Since $\frac{d}{dx}a^x = f'(0)\cdot a^x$, if $f(x)=e^x$, then $\frac{d}{dx}e^x = f'(0)\cdot e^x = 1\cdot e^x = e^x$.

THEOREM Derivative of the Exponential Function $y=e^x$

The derivative of the exponential function $y=e^x$ is

$$y' = \frac{d}{dx}e^x = e^x \qquad (1)$$

EXAMPLE 7 Differentiating an Expression Involving $y=e^x$ (CALC CLIP)

Find the derivative of $f(x)=4e^x+x^3$.

Solution

The function f is the sum of $4e^x$ and x^3. Then

$$f'(x)=\frac{d}{dx}(4e^x+x^3) \underset{\text{Sum Rule}}{=} \frac{d}{dx}(4e^x)+\frac{d}{dx}x^3 \underset{\text{Constant Multiple Rule; Simple Power Rule}}{=} 4\frac{d}{dx}e^x+3x^2 \underset{\text{Use (1).}}{=} 4e^x+3x^2$$

NOTE We have not forgotten $y=\ln x$. Here is its derivative:

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

Use this result for now. We do not have the necessary mathematics to prove it until Chapter 3.

NOW WORK Problem 25 and AP® Practice Problems 4 and 9.

Now we know $\frac{d}{dx}e^x = e^x$. To find the derivative of $f(x)=a^x$, $a>0$ and $a\neq 1$, we need more information. See Chapter 3.

2.3 Assess Your Understanding

Concepts and Vocabulary

PAGE 185 1. $\frac{d}{dx}\pi^2 =$ ____ ; $\frac{d}{dx}x^3 =$ ____ .

2. When n is a positive integer, the Simple Power Rule states that $\frac{d}{dx}x^n =$ ____ .

3. *True or False* The derivative of a power function of degree greater than 1 is also a power function.

4. If k is a constant and f is a differentiable function, then $\frac{d}{dx}[kf(x)] =$ ____ .

5. The derivative of $f(x)=e^x$ is ____ .

6. *True or False* The derivative of an exponential function $f(x)=a^x$, where $a>0$ and $a\neq 1$, is always a constant multiple of a^x.

Skill Building

In Problems 7–26, find the derivative of each function using the formulas of this section. (a, b, c, and d, when they appear, are constants.)

PAGE 187 7. $f(x)=3x+\sqrt{2}$

8. $f(x)=5x-\pi$

9. $f(x)=x^2+3x+4$

10. $f(x)=4x^4+2x^2-2$

11. $f(u)=8u^5-5u+1$

12. $f(u)=9u^3-2u^2+4u+4$

13. $f(s)=as^3+\frac{3}{2}s^2$

14. $f(s)=4-\pi s^2$

15. $f(t)=\frac{1}{3}(t^5-8)$

16. $f(x)=\frac{1}{5}(x^7-3x^2+2)$

17. $f(t)=\frac{t^3+2}{5}$

18. $f(x)=\frac{x^7-5x}{9}$

19. $f(x)=\frac{x^3+2x+1}{7}$

20. $f(x)=\frac{1}{a}(ax^2+bx+c)$, $a\neq 0$

21. $f(x)=ax^2+bx+c$

22. $f(x)=ax^3+bx^2+cx+d$

23. $f(x)=4e^x$

24. $f(x)=-\frac{1}{2}e^x$

PAGE 190 25. $f(u)=5u^2-2e^u$

26. $f(u)=3e^u+10$

In Problems 27–32, find each derivative.

27. $\frac{d}{dt}\left(\sqrt{3}\,t+\frac{1}{2}\right)$

28. $\frac{d}{dt}\left(\frac{2t^4-5}{8}\right)$

29. $\frac{dA}{dR}$ if $A(R)=\pi R^2$

30. $\frac{dC}{dR}$ if $C=2\pi R$

PAGE 186 31. $\frac{dV}{dr}$ if $V=\frac{4}{3}\pi r^3$

32. $\frac{dP}{dT}$ if $P=0.2T$

In Problems 33–36:

(a) Find the slope of the tangent line to the graph of each function f at the indicated point.

(b) Find an equation of the tangent line at the point.

(c) Find an equation of the normal line at the point.

(d) Graph f and the tangent line and normal line found in (b) and (c) on the same set of axes.

PAGE 188 33. $f(x)=x^3+3x-1$ at $(0,-1)$

34. $f(x)=x^4+2x-1$ at $(1,2)$

35. $f(x)=e^x+5x$ at $(0,1)$

36. $f(x)=4-e^x$ at $(0,3)$

In Problems 37–42:

(a) Find the points, if any, at which the graph of each function f has a horizontal tangent line.

(b) Find an equation for each horizontal tangent line.

(c) Solve the inequality $f'(x) > 0$.

(d) Solve the inequality $f'(x) < 0$.

(e) Graph f and any horizontal lines found in (b) on the same set of axes.

(f) Describe the graph of f for the results obtained in parts (c) and (d).

37. $f(x) = 3x^2 - 12x + 4$ **38.** $f(x) = x^2 + 4x - 3$

39. $f(x) = x + e^x$ **40.** $f(x) = 2e^x - 1$

41. $f(x) = x^3 - 3x + 2$ **42.** $f(x) = x^4 - 4x^3$

43. Rectilinear Motion At t seconds, an object in rectilinear motion is s meters from the origin, where $s(t) = t^3 - t + 1$. Find the velocity of the object at $t = 0$ and at $t = 5$.

44. Rectilinear Motion At t seconds, an object in rectilinear motion is s meters from the origin, where $s(t) = t^4 - t^3 + 1$. Find the velocity of the object at $t = 0$ and at $t = 1$.

Rectilinear Motion *In Problems 45 and 46, each position function gives the signed distance s from the origin at time t of an object in rectilinear motion:*

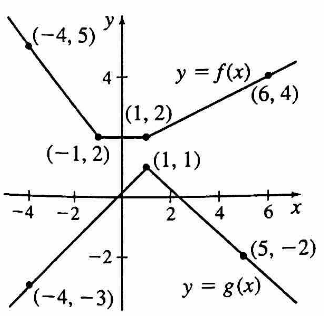
(a) Find the velocity v of the object at any time t.

(b) When is the velocity of the object 0?

45. $s(t) = 2 - 5t + t^2$ **46.** $s(t) = t^3 - \frac{9}{2}t^2 + 6t + 4$

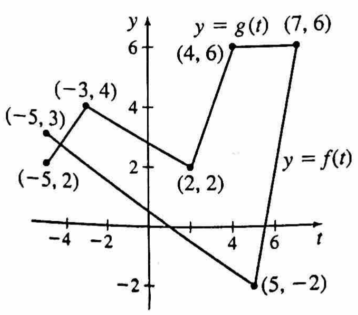
In Problems 47 and 48, use the graphs to find each derivative.

47. Let $u(x) = f(x) + g(x)$ and $v(x) = f(x) - g(x)$.



(a) $u'(0)$ **(b)** $u'(4)$
(c) $v'(-2)$ **(d)** $v'(6)$
(e) $3u'(5)$ **(f)** $-2v'(3)$

48. Let $F(t) = f(t) + g(t)$ and $G(t) = g(t) - f(t)$.



(a) $F'(0)$ **(b)** $F'(3)$
(c) $F'(-4)$ **(d)** $G'(-2)$
(e) $G'(-1)$ **(f)** $G'(6)$

In Problems 49 and 50, for each function f:

(a) Find $f'(x)$ by expanding $f(x)$ and differentiating the polynomial.

CAS *(b) Find $f'(x)$ using a CAS.*

(c) Show that the results found in parts (a) and (b) are equivalent.

49. $f(x) = (2x - 1)^3$ **50.** $f(x) = (x^2 + x)^4$

Applications and Extensions

In Problems 51–56, find each limit.

51. $\lim_{h \to 0} \dfrac{5\left(\frac{1}{2} + h\right)^8 - 5\left(\frac{1}{2}\right)^8}{h}$ **52.** $\lim_{h \to 0} \dfrac{6(2+h)^5 - 6 \cdot 2^5}{h}$

53. $\lim_{h \to 0} \dfrac{\sqrt{3}(8+h)^5 - \sqrt{3} \cdot 8^5}{h}$ **54.** $\lim_{h \to 0} \dfrac{\pi(1+h)^{10} - \pi}{h}$

55. $\lim_{h \to 0} \dfrac{a(x+h)^3 - ax^3}{h}$ **56.** $\lim_{h \to 0} \dfrac{b(x+h)^n - bx^n}{h}$

In Problems 57–62, find an equation of the tangent line(s) to the graph of the function f that is (are) parallel to the line L.

57. $f(x) = 3x^2 - x$; $L: y = 5x$

58. $f(x) = 2x^3 + 1$; $L: y = 6x - 1$

59. $f(x) = e^x$; $L: y - x - 5 = 0$

60. $f(x) = -2e^x$; $L: y + 2x - 8 = 0$

61. $f(x) = \frac{1}{3}x^3 - x^2$; $L: y = 3x - 2$

62. $f(x) = x^3 - x$; $L: x + y = 0$

63. Tangent Lines Let $f(x) = 4x^3 - 3x - 1$.

(a) Find an equation of the tangent line to the graph of f at $x = 2$.

(b) Find the coordinates of any points on the graph of f where the tangent line is parallel to $y = x + 12$.

(c) Find an equation of the tangent line to the graph of f at any points found in (b).

(d) Graph f, the tangent line found in (a), the line $y = x + 12$, and any tangent lines found in (c) on the same screen.

64. Tangent Lines Let $f(x) = x^3 + 2x^2 + x - 1$.

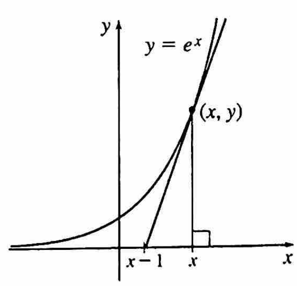
(a) Find an equation of the tangent line to the graph of f at $x = 0$.

(b) Find the coordinates of any points on the graph of f where the tangent line is parallel to $y = 3x - 2$.

(c) Find an equation of the tangent line to the graph of f at any points found in (b).

(d) Graph f, the tangent line found in (a), the line $y = 3x - 2$, and any tangent lines found in (c) on the same screen.

65. Tangent Line Show that the line perpendicular to the x-axis and containing the point (x, y) on the graph of $y = e^x$ and the tangent line to the graph of $y = e^x$ at the point (x, y) intersect the x-axis 1 unit apart. See the figure.



66. Tangent Line Show that the tangent line to the graph of $y=x^n$, $n \geq 2$ an integer, at $(1, 1)$ has y-intercept $1-n$.

67. Tangent Lines If n is an odd positive integer, show that the tangent lines to the graph of $y=x^n$ at $(1, 1)$ and at $(-1, -1)$ are parallel.

68. Tangent Line If the line $3x-4y=0$ is tangent to the graph of $y=x^3+k$ in the first quadrant, find k.

69. Tangent Line Find the constants a, b, and c so that the graph of $y=ax^2+bx+c$ contains the point $(-1, 1)$ and is tangent to the line $y=2x$ at $(0, 0)$.

70. Tangent Line Let T be the tangent line to the graph of $y=x^3$ at the point $\left(\frac{1}{2}, \frac{1}{8}\right)$. At what other point Q on the graph of $y=x^3$ does the line T intersect the graph? What is the slope of the tangent line at Q?

71. Military Tactics A dive bomber is flying from right to left along the graph of $y=x^2$. When a rocket bomb is released, it follows a path that is approximately along the tangent line. Where should the pilot release the bomb if the target is at $(1, 0)$?

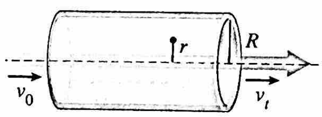
72. Military Tactics Answer the question in Problem 71 if the plane is flying from right to left along the graph of $y=x^3$.

73. Fluid Dynamics The velocity v of a liquid flowing through a cylindrical tube is given by the **Hagen–Poiseuille equation** $v=k(R^2-r^2)$, where R is the radius of the tube, k is a constant that depends on the length of the tube and the velocity of the liquid at its ends, and r is the variable distance of the liquid from the center of the tube. See the figure below.

(a) Find the rate of change of v with respect to r at the center of the tube.

(b) What is the rate of change halfway from the center to the wall of the tube?

(c) What is the rate of change at the wall of the tube?



74. Rate of Change Water is leaking out of a swimming pool that measures 20 ft by 40 ft by 6 ft. The amount of water in the pool at a time t is $W(t)=35{,}000-20t^2$ gallons, where t equals the number of hours since the pool was last filled. At what rate is the water leaking when $t=2$ h?

75. Luminosity of the Sun The luminosity L of a star is the rate at which it radiates energy. This rate depends on the temperature T and surface area A of the star's photosphere (the gaseous surface that emits the light). Luminosity is modeled by the equation $L=\sigma AT^4$, where σ is a constant known as the **Stefan–Boltzmann constant**, and T is expressed in the absolute (Kelvin) scale for which 0 K is absolute zero. As with most stars, the Sun's temperature has gradually increased over the 6 billion years of its existence, causing its luminosity to slowly increase.

(a) Find the rate at which the Sun's luminosity changes with respect to the temperature of its photosphere. Assume that the surface area A remains constant.

(b) Find the rate of change at the present time. The temperature of the photosphere is currently 5800 K (10,000 °F), the radius of the photosphere is $r=6.96\times 10^8$ m, and $\sigma=5.67\times 10^{-8}\ \frac{\text{W}}{\text{m}^2\ \text{K}^4}$.

(c) Assuming that the rate found in (b) remains constant, how much would the luminosity change if its photosphere temperature increased by 1 K (1 °C or 1.8 °F)? Compare this change to the present luminosity of the Sun.

76. Medicine: Poiseuille's Equation The French physician Poiseuille discovered that the volume V of blood (in cubic centimeters per unit time) flowing through an artery with inner radius R (in centimeters) can be modeled by

$$V(R)=kR^4$$

where $k=\frac{\pi}{8\nu l}$ is constant (here ν represents the viscosity of blood and l is the length of the artery).

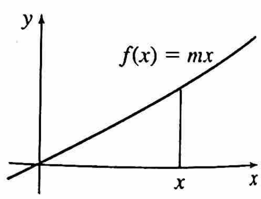
(a) Find the rate of change of the volume V of blood flowing through the artery with respect to the radius R.

(b) Find the rate of change when $R=0.03$ and when $R=0.04$.

(c) If the radius of a partially clogged artery is increased from 0.03 to 0.04 cm, estimate the effect on the rate of change of the volume V with respect to R of the blood flowing through the enlarged artery.

(d) How do you interpret the results found in (b) and (c)?

77. Derivative of an Area Let $f(x)=mx$, $m>0$. Let $F(x)$, $x>0$, be defined as the area of the shaded region in the figure. Find $F'(x)$.



78. The Difference Rule Prove that if f and g are differentiable functions and if $F(x)=f(x)-g(x)$, then

$$F'(x)=f'(x)-g'(x)$$

79. Simple Power Rule Let $f(x)=x^n$, where n is a positive integer. Use a factoring principle to show that

$$f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=nc^{n-1}$$

80. Normal Lines For what nonnegative number b is the line given by $y=-\frac{1}{3}x+b$ normal to the graph of $y=x^3$?

81. Normal Lines Let N be the normal line to the graph of $y=x^2$ at the point $(-2, 4)$. At what other point Q does N meet the graph?

Challenge Problems

82. **Tangent Line** Find a, b, c, d so that the tangent line to the graph of the cubic $y = ax^3 + bx^2 + cx + d$ at the point $(1, 0)$ is $y = 3x - 3$ and at the point $(2, 9)$ is $y = 18x - 27$.

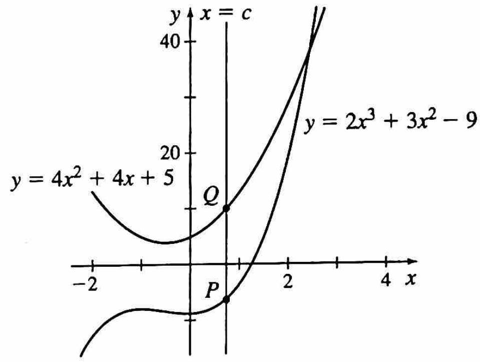
83. **Tangent Line** Find the fourth degree polynomial that contains the origin and to which the line $x + 2y = 14$ is tangent at both $x = 4$ and $x = -2$.

84. **Tangent Lines** Find equations for all the lines containing the point $(1, 4)$ that are tangent to the graph of $y = x^3 - 10x^2 + 6x - 2$. At what points do each of the tangent lines touch the graph?

85. The line $x = c$, where $c > 0$, intersects the cubic $y = 2x^3 + 3x^2 - 9$ at the point P and intersects the parabola $y = 4x^2 + 4x + 5$ at the point Q, as shown in the figure on the right.

    (a) If the line tangent to the cubic at the point P is parallel to the line tangent to the parabola at the point Q, find the number c.

    (b) Write an equation for each of the two tangent lines described in (a).



86. $f(x) = Ax^2 + B$, $A > 0$.

    (a) Find c, $c > 0$, in terms of A so that the tangent lines to the graph of f at $(c, f(c))$ and $(-c, f(-c))$ are perpendicular.

    (b) Find the slopes of the tangent lines in (a).

    (c) Find the coordinates, in terms of A and B, of the point of intersection of the tangent lines in (a).

Preparing for the AP® Exam

AP® Practice Problems

PAGE 185 1. If $g(x) = x$, then $g'(7) =$

(A) 0 (B) 1 (C) 7 (D) $\frac{49}{2}$

PAGE 188 2. The line $x + y = k$, where k is a constant, is a tangent line to the graph of the function $f(x) = x^2 - 5x + 2$. What is the value of k?

(A) -1 (B) 2 (C) -2 (D) -4

PAGE 189 3. An object moves along the x-axis so that its position at time t is $x(t) = 3t^2 - 9t + 7$. For what time t is the velocity of the object zero?

(A) -3 (B) 3 (C) $\frac{3}{2}$ (D) 7

PAGE 190 4. If $f(x) = e^x$, then $\ln(f'(3)) =$

(A) 3 (B) 0 (C) e^3 (D) $\ln 3$

PAGE 188 5. An equation of the normal line to the graph of $g(x) = x^3 + 2x^2 - 2x + 1$ at the point where $x = -2$ is

(A) $x + 2y = 12$ (B) $x - 2y = 8$

(C) $2x + y = -9$ (D) $x + 2y = 8$

PAGE 187 6. The line $9x - 16y = 0$ is tangent to the graph of $f(x) = 3x^3 + k$, where k is a constant, at a point in the first quadrant. Find k.

(A) $\frac{3}{32}$ (B) $\frac{3}{16}$ (C) $\frac{3}{64}$ (D) $\frac{9}{64}$

7. If $f(x) = 1 + |x - 4|$, find $f'(4)$.

(A) -1 (B) 0 (C) 1 (D) $f'(4)$ does not exist.

PAGE 187 8. The cost C (in dollars) of manufacturing x units of a product is $C(x) = 0.3x^2 + 4.02x + 3500$.
What is the rate of change of C when $x = 1000$ units?

(A) 307.52 (B) 0.60402 (C) 604.02 (D) 1020

PAGE 190 9. $\frac{d}{dx}(5 \ln x) =$

(A) $\frac{1}{5x}$ (B) $5e^x$ (C) $-\frac{5}{\ln x}$ (D) $\frac{5}{x}$

PAGE 188 10. For the function $f(x) = x^2 + 4$

(a) Find $f'(1)$.

(b) Find an equation of the tangent line to the graph of f at $x = 1$.

(c) Find $f'(-4)$.

(d) Find an equation of the tangent line to the graph of f at $x = -4$.

(e) Find the point of intersection of the two tangent lines found in (b) and (d).

PAGE 188 11. Which is an equation of the tangent line to the graph of $f(x) = x^4 + 3x^2 + 2$ at the point where $f'(x) = 2$?

(A) $y = 2x + 2$ (B) $y = 2x + 2.929$

(C) $y = 2x + 1.678$ (D) $y = 2x - 2.929$