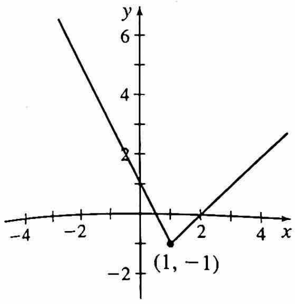


Figure 22 $g(x) = \begin{cases} 1-2x & \text{if } x \le 1 \\ x-2 & \text{if } x > 1 \end{cases}$

(b) See Figure 22. The function g is continuous at 1, which you should verify. To determine whether g is differentiable at 1, examine the one-sided limits at 1 using Form (1).

For $x < 1$,

$$\lim_{x\to 1^-} \frac{g(x)-g(1)}{x-1} = \lim_{x\to 1^-} \frac{(1-2x)-(-1)}{x-1} = \lim_{x\to 1^-} \frac{2-2x}{x-1}$$
$$= \lim_{x\to 1^-} \frac{-2(x-1)}{x-1} = \lim_{x\to 1^-} (-2) = -2$$

For $x > 1$,

$$\lim_{x\to 1^+} \frac{g(x)-g(1)}{x-1} = \lim_{x\to 1^+} \frac{(x-2)-(-1)}{x-1} = \lim_{x\to 1^+} \frac{x-1}{x-1} = 1$$

The one-sided limits are not equal, so $\lim_{x\to 1} \frac{g(x)-g(1)}{x-1}$ does not exist. That is, g is not differentiable at 1. ■

Notice in Figure 21 the tangent lines to the graph of f turn smoothly around the origin. On the other hand, notice in Figure 22 the tangent lines to the graph of g change abruptly at the point $(1, -1)$, where the graph of g has a corner.

NOW WORK Problem 41 and AP® Practice Problems 3, 4, 6, and 7.

2.2 Assess Your Understanding

Concepts and Vocabulary

1. *True or False* The domain of a function f and the domain of its derivative function f' are always equal.
2. *True or False* If a function is continuous at a number c, then it is differentiable at c.
3. *Multiple Choice* If f is continuous at a number c and if $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ is infinite, then the graph of f has

   [**(a)** a horizontal **(b)** a vertical **(c)** no]

   tangent line at c.
4. The instruction, "Differentiate f," means to find the ____ of f.

Skill Building

In Problems 5–10, find the derivative of each function f at any real number c. Use Form (1) on page 171.

5. $f(x) = 10$
6. $f(x) = -4$
7. $f(x) = 2x + 3$
8. $f(x) = 3x - 5$
9. $f(x) = 2 - x^2$
10. $f(x) = 2x^2 + 4$

In Problems 11–16, differentiate each function f and determine the domain of f′. Use Form (2) on page 172.

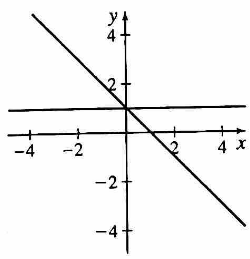
11. $f(x) = 5$
12. $f(x) = -2$
13. $f(x) = 3x^2 + x + 5$
14. $f(x) = 2x^2 - x - 7$
15. $f(x) = 5\sqrt{x-1}$ (PAGE 173)
16. $f(x) = 4\sqrt{x+3}$

In Problems 17–22, differentiate each function f. Graph $y = f(x)$ and $y = f'(x)$ on the same set of coordinate axes.

17. $f(x) = \frac{1}{3}x + 1$
18. $f(x) = -4x - 5$
19. $f(x) = 2x^2 - 5x$ (PAGE 174)
20. $f(x) = -3x^2 + 2$
21. $f(x) = x^3 - 8x$
22. $f(x) = -x^3 - 8$

In Problems 23–26, for each figure determine if the graphs represent a function f and its derivative f′. If they do, indicate which is the graph of f and which is the graph of f′.

23.



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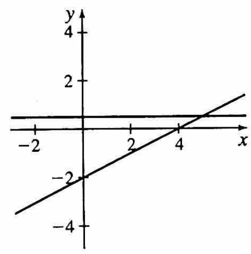

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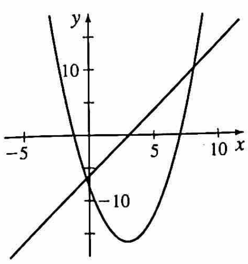

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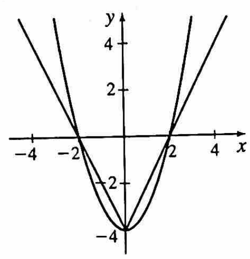
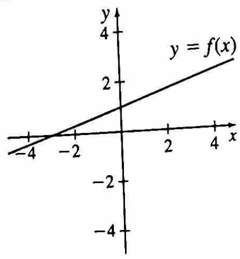


In Problems 27–30, use the graph of f to obtain the graph of f′.

27.



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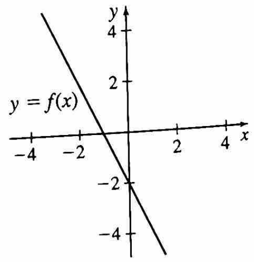
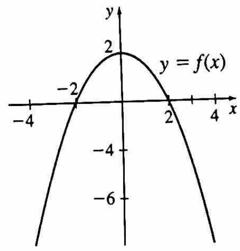


PAGE 175 **29.**



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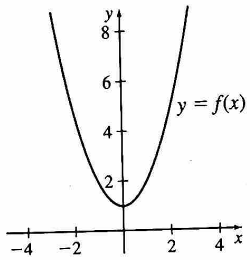
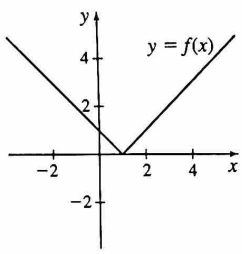


In Problems 31–34, the graph of a function f is given. Match each graph to the graph of its derivative f′ in A–D.

31.



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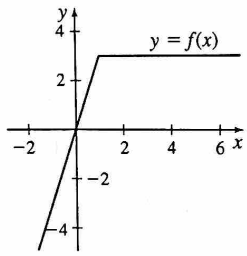

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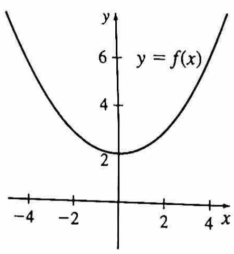

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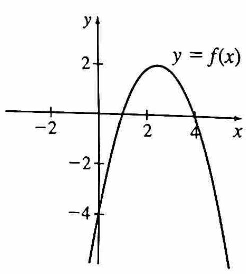
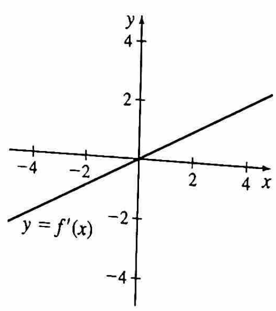
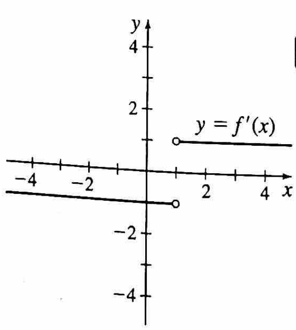
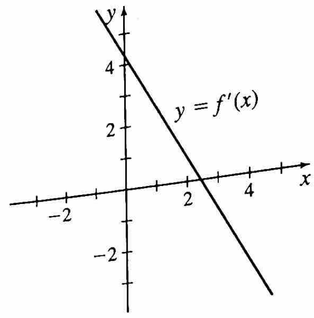




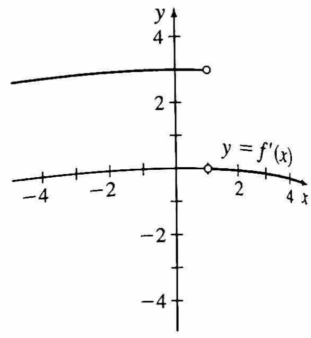
(A)



(B)



(C)



(D)

In Problems 35–44, determine whether each function f has a derivative at c. If it does, what is $f'(c)$? If it does not, give the reason why.

PAGE 176 **35.** $f(x)=x^{2/3}$ at $c=-8$

36. $f(x)=2x^{1/3}$ at $c=0$

37. $f(x)=|x^2-4|$ at $c=2$

38. $f(x)=|x^2-4|$ at $c=-2$

PAGE 176 **39.** $f(x)=\begin{cases} 2x+3 & \text{if } x<1 \\ x^2+4 & \text{if } x\geq 1 \end{cases}$ at $c=1$

40. $f(x)=\begin{cases} 3-4x & \text{if } x<-1 \\ 2x+9 & \text{if } x\geq -1 \end{cases}$ at $c=-1$

PAGE 179 **41.** $f(x)=\begin{cases} -4+2x & \text{if } x\leq \frac{1}{2} \\ 4x^2-4 & \text{if } x>\frac{1}{2} \end{cases}$ at $c=\frac{1}{2}$

42. $f(x)=\begin{cases} 2x^2+1 & \text{if } x<-1 \\ -1-4x & \text{if } x\geq -1 \end{cases}$ at $c=-1$

PAGE 178 **43.** $f(x)=\begin{cases} 2x^2+1 & \text{if } x<-1 \\ 2+2x & \text{if } x\geq -1 \end{cases}$ at $c=-1$

44. $f(x)=\begin{cases} 5-2x & \text{if } x<2 \\ x^2 & \text{if } x\geq 2 \end{cases}$ at $c=2$

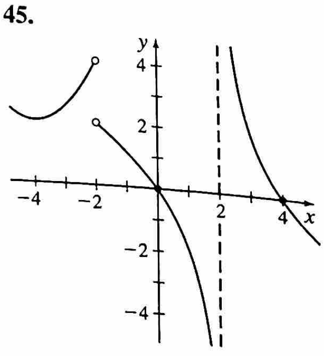
In Problems 45–48, each function f is continuous for all real numbers, and the graph of $y=f'(x)$ is given.

(a) *Does the graph of f have any horizontal tangent lines? If yes, explain why and identify where they occur.*

(b) *Does the graph of f have any vertical tangent lines? If yes, explain why, identify where they occur, and determine whether the point is a cusp of f.*

(c) *Does the graph of f have any corners? If yes, explain why and identify where they occur.*

PAGE 177 **45.**



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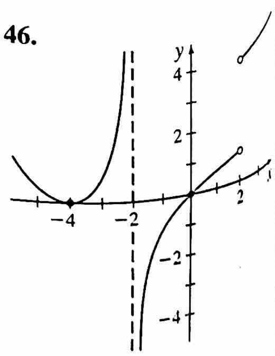

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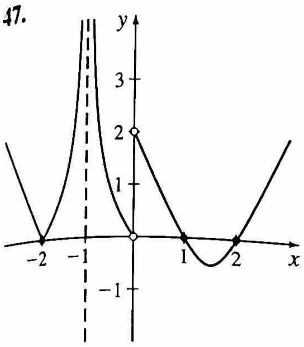

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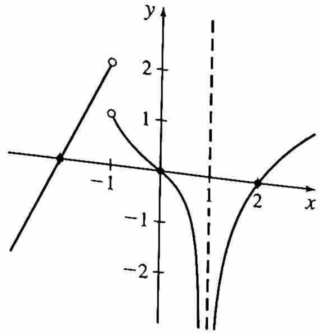
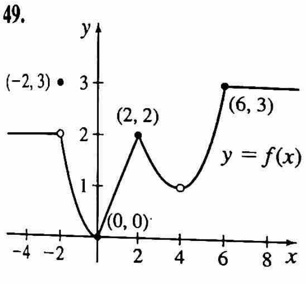


In Problems 49 and 50, use the given points $(c, f(c))$ on the graph of the function f.

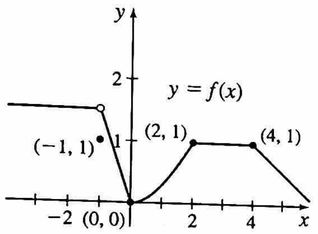
(a) For which numbers c does $\lim_{x\to c} f(x)$ exist but f is not continuous at c?

(b) For which numbers c is f continuous at c but not differentiable at c?

49.



50.



In Problems 51–54, find the derivative of each function.

51. $f(x)=mx+b$

52. $f(x)=ax^2+bx+c$

53. $f(x)=\dfrac{1}{x^2}$

54. $f(x)=\dfrac{1}{\sqrt{x}}$

Applications and Extensions

In Problems 55–66, each limit represents the derivative of a function f at some number c. Determine f and c in each case.

55. $\lim_{h\to 0}\dfrac{(2+h)^2-4}{h}$

56. $\lim_{h\to 0}\dfrac{(2+h)^3-8}{h}$

57. $\lim_{x\to 1}\dfrac{x^2-1}{x-1}$

58. $\lim_{x\to 1}\dfrac{x^4-1}{x-1}$

59. $\lim_{h\to 0}\dfrac{\sqrt{9+h}-3}{h}$

60. $\lim_{h\to 0}\dfrac{(8+h)^{1/3}-2}{h}$

61. $\lim_{x\to\pi/6}\dfrac{\sin x-\frac{1}{2}}{x-\frac{\pi}{6}}$

62. $\lim_{x\to\pi/4}\dfrac{\cos x-\frac{\sqrt{2}}{2}}{x-\frac{\pi}{4}}$

63. $\lim_{x\to 0}\dfrac{2(x+2)^2-(x+2)-6}{x}$

64. $\lim_{x\to 0}\dfrac{3x^3-2x}{x}$

65. $\lim_{h\to 0}\dfrac{(3+h)^2+2(3+h)-15}{h}$

66. $\lim_{h\to 0}\dfrac{3(h-1)^2+h-3}{h}$

67. **Units** The volume V (in cubic feet) of a balloon is expanding according to $V=V(t)=4t$, where t is the time (in seconds). Find the rate of change of the volume of the balloon with respect to time. What are the units of $V'(t)$?

68. **Units** The area A (in square miles) of a circular patch of oil is expanding according to $A=A(t)=2t$, where t is the time (in hours). At what rate is the area changing with respect to time? What are the units of $A'(t)$?

69. **Units** A manufacturer of precision digital switches has a daily cost C (in dollars) of $C(x)=10{,}000+3x$, where x is the number of switches produced daily. What is the rate of change of cost with respect to x? What are the units of $C'(x)$?

70. **Units** A manufacturer of precision digital switches has daily revenue R (in dollars) of $R(x)=5x-\dfrac{x^2}{2000}$, where x is the number of switches produced daily. What is the rate of change of revenue with respect to x? What are the units of $R'(x)$?

71. $f(x)=\begin{cases} x^3 & \text{if } x\le 0\\ x^2 & \text{if } x>0\end{cases}$

(a) Determine whether f is continuous at 0.
(b) Determine whether $f'(0)$ exists.
(c) Graph the function f and its derivative f'.

72. For the function $f(x)=\begin{cases} 2x & \text{if } x\le 0\\ x^2 & \text{if } x>0\end{cases}$

(a) Determine whether f is continuous at 0.
(b) Determine whether $f'(0)$ exists.
(c) Graph the function f and its derivative f'.

73. **Velocity** The distance s (in feet) of an automobile from the origin at time t (in seconds) is given by the position function

$$s=s(t)=\begin{cases} t^3 & \text{if } 0\le t<5\\ 125 & \text{if } t\ge 5\end{cases}$$

(This could represent a crash test in which a vehicle is accelerated until it hits a brick wall at $t=5$ s.)

(a) Find the velocity just before impact (at $t=4.99$ s) and just after impact (at $t=5.01$ s).
(b) Is the velocity function $v=s'(t)$ continuous at $t=5$?
(c) How do you interpret the answer to (b)?

74. **Population Growth** A simple model for population growth states that the rate of change of population size P with respect to time t is proportional to the population size. Express this statement as an equation involving a derivative.

75. **Atmospheric Pressure** Atmospheric pressure p decreases as the distance x from the surface of Earth increases, and the rate of change of pressure with respect to altitude is proportional to the pressure. Express this law as an equation involving a derivative.

76. **Electrical Current** Under certain conditions, an electric current I will die out at a rate (with respect to time t) that is proportional to the current remaining. Express this law as an equation involving a derivative.

77. **Tangent Line** Let $f(x)=x^2+2$. Find all points on the graph of f for which the tangent line passes through the origin.

78. **Tangent Line** Let $f(x)=x^2-2x+1$. Find all points on the graph of f for which the tangent line passes through the point $(1, -1)$.

79. Area and Circumference of a Circle A circle of radius r has area $A = \pi r^2$ and circumference $C = 2\pi r$. If the radius changes from r to $r + \Delta r$, find the:

(a) Change in area.
(b) Change in circumference.
(c) Average rate of change of area with respect to radius.
(d) Average rate of change of circumference with respect to radius.
(e) Rate of change of circumference with respect to radius.

80. Volume of a Sphere The volume V of a sphere of radius r is $V = \frac{4\pi r^3}{3}$. If the radius changes from r to $r + \Delta r$, find the:

(a) Change in volume.
(b) Average rate of change of volume with respect to radius.
(c) Rate of change of volume with respect to radius.

81. Use the definition of the derivative to show that $f(x) = |x|$ is not differentiable at 0.

82. Use the definition of the derivative to show that $f(x) = \sqrt[3]{x}$ is not differentiable at 0.

83. If f is an even function that is differentiable at c, show that its derivative function is odd. That is, show $f'(-c) = -f'(c)$.

84. If f is an odd function that is differentiable at c, show that its derivative function is even. That is, show $f'(-c) = f'(c)$.

85. Tangent Lines and Derivatives Let f and g be two functions, each with derivatives at c. State the relationship between their tangent lines at c if:

(a) $f'(c) = g'(c)$ (b) $f'(c) = -\frac{1}{g'(c)}$ $g'(c) \neq 0$

Challenge Problems

86. Let f be a function defined for all real numbers x. Suppose f has the following properties:

$$f(u+v) = f(u)f(v) \qquad f(0) = 1 \qquad f'(0) \text{ exists}$$

(a) Show that $f'(x)$ exists for all real numbers x.
(b) Show that $f'(x) = f'(0)f(x)$.

87. A function f is defined for all real numbers and has the following three properties:

$$f(1) = 5 \qquad f(3) = 21 \qquad f(a+b) - f(a) = kab + 2b^2$$

for all real numbers a and b where k is a fixed real number independent of a and b.

(a) Use $a = 1$ and $b = 2$ to find k.
(b) Find $f'(3)$.
(c) Find $f'(x)$ for all real x.

88. A function f is **periodic** if there is a positive number p so that $f(x + p) = f(x)$ for all x. Suppose f is differentiable. Show that if f is periodic with period p, then f' is also periodic with period p.

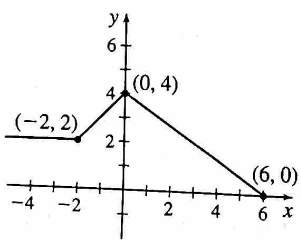
AP® Practice Problems

Preparing for the AP® Exam

PAGE 176 **1.** The function $f(x) = \begin{cases} x^2 - ax & \text{if } x \le 1 \\ ax + b & \text{if } x > 1 \end{cases}$, where a and b are constants. If f is differentiable at $x = 1$, then $a + b =$

(A) -3 (B) -2 (C) 0 (D) 2

PAGE 172 **2.** The graph of the function f, given below, consists of three line segments. Find $\lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$.



(A) -1 (B) $-\frac{2}{3}$ (C) $-\frac{3}{2}$ (D) does not exist

PAGE 179 **3.** If $f(x) = \begin{cases} \frac{x^2 - 25}{x - 5} & \text{if } x \neq 5 \\ 5 & \text{if } x = 5 \end{cases}$

which of the following statements about f are true?

I. $\lim_{x \to 5} f$ exists.
II. f is continuous at $x = 5$.
III. f is differentiable at $x = 5$.

(A) I only (B) I and II only
(C) I and III only (D) I, II, and III

PAGE 179 **4.** Suppose f is a function that is differentiable on the open interval $(-2, 8)$. If $f(0) = 3$, $f(2) = -3$, and $f(7) = 3$, which of the following must be true?

I. f has at least 2 zeros.
II. f is continuous on the closed interval $[-1, 7]$.
III. For some c, $0 < c < 7$, $f(c) = -2$.

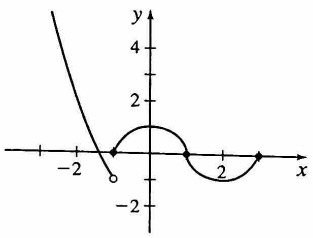
(A) I only (B) I and II only
(C) II and III only (D) I, II, and III

PAGE 176 **5.** If $f(x) = |x|$, which of the following statements about f are true?

I. f is continuous at 0.
II. f is differentiable at 0.
III. $f(0) = 0$.

(A) I only (B) III only
(C) I and III only (D) I, II, and III

PAGE 179 **6.** The graph of the function f shown in the figure has horizontal tangent lines at the points $(0, 1)$ and $(2, -1)$ and a vertical tangent line at the point $(1, 0)$. For what numbers x in the open interval $(-2, 3)$ is f not differentiable?



(A) -1 only (B) -1 and 1 only

(C) $-1, 0,$ and 2 only (D) $-1, 0, 1,$ and 2

PAGE 179 **7.** Let f be a function for which $\lim\limits_{h\to 0} \dfrac{f(1+h) - f(1)}{h} = -3$. Which of the following must be true?

I. f is continuous at 1.
II. f is differentiable at 1.
III. f' is continuous at 1.

(A) I only (B) II only

(C) I and II only (D) I, II, and III

8. At what point on the graph of $f(x) = x^2 - 4$ is the tangent line parallel to the line $6x - 3y = 2$?

(A) $(1, -3)$ (B) $(1, 2)$ (C) $(2, 0)$ (D) $(2, 4)$

PAGE 176 **9.** At $x = 2$, the function $f(x) = \begin{cases} 4x + 1 & \text{if } x \le 2 \\ 3x^2 - 3 & \text{if } x > 2 \end{cases}$ is

(A) Both continuous and differentiable.
(B) Continuous but not differentiable.
(C) Differentiable but not continuous.
(D) Neither continuous nor differentiable.

PAGE 173 **10.** Oil is leaking from a tank. The amount of oil, in gallons, in the tank is given by $G(t) = 4000 - 3t^2$, where t, $0 \le t \le 24$ is the number of hours past midnight.

(a) Find $G'(5)$ using the definition of the derivative.
(b) Using appropriate units, interpret the meaning of $G'(5)$ in the context of the problem.

PAGE 173 **11.** A rod of length 12 cm is heated at one end. The table below gives the temperature $T(x)$ in degrees Celsius at selected numbers x cm from the heated end.

x	0	2	5	7	9	12
$T(x)$	80	71	66	60	54	50

(a) Use the table to approximate $T'(8)$.
(b) Using appropriate units, interpret $T'(8)$ in the context of the problem.

2.3 The Derivative of a Polynomial Function; The Derivative of $y = e^x$

OBJECTIVES ***When you finish this section, you should be able to:***

1. **Differentiate a constant function (p. 184)**
2. **Differentiate a power function (p. 184)**
3. **Differentiate the sum and the difference of two functions (p. 186)**
4. **Differentiate the exponential function $y = e^x$ (p. 189)**

Finding the derivative of a function from the definition can become tedious, especially if the function f is complicated. Just as we did for limits, we derive some basic derivative formulas and some properties of derivatives that make finding a derivative simpler.

Before getting started, we introduce other notations commonly used for the derivative $f'(x)$ of a function $y = f(x)$. The most common ones are

$$y' \qquad \frac{dy}{dx} \qquad Df(x)$$

Leibniz notation $\dfrac{dy}{dx}$ may be written in several equivalent ways as

$$\frac{dy}{dx} = \frac{d}{dx}y = \frac{d}{dx}f(x)$$

where $\dfrac{d}{dx}$ is an instruction to find the derivative (with respect to the independent variable x) of the function $y = f(x)$.

In **operator notation** $Df(x)$, D is said to *operate* on the function, and the result is the derivative of f. To emphasize that the operation is performed with respect to the independent variable x, it is sometimes written $Df(x) = D_x f(x)$.

We use prime notation or Leibniz notation, or sometimes a mixture of the two, depending on which is more convenient. We do not use the notation $Df(x)$ in this book.