'(-2) = \lim_{x \to -2} \frac{f(x) - f(-2)}{x - (-2)}$$

$$= \lim_{x \to -2} \frac{x^3 + 2x^2 + 1 - 1}{x + 2} = \lim_{x \to -2} \frac{x^2\cancel{(x+2)}}{\cancel{(x+2)}} = (-2)^2 = 4$$

$\boxed{f'(-2) = 4}$

Determine Differentiability by finding $\lim_{x \to c^-} f'(x)$ and $\lim_{x \to c^+} f'(x)$ to determine if $\lim_{x \to c^-} f'(x) = \lim_{x \to c^+} f'(x)$

$\lim_{x \to c^-} f'(x) = \lim_{x \to c^+} f'(x)$

85) $f(x) = |x - 1| = \begin{cases} x - 1 & \text{if } x > 1 \\ -(x-1) & \text{if } x < 1 \end{cases}$

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \frac{-(x-1) - 0}{x - 1} = -1$$

$$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \frac{x - 1 - 0}{x - 1} = 1$$

since $\lim_{x \to 1^-} f'(x) \neq \lim_{x \to 1^+} f'(x)$, $f(x)$ is not differentiable at $x = 1$

87) $f(x) = \begin{cases} (x-1)^3, & x \leq 1 \\ (x-1)^2, & x > 1 \end{cases}$

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \frac{(x-1)^3 - 0}{x - 1} = 0$$

$$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \frac{(x-1)^2 - 0}{x - 1} = 0$$

since $\lim_{x \to 1^-} f'(x) = \lim_{x \to 1^+} f'(x)$, f is differentiable at $x = 1$

89) Determine differentiability at $x=2$

$$f(x)=\begin{cases} x^2+1, & x \le 2 \\ 4x-3, & x > 2 \end{cases}$$

$*$ Test if $\lim_{x\to 2^-} f'(x) = \lim_{x\to 2^+} f'(x)$

$$\lim_{x\to 2^-} \frac{f(x)-f(2)}{x-2} = \frac{x^2+1-5}{x-2} = \lim_{x\to 2^-} \frac{x^2-4}{x-2} = \lim_{x\to 2^-} \frac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}} = 4$$

$$\lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2} = \frac{4x-3-5}{x-2} = \lim_{x\to 2^+} \frac{4x-8}{x-2} = \lim_{x\to 2^+} \frac{4\cancel{(x-2)}}{\cancel{x-2}} = 4$$

✓

$$\boxed{\text{Since } \lim_{x\to 2^-} f'(x) = \lim_{x\to 2^+} f'(x),\ f \text{ is differentiable at } x=2\ (f'(2)=4)}$$