EXAMPLE 8 **Approximating the Derivative of a Function Defined by a Table**

The table below lists several values of a function $y = f(x)$ that is continuous on the interval $[-1, 5]$ and has a derivative at each number in the interval $(-1, 5)$. Approximate the derivative of f at 2.

x	0	1	2	3	4
$f(x)$	0	3	12	33	72

Solution

There are several ways to approximate the derivative of a function defined by a table. Each uses an average rate of change to approximate the rate of change of f at 2, which is the derivative of f at 2.

- Using the average rate of change from 2 to 3, we have
$$\frac{f(3)-f(2)}{3-2} = \frac{33-12}{1} = 21$$
With this choice, $f'(2)$ is approximately 21.
- Using the average rate of change from 1 to 2, we have
$$\frac{f(2)-f(1)}{2-1} = \frac{12-3}{1} = 9$$
With this choice, $f'(2)$ is approximately 9.
- A third approximation can be found by averaging the above two approximations. Then $f'(2)$ is approximately $\frac{21+9}{2} = 15$. ■

NOW WORK Problem 51 and AP® Practice Problem 8.

2.1 Assess Your Understanding

Concepts and Vocabulary

1. *True or False* The derivative is used to find instantaneous velocity.
2. *True or False* The derivative can be used to find the rate of change of a function.
3. The notation $f'(c)$ is read f ____ of c; $f'(c)$ represents the ____ of the tangent line to the graph of f at the point ____.
4. *True or False* If it exists, $\lim\limits_{x\to 3}\frac{f(x)-f(3)}{x-3}$ is the derivative of the function f at 3.
5. If $f(x) = 6x - 3$, then $f'(3) =$ ________.
6. The velocity of an object, the slope of a tangent line, and the rate of change of a function are three different interpretations of the mathematical concept called the ________.

Skill Building

In Problems 7–16,

(a) Find an equation for the tangent line to the graph of each function at the indicated point.

(b) Find an equation of the normal line to each function at the indicated point.

(c) Graph the function, the tangent line, and the normal line at the indicated point on the same set of coordinate axes.

7. $f(x) = 3x^2$ at $(-2, 12)$
8. $f(x) = x^2 + 2$ at $(-1, 3)$
9. $f(x) = x^3$ at $(-2, -8)$
10. $f(x) = x^3 + 1$ at $(1, 2)$
11. $f(x) = \frac{1}{x}$ at $(1, 1)$ (PAGE 163)
12. $f(x) = \sqrt{x}$ at $(4, 2)$
13. $f(x) = \frac{1}{x+5}$ at $\left(1, \frac{1}{6}\right)$
14. $f(x) = \frac{2}{x+4}$ at $\left(1, \frac{2}{5}\right)$
15. $f(x) = \frac{1}{\sqrt{x}}$ at $(1, 1)$ (PAGE 167)
16. $f(x) = \frac{1}{x^2}$ at $(1, 1)$

In Problems 17–20, find the rate of change of f at the indicated numbers.

17. $f(x) = 5x - 2$ at **(a)** $c = 0$, **(b)** $c = 2$ (PAGE 164)
18. $f(x) = x^2 - 1$ at **(a)** $c = -1$, **(b)** $c = 1$
19. $f(x) = \frac{x^2}{x+3}$ at **(a)** $c = 0$, **(b)** $c = 1$
20. $f(x) = \frac{x}{x^2-1}$ at **(a)** $c = 0$, **(b)** $c = 2$

In Problems 21–30, find the derivative of each function at the given number.

21. $f(x) = 2x + 3$ at 1
22. $f(x) = 3x - 5$ at 2
23. $f(x) = x^2 - 2$ at 0 (PAGE 167)
24. $f(x) = 2x^2 + 4$ at 1
25. $f(x) = 3x^2 + x + 5$ at -1
26. $f(x) = 2x^2 - x - 7$ at -1
27. $f(x) = \sqrt{x}$ at 4
28. $f(x) = \frac{1}{x^2}$ at 2
29. $f(x) = \frac{2-5x}{1+x}$ at 0
30. $f(x) = \frac{2+3x}{2+x}$ at 1

PAGE 166 **31. Approximating Velocity** An object in rectilinear motion moves according to the position function $s(t) = 10t^2$ (s in centimeters and t in seconds). Approximate the velocity of the object at time $t_0 = 3$ s by letting Δt first equal 0.1 s, then 0.01 s, and finally 0.001 s. What limit does the velocity appear to be approaching? Organize the results in a table.

32. Approximating Velocity An object in rectilinear motion moves according to the position function $s(t) = 5 - t^2$ (s in centimeters and t in seconds). Approximate the velocity of the object at time $t_0 = 1$ by letting Δt first equal 0.1, then 0.01, and finally 0.001. What limit does the velocity appear to be approaching? Organize the results in a table.

PAGE 166 **33. Rectilinear Motion** As an object in rectilinear motion moves, its signed distance s (in meters) from the origin after t seconds is given by the position function $s = f(t) = 3t^2 + 4t$. Find the velocity v at $t_0 = 0$. At $t_0 = 2$. At any time t_0.

34. Rectilinear Motion As an object in rectilinear motion moves, its signed distance s (in meters) from the origin after t seconds is given by the position function $s = f(t) = 2t^3 + 4$. Find the velocity v at $t_0 = 0$. At $t_0 = 3$. At any time t_0.

35. Rectilinear Motion As an object in rectilinear motion moves, its signed distance s from the origin at time t is given by the position function $s = s(t) = 3t^2 - \frac{1}{t}$, where s is in centimeters and t is in seconds. Find the velocity v of the object at $t_0 = 1$ and $t_0 = 4$.

36. Rectilinear Motion As an object in rectilinear motion moves, its signed distance s from the origin at time t is given by the position function $s = s(t) = 2\sqrt{t}$, where s is in centimeters and t is in seconds. Find the velocity v of the object at $t_0 = 1$ and $t_0 = 4$.

37. The Princeton Dinky is the shortest rail line in the country. It runs for 2.7 miles, connecting Princeton University to the Princeton Junction railroad station. The Dinky starts from the university and moves north toward Princeton Junction. Its distance from Princeton is shown in the graph (top, right), where the time t is in minutes and the distance s of the Dinky from Princeton University is in miles.

MancusoMichael/AP Images

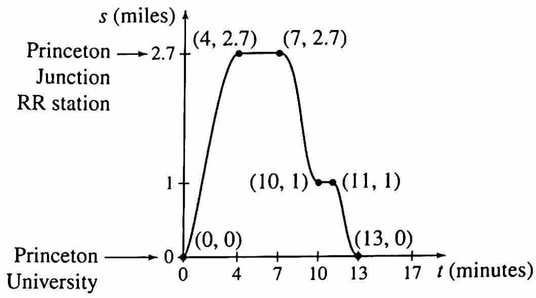
(a) When is the Dinky headed toward Princeton University?

(b) When is it headed toward Princeton Junction?

(c) When is the Dinky stopped?

(d) Find its average velocity on a trip from Princeton to Princeton Junction.

(e) Find its average velocity for the round-trip shown in the graph, that is, from $t = 0$ to $t = 13$.



38. Barbara walks to the deli, which is six blocks east of her house. After walking two blocks, she realizes she left her phone on her desk, so she runs home. After getting the phone, and closing and locking the door, Barbara starts on her way again. At the deli, she waits in line to buy a bottle of **vitaminwater**™, and then she jogs home. The graph below represents Barbara's journey. The time t is in minutes, and s is Barbara's distance, in blocks, from home.

(a) At what times is she headed toward the deli?

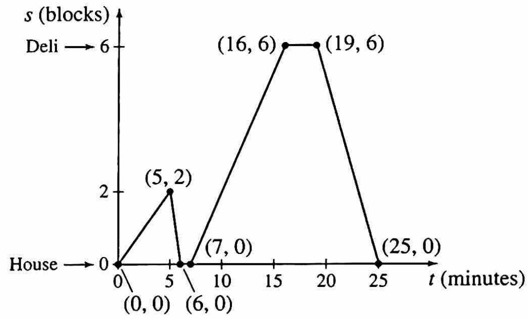
(b) At what times is she headed home?

(c) When is the graph horizontal? What does this indicate?

(d) Find Barbara's average velocity from home until she starts back to get her phone.

(e) Find Barbara's average velocity from home to the deli after getting her phone.

(f) Find her average velocity from the deli to home.



Applications and Extensions

39. Slope of a Tangent Line An equation of the tangent line to the graph of a function f at $(2, 6)$ is $y = -3x + 12$. What is $f'(2)$?

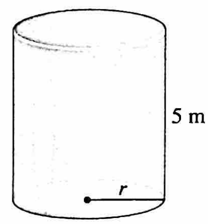
40. Slope of a Tangent Line An equation of the tangent line of a function f at $(3, 2)$ is $y = \frac{1}{3}x + 1$. What is $f'(3)$?

41. Tangent Line Does the tangent line to the graph of $y = x^2$ at $(1, 1)$ pass through the point $(2, 5)$?

42. Tangent Line Does the tangent line to the graph of $y = x^3$ at $(1, 1)$ pass through the point $(2, 5)$?

PAGE 164 **43. Respiration Rate** A human being's respiration rate R (in breaths per minute) is given by $R = R(p) = 10.35 + 0.59p$, where p is the partial pressure of carbon dioxide in the lungs. Find the rate of change in respiration when $p = 50$.

44. Instantaneous Rate of Change The volume V of the right circular cylinder of height 5 m and radius r m shown in the figure is $V = V(r) = 5\pi r^2$. Find the instantaneous rate of change of the volume with respect to the radius when $r = 3$ m.



45. Market Share During a month-long advertising campaign, the total sales S of a magazine is modeled by the function $S(x) = 5x^2 + 100x + 10{,}000$, where x, $0 \le x \le 30$, represents the number of days since the campaign began.

(a) What is the average rate of change of sales from $x = 10$ to $x = 20$ days?

(b) What is the instantaneous rate of change of sales when $x = 10$ days?

46. Demand Equation The demand equation for an item is $p = p(x) = 90 - 0.02x$, where p is the price in dollars and x is the number of units (in thousands) made.

(a) Assuming all units made can be sold, find the revenue function $R(x) = xp(x)$.

(b) **Marginal Revenue** Marginal revenue is defined as the additional revenue earned by selling an additional unit. If we use $R'(x)$ to measure the marginal revenue, find the marginal revenue when 1 million units are sold.

47. Gravity If a ball is dropped from the top of the Empire State Building, 1002 ft above the ground, the distance s (in feet) it falls after t seconds is $s(t) = 16t^2$.

(a) What is the average velocity of the ball for the first 2 s?

(b) How long does it take for the ball to hit the ground?

(c) What is the average velocity of the ball during the time it is falling?

(d) What is the velocity of the ball when it hits the ground?

48. Velocity A ball is thrown upward. Its height h in feet is given by $h(t) = 100t - 16t^2$, where t is the time elapsed in seconds.

(a) What is the velocity v of the ball at $t = 0$ s, $t = 1$ s, and $t = 4$ s?

(b) At what time t does the ball strike the ground?

(c) At what time t does the ball reach its highest point? *Hint:* At the time the ball reaches its maximum height, it is stationary. So, its velocity $v = 0$.

49. Gravity A rock is dropped from a height of 88.2 m and falls toward Earth in a straight line. In t seconds the rock falls $4.9t^2$ m.

(a) What is the average velocity of the rock for the first 2 s?

(b) How long does it take for the rock to hit the ground?

(c) What is the average velocity of the rock during its fall?

(d) What is the velocity v of the rock when it hits the ground?

50. Velocity At a certain instant, the speedometer of an automobile reads V mi/h. During the next $\frac{1}{4}$ s the automobile travels 20 ft. Approximate V from this information.

PAGE 168 **51.** A tank is filled with 80 liters of water at 7 a.m. ($t = 0$). Over the next 12 h the water is continuously used. The table below gives the amount of water $A(t)$ (in liters) remaining in the tank at selected times t, where t measures the number of hours after 7 a.m.

t	0	2	5	7	9	12
$A(t)$	80	71	66	60	54	50

(a) Use the table to approximate $A'(5)$.

(b) Using appropriate units, interpret $A'(5)$ in the context of the problem.

52. The table below lists the outside temperature T, in degrees Fahrenheit, in Naples, Florida, on a certain day in January, for selected times x, where x is the number of hours since 12 a.m.

x	5	7	9	11	12	13	14	16	17
$T(x)$	62	71	74	78	81	83	84	85	78

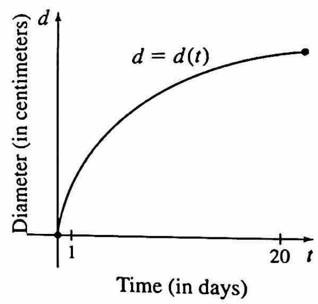
(a) Use the table to approximate $T'(11)$.

(b) Using appropriate units, interpret $T'(11)$ in the context of the problem.

53. Rate of Change Show that the rate of change of a linear function $f(x) = mx + b$ is the slope m of the line $y = mx + b$.

54. Rate of Change Show that the rate of change of a quadratic function $f(x) = ax^2 + bx + c$ is a linear function of x.

55. Agriculture The graph represents the diameter d (in centimeters) of a maturing peach as a function of the time t (in days) it is on the tree.

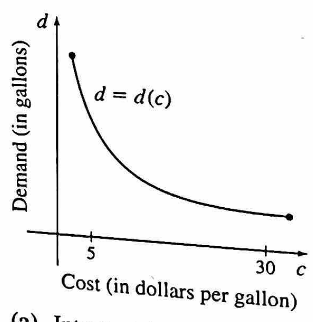


(a) Interpret the derivative $d'(t)$ as a rate of change.

(b) Which is larger, $d'(1)$ or $d'(20)$?

(c) Interpret both $d'(1)$ and $d'(20)$.

56. Business The graph represents the demand d (in gallons) for olive oil as a function of the cost c (in dollars per gallon) of the oil.



(a) Interpret the derivative $d'(c)$.

(b) Which is larger, $d'(5)$ or $d'(30)$? Give an interpretation to $d'(5)$ and $d'(30)$.

57. Volume of a Cube A metal cube with each edge of length x centimeters is expanding uniformly as a consequence of being heated.

(a) Find the average rate of change of the volume of the cube with respect to an edge as x increases from 2.00 to 2.01 cm.

(b) Find the instantaneous rate of change of the volume of the cube with respect to an edge at the instant when $x = 2$ cm.

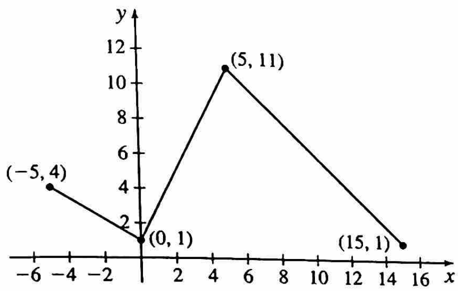
AP® Practice Problems

Preparing for the AP® Exam

PAGE 163 **1.** The line $x + y = 5$ is tangent to the graph of $y = f(x)$ at the point where $x = 2$. The values $f(2)$ and $f'(2)$ are:

(A) $f(2) = 2;\ f'(2) = -1$ (B) $f(2) = 3;\ f'(2) = -1$
(C) $f(2) = 2;\ f'(2) = 1$ (D) $f(2) = 3;\ f'(2) = 2$

PAGE 167 **2.** The graph of the function f, given below, consists of three line segments. Find $f'(3)$.



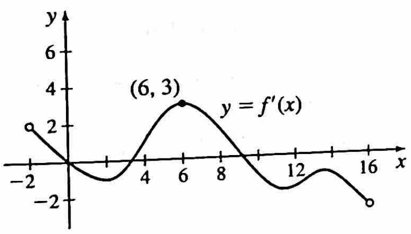
Graph of f

(A) 1 (B) 2 (C) 3 (D) $f'(3)$ does not exist

PAGE 164 **3.** What is the instantaneous rate of change of the function $f(x) = 3x^2 + 5$ at $x = 2$?

(A) 5 (B) 7 (C) 12 (D) 17

PAGE 167 **4.** The function f is defined on the closed interval $[-2, 16]$. The graph of the derivative of f, $y = f'(x)$, is given below.



The point $(6, -2)$ is on the graph of $y = f(x)$. An equation of the tangent line to the graph of f at $(6, -2)$ is

(A) $y = 3$ (B) $y + 2 = 6(x + 3)$
(C) $y + 2 = 6x$ (D) $y + 2 = 3(x - 6)$

PAGE 163 **5.** If $x - 3y = 13$ is an equation of the normal line to the graph of f at the point $(2, 6)$, then $f'(2) =$

(A) $-\frac{1}{3}$ (B) $\frac{1}{3}$ (C) -3 (D) $-\frac{13}{3}$

PAGE 167 **6.** If f is a function for which $\lim_{x\to -3} \frac{f(x) - f(-3)}{x + 3} = 0$, then which of the following statements must be true?

(A) $x = -3$ is a vertical asymptote of the graph.
(B) The derivative of f at $x = -3$ exists.
(C) The function f is continuous at $x = 3$.
(D) f is not defined at $x = -3$.

PAGE 165 **7.** If the position of an object on the x-axis at time t is $4t^2$, then the average velocity of the object over the interval $0 \le t \le 5$ is

(A) 5 (B) 20 (C) 40 (D) 100

PAGE 168 **8.** A tank is filled with 80 liters of water at 7 a.m. ($t = 0$). Over the next 12 hours the water is continuously used and no water is added to replace it. The table below gives the amount of water $A(t)$ (in liters) remaining in the tank at selected times t, where t measures the number of hours after 7 a.m.

t	0	2	5	7	9	12
$A(t)$	80	71	66	60	54	50

Use the table to approximate $A'(5)$.

2.2 The Derivative as a Function; Differentiability

OBJECTIVES ***When you finish this section, you should be able to:***

1 Define the derivative function (p. 171)
2 Graph the derivative function (p. 173)
3 Identify where a function is not differentiable (p. 175)

1 Define the Derivative Function

The derivative of f at a real number c has been defined as the real number

$$f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c} \qquad (1)$$

provided the limit exists. We refer to this representation of the derivative as Form (1).