Key

A.P. Calculus AB 2.2-2.5 Review Session Problems (WS #4)

1) Consider the curve given by $x^2 - xy + y^2 = 4$.

a) Find the two points on the curve at x = 0

$(0)^2 - (0)y + y^2 = 4$

$y^2 = 4$

$\sqrt{y^2} = \pm\sqrt{4}$

$y = 2, -2$

$\boxed{\text{The 2 points are } (0,2) \text{ and } (0,-2)}$

b) Find $\frac{dy}{dx}$ by differentiating implicitly.

* product rule

$x^2 - (\underbrace{x}_{f}\underbrace{y}_{g}) + y^2 = 4$

$2x - \left(\underbrace{(1)}_{f'}\underbrace{(y)}_{g} + \underbrace{(x)}_{f}\underbrace{(1)\left(\frac{dy}{dx}\right)}_{g'}\right) + 2y\left(\frac{dy}{dx}\right) = 0$

$2x - y - x\left(\frac{dy}{dx}\right) + 2y\left(\frac{dy}{dx}\right) = 0$

$-x\left(\frac{dy}{dx}\right) + 2y\left(\frac{dy}{dx}\right) = -2x + y$

$\frac{dy}{dx}(-x + 2y) = -2x + y$

$\boxed{\frac{dy}{dx} = \frac{-2x+y}{-x+2y}}$

c) Use $\frac{dy}{dx}$ to find the slope of the lines tangent to the curve at the points found in part a.

i) point: $(0,2)$

slope: $\left.\frac{dy}{dx}\right|_{(0,2)} = \frac{-2(0)+2}{-0+2(2)} = \frac{2}{4}$

$\boxed{\text{slope: } m = 1/2}$

$\boxed{y - 2 = \frac{1}{2}(x - 0)}$

ii) point: $(0,-2)$

slope: $\left.\frac{dy}{dx}\right|_{(0,-2)} = \frac{-2(0)-2}{-0+2(-2)} = \frac{-2}{-4}$

$\boxed{m = 1/2}$

$\boxed{y + 2 = \frac{1}{2}(x - 0)}$

d) Write the equation of the line tangent to the curve at the points above

$\boxed{y - 2 = \frac{1}{2}(x - 0)}$ $\boxed{y + 2 = \frac{1}{2}(x - 0)}$

e) Detail how you would set up (don't solve) in order to find vertical and horizontal tangent:

horizontal tangent: set numerator of $\frac{dy}{dx} = 0$ → $\boxed{-2x + y = 0}$

vertical tangent: set denominator of $\frac{dy}{dx} = 0$ → $\boxed{-x + 2y = 0}$

2) If g is differentiable everywhere and $g(x) = \begin{cases} 6x^3 - 8x^2 + 8, & x < -2 \\ ax + b, & x \ge -2 \end{cases}$, find a and b

(Involve derivatives in your work)

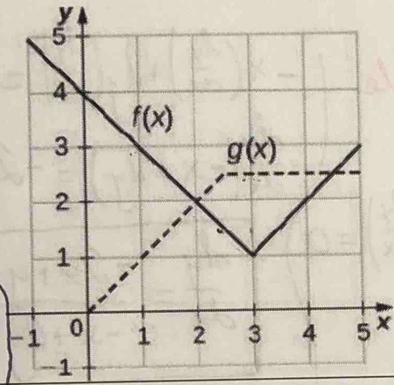
continuous property: set equations equal (at $x=-2$)

$$6x^3 - 8x^2 + 8 = ax + b$$
$$6(-2)^3 - 8(-2)^2 + 8 = a(-2) + b$$
$$-72 = -2a + b$$
$$-72 = -2(104) + b$$
$$\boxed{136 = b}$$

differentiable property: set derivatives equal (at $x=-2$)

$$18x^2 - 16x = a$$
$$18(-2)^2 - 16(-2) = a$$
$$\boxed{a = 104}$$

3)



a) If $h(x) = f(x) \cdot g(x)$, find $h'(2)$ *product rule

$$h'(x) = f'(x)g(x) + f(x)g'(x)$$
$$h'(2) = f'(2)g(2) + f(2)g'(2)$$
$$h'(2) = (-1)(2) + (2)(1)$$
$$\boxed{h'(2) = 0}$$

b) If $w(x) = f(g(x))$, find $w'(3)$ *chain rule

$$w'(x) = f'[g(x)] \cdot g'(x)$$
$$w'(3) = f'[g(3)] \cdot g'(3)$$
$$w'(3) = f'[2.5] \cdot g'(3)$$
$$w'(3) = (-1)(0)$$
$$\boxed{w'(3) = 0}$$

*chain rule
out: $2[\ \ ]^3$
in: $f(x)$

c) If $z(x) = 2[f(x)]^3$ find $z'(0)$

$$z'(x) = 6(\ \ )^2 \cdot f'(x)$$
$$z'(x) = 6[f(x)]^2 \cdot f'(x)$$
$$z'(0) = 6[f(0)]^2 \cdot f'(0)$$
$$z'(0) = 6[4]^2 \cdot (-1)$$
$$\boxed{z'(0) = -96}$$

d) If $k(x) = f(f(x))$, find $k'(4)$ ← chain rule

$$k'(x) = f'[f(x)] \cdot f'(x)$$
$$k'(4) = f'[f(4)] \cdot f'(4)$$
$$k'(4) = f'[2] \cdot f'(4)$$
$$k'(4) = (-1)(1)$$
$$\boxed{k'(4) = -1}$$

e) If $p(x) = \frac{g(x)}{f(x)}$ find $p'(1)$ *quotient rule

$$p'(x) = \frac{g'(x)f(x) - g(x)f'(x)}{[f(x)]^2}$$
$$p'(1) = \frac{g'(1)f(1) - g(1)f'(1)}{[f(1)]^2}$$
$$p'(1) = \frac{(1)(3) - (1)(-1)}{3^2}$$
$$p'(1) = \frac{3+1}{9} = \boxed{\frac{4}{9}}$$

4) Find $\frac{dy}{dx}$ for $y = 2\left(\frac{5x^3-2}{3-x^2}\right)^7$ (Write derivative as a simplified rational expression)

*i) chain rule
ii) quotient rule

out: $2[\ \ ]^7$

in: $\frac{5x^3-2}{3-x^2}$ ↑ quotient rule

$$y' = 14\left[\frac{5x^3-2}{3-x^2}\right]^6 \cdot \left[\frac{\overbrace{15x^2}^{f'}\overbrace{(3-x^2)}^{g} - \overbrace{(5x^3-2)}^{f}\overbrace{(-2x)}^{g'}}{\underbrace{(3-x^2)^2}_{}}\right]$$

$45x^2 - 15x^4 + 10x^4 - 4x$

$$y' = 14\left[\frac{5x^3-2}{3-x^2}\right]^6 \left[\frac{-5x^4 + 45x^2 - 4x}{(3-x^2)^2}\right] \quad g^2$$

$$\boxed{y' = \frac{14(5x^3-2)^6(-5x^4 + 45x^2 - 4x)}{(3-x^2)^8}}$$

5) A particle moves along a straight line according to the given equation: x(t) = $\frac{t^4}{4} - t^3 - 2t^2 + 1$, for all real numbers in **meters per minute**

a) Find the velocity and acceleration function

$v(t) = t^3 - 3t^2 - 4t$

$a(t) = 3t^2 - 6t - 4$

b.) Find when the particle changes direction (justify with because statement)

$0 = t(t^2 - 3t - 4)$

$0 = t(t-4)(t+1)$

$t = 0, -1, 4$

$v(t)$

− + − +

−2 −1 −1/2 0 1 4 5

Direction change at $t = -1, 0, 4$ b/c $v(t)$ change signs

c) Determine interval when particle is moving left (justify with because statement)

particle moves left $(-\infty, -1) \cup (0, 4)$ b/c $v(t) < 0$

d) Determine interval when particle is moving left (justify with because statement)

particle moves right $(-1, 0) \cup (4, \infty)$ b/c $v(t) > 0$

e) Find the average velocity of particle in interval [1, 3] (show work, include units)

Avg. velocity $= \frac{\text{change in position}}{\text{change in time}} \rightarrow \frac{x(3) - x(1)}{3 - 1}$

$x(3) = -23.75$

$x(1) = -1.75$

$\frac{-23.75 - (-1.75)}{3 - 1} = \boxed{-11 \text{ meters/min}}$

f) Find the average acceleration of particle in interval [1, 3] (show work, include units)

Avg. acceleration $= \frac{\text{change in velocity}}{\text{change in time}} \rightarrow \frac{v(3) - v(1)}{3 - 1}$

$v(3) = -12$

$v(1) = -6$

$\frac{-12 - (-6)}{3 - 1} = \boxed{-3 \text{ meters/min}^2}$

g) Find particle's displacement from t = 1 to t = 6 (Show your work)

displacement = final position − initial position

displacement $= x(6) - x(1)$

$= 37 - (-1.75)$

$x(6) = 37$

$x(1) = -1.75$

$\boxed{\text{displacement is } 38.75\text{m}}$

h) Find particle's distance from t = 1 to t = 6 (Show your work) *count the distance of endpoints to locations of direction change

(*direction change at $t = 4$)

$x(1) = -1.75$

$x(4) = -31$

$x(6) = 37$

$x(t)$

−31 −1.75 37

29.25

68

$\boxed{\text{Total distance is } 29.25 + 68 = 97.25\text{m}}$

i) At t = 2, is the speed increasing or decreasing? Provide justification for your answer.

$v(2) = -12$ $a(2) = -4$

$\boxed{\text{Speed is increasing since } v(t) \text{ and } a(t) \text{ have same signs at } t = 2}$

j) At t = 5, is the velocity increasing or decreasing? Provide justification for your answer.

$a(5) = 41$

*This phrasing is describing acceleration, not velocity

$\boxed{\text{Velocity is increasing at } t = 5 \text{ since acceleration is positive}}$