AP Calculus PVA (Position-Velocity-Acceleration) Notes

Instantaneous velocity, v(t), of the object is the derivative of the position function s(t) with respect to time

$v(t) = s'(t)$

Acceleration, a(t), is the derivative of velocity with respect to time

$a(t) = v'(t) = s''(t)$

AVERAGE rate of change of f(x) from a to b = slope of secant = $\dfrac{f(b)-f(a)}{b-a}$

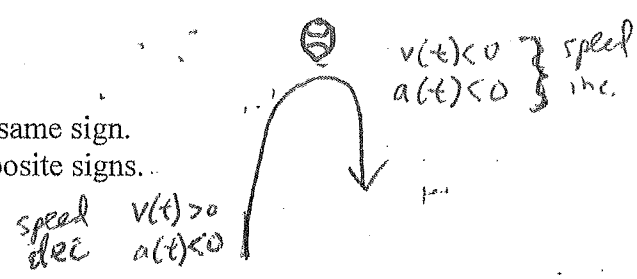
INSTANTANEOUS rate of change of f(x) at x = c = slope of tangent = $f'(c)$

Speed = |velocity|

Displacement = how far you are from where you started
Distance = total amount you have traveled

Ex) If I travel 10 feet to the right and then turn around and travel 3 feet back to the left, my distance is 13 feet but my displacement is 7 feet.

Speed is increasing when velocity and acceleration have the same sign.
Speed is decreasing when velocity and acceleration have opposite signs.



Particle Motion

Particle motion (linear motion) describes the object moving along a line (usually along a horizontal line)

x(t) = Position function
v(t) = velocity function
a(t) = acceleration function

Positive velocity indicates particle moving in positive direction (usually right)

Negative velocity indicates " " " negative direction (usually left)

When v(t) = 0, this indicates particle is at rest

A.P. Calculus PVA **Worksheet 2-2a**

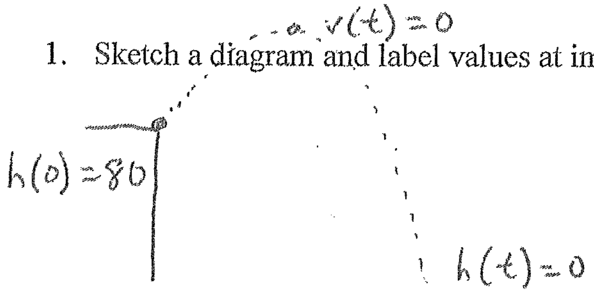
A ball is thrown vertically upwards from the edge of a building and it eventually hits the ground next to the building. If the height of the ball at any given time, $t \geq 0$ (seconds), is $h(t) = -16t^2 + 64t + 80$ (feet), answer the following:

$h(t) = -16t^2 + 64t + 80$

$v(t) = -32t + 64$

$a(t) = -32$

1. Sketch a diagram and label values at important places



2. How tall is the building?

$h(0) = 80$ ft.

3. When does the ball reach maximum height?

find t when $v(t) = 0$

$0 = -32t + 64$

$32t = 64$

$\boxed{t = 2 \text{ sec}}$

4. What is the maximum height?

find $h(2)$

$h(2) = -16(2)^2 + 64(2) + 80 = \boxed{144 \text{ ft}}$

5. How long does it take to hit the ground?

set $h(t) = 0$

$0 = -16t^2 + 64t + 80$

$0 = 16t^2 - 64t - 80$

$0 = t^2 - 4t - 5$

$0 = (t-5)(t+1)$

$t = 5$, $\cancel{t = -1}$ $\quad t = 5$ secs.

6. What was the initial velocity?

$v(0) = -32(0) + 64 = 64$

7. What is the velocity at $t = 1$ second? At $t = 2$ seconds?

$v(1) = -32 + 64 = 32$ ft/s

$v(2) = -32(2) + 64 = 0$ ft/s

8. What is the height at $t = 3$ seconds?

$h(3) = -16(3)^2 + 64(3) + 80 = 128$ ft.

9. What is the speed when it hits the ground?

$v(5) = -32(5) + 64 = -96 \rightarrow 96$ ft/s

10. What is the acceleration at $t = 1$ second? At $t = 2$ seconds?

$a(1) = -32$ ft/s^2

$a(2) = -32$ ft/s^2

11) Avg. velocity $[0, 2]$

$$\frac{h(2) - h(0)}{2 - 0} = \frac{144 - 80}{2 - 0} = 32 \text{ ft/s}$$

12) Avg. acceleration $[1, 2]$

$$\frac{v(2) - v(1)}{2 - 1} = \frac{0 - 32}{1} = -32 \text{ ft/s}^2$$

13) inc. or dec. speed at $t = 1$ sec.

$v(1) = 32$ ft/s

$a(1) = -32$ ft/s

dec. speed

A.P. Calculus PVA   Worksheet 2-2b   Linear Motion Problems

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1. An object is traveling at 20 m/sec to the left. What is its speed and velocity?

speed : 20m/s

velocity = -20m/s

2. Which has the greater speed and velocity: object A with a velocity of -20 m/sec or object B with a velocity of -10 m/sec?

greater velocity → object B (-10m/s)

greater speed: object A (20 m/s)

3. A billiard ball is hit and travels in a straight line. If x centimeters is the distance of the ball from its initial position at t seconds, then $x(t) = 5t^2 - 4t$. If the ball hits a cushion that is 12 cm from its initial position, at what velocity does it hit the cushion?

$12 = x(t)$

$12 = 5t^2 - 4t$

$5t^2 - 4t - 12 = 0$

$(5t+6)(t-2) = 0 \quad t = -6/5, 2$

$v(t) = 10t - 4$

$v(2) = 10(2) - 4 = 16\text{cm/s}$

4. If a particle moves along a line according to the equation $s(t) = t^5 - 5t^4$ for all real numbers, t, then how many times does the particle reverse its direction?

$v(t) = 5t^4 - 20t^3$

$0 = 5t^3(t-4)$

$t = 0, 4$

$+ \quad | \quad - \quad | \quad +$

$0 \quad 4$

$\boxed{\text{twice, at } t=0, t=4s}$

5. The position in meters of a particle moving on the x-axis is given by $x(t) = 2t^3 - 2t + 1$ at all times t, $t > 0$. Find the acceleration when the velocity is 4 m/sec.

$v(t) = 6t^2 - 2$

$a(t) = 12t$

$4 = 6t^2 - 2$

$6 = 6t^2$

$t = 1$

$a(1) = 12(1) = \boxed{12\text{m/s}^2}$

6. If $x(t) = \dfrac{t}{t^2+5}$ is the position function of a moving particle for $t > 0$, at what instant of time will the particle start to reverse its direction of motion, and where is it at that instant?

$v(t) = \dfrac{1(t^2+5) - t(2t)}{(t^2+5)^2}$

$= \dfrac{t^2+5-2t^2}{} = \dfrac{5-t^2}{} = 0 \quad \boxed{t = \sqrt{5}}$

$+ \quad | \quad -$

$\sqrt{5}$

7. The position function of a particle moving on a coordinate line is given by: $x(t) = 2t^3 - 21t^2 + 60t + 3$, where x is in feet and t is in seconds.

$v(t) = 6t^2 - 42t + 60 = 6(t^2 - 7t + 10)$

a) When is the particle at rest?

$0 = 6(t-5)(t-2)$

$t = 5, 2$ secs.

b) When does the particle reverse direction?

+ − +

2 5

$t = 2, 5$ secs.

c) What is the velocity when the acceleration is zero?

$a(t) = 12t - 42$

$a(t) = 6(2t - 7)$

$t = 7/2$

$v(7/2) = -13.5$ ft/s

d) What is the speed when the acceleration is 6 ft/sec²?

$6 = 6(2t - 7)$

$1 = 2t - 7$

$2t = 8 \quad t = 4$

$v(4) = -12$

Speed $= 12$ ft/s

e) What is the displacement from $t = 1$ to $t = 3$?

$x(1) = 44$

$x(3) = 48$

$48 - 44 = 4$ ft.

f) What is the total distance moved from $t = 1$ to $t = 3$?

$x(1) = 44$ > 11

$x(2) = 55$

> 7

$x(3) = 48$

$11 + 7 = 18$ ft

8. If $v(t) = (t-5)(t-3)^2(t-1)$ represents the velocity of a particle moving along a line,

a) When will the particle be at rest?

b) When will the particle move to the left?

c) When will the particle change direction?

a) $t = 5, 3, 1$

b) $(1, 3) \cup (3, 5)$

c) $t = 1, 5$ sec.

+ − − +

1 3 5

g)

9. A ball is thrown vertically upwards from the edge at the top of a building 160 ft tall with an initial velocity of 24 ft/sec. If the height of the ball (measured from the ground) is given by the function: $h(t) = -16t^2 + bt + c$,

a) Find the values of b and c.

$b = 24$ ft/s $\quad c = 160$

$h(t) = -16t^2 + 24t + 160$

$= -8(2t^2 - 3t - 20)$

$-8(2t + 5)(t - 4)$

b) How long does it take the ball to reach its maximum height?

$v(t) = -32t + 24 \quad 0 = -32t + 24 \quad t = 3/4$ sec.

c) What is the maximum height of the ball?

$h(3/4) = 169$ ft.

d) How long before the ball passes the top of the building on the way down?

$\cancel{160} = -16t^2 + 24t + \cancel{160} \quad 0 = -8t(2t - 3) \quad t = \frac{3}{2}$ sec.

e) How long does it take for the ball to hit the ground?

$h(t) = -8(2t + 5)(t - 4) \quad t = 4$ sec.

f) What is the speed of the ball when it hits the ground?

$v(4) = -104 \quad 104$ ft/s

g) What is the speed of the ball at $t = 1$ second?

$v(1) = -8 \quad 8$ ft/s

 Solution Key 31

Vertical Motion In Exercises 97 and 98, use the position function $s(t) = -16t^2 + v_0 t + s_0$ for free-falling objects.

97. A silver dollar is dropped from the top of a building that is 1362 feet tall.

$V_0 = 0 \qquad S_0 = 1362$

(a) Determine the position and velocity functions for the coin.

(b) Determine the average velocity on the interval [1, 2].

(c) Find the instantaneous velocities when $t = 1$ and $t = 2$.

(d) Find the time required for the coin to reach ground level. * set $s(t) = 0$

(e) Find the velocity of the coin at impact.

a) $s(t) = -16t^2 + 0t + 1362$

$v(t) = -32t$

b) avg. velocity $= \dfrac{s(2) - s(1)}{2 - 1} = \dfrac{1298 - 1346}{2 - 1} = \boxed{-48 \text{ ft/s}}$

$s(1) = 1346$

$s(2) = 1298$

c) $v(1) = -32$ ft/s

$v(2) = -64$ ft/s

d) $0 = -16t^2 + 1362$

$16t^2 = 1362$

$t^2 = \dfrac{1362}{16} \qquad t = \sqrt{\dfrac{1362}{16}} \qquad \boxed{t \approx 9.226 \text{ secs}}$

e) $v(9.226) \approx \boxed{-295.242 \text{ ft/s}}$

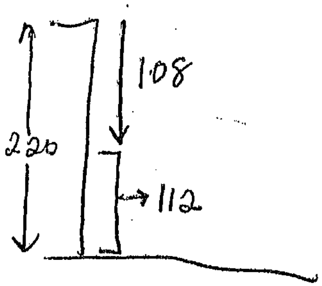
Vertical Motion In Exercises 97 and 98, use the position function $s(t) = -16t^2 + v_0 t + s_0$ for free-falling objects.

$V_0 = -22$ ft/s $\quad S_0 = 220$

98. A ball is thrown straight down from the top of a 220-foot building with an initial velocity of −22 feet per second. What is its velocity after 3 seconds? What is its velocity after falling 108 feet?

height/position = 220 − 108 = 112 ft.

* height is measured from the ground up.



$s(t) = -16t^2 - 22t + 220$

$v(t) = -32t - 22$

$v(3) = -32(3) - 22 = -118$ ft/s

* Find t when $h(t) = 112$, then find $v(t)$

$112 = -16t^2 - 22t + 220$

$0 = -16t^2 - 22t + 108$

$0 = -2(8t^2 + 11t - 54)$

$-2(t - 2)(8t + 27) = 0$

$\boxed{t = 2} \qquad t = \dfrac{-27}{8}$

$v(2) = -32(2) - 22$

$= \boxed{-86 \text{ ft/s}}$

Vertical Motion In Exercises 99 and 100, use the position function $s(t) = -4.9t^2 + v_0t + s_0$ for free-falling objects.

$V_0 = 120$ m/s

$S_0 = 0$

99. A projectile is shot upward from the surface of Earth with an initial velocity of 120 meters per second. What is its velocity after 5 seconds? After 10 seconds?

$s(t) = -4.9t^2 + 120t + 0$

$s'(t) = -9.8t + 120$

$s'(5) = -9.8(5) + 120 = \boxed{71 \text{ m/s}}$

$s'(10) = -9.8(10) + 120 = \boxed{22 \text{ m/s}}$

Vertical Motion In Exercises 99 and 100, use the position function $s(t) = -4.9t^2 + v_0t + s_0$ for free-falling objects.

$V_0 = 0$

$S_0 = S_0$

100. To estimate the height of a building, a stone is dropped from the top of the building into a pool of water at ground level. The splash is seen 5.6 seconds after the stone is dropped. What is the height of the building?

$s(t) = -4.9t^2 + 0t + S_0$

* $s(t) = 0$ when $t = 5.6$ sec.

$0 = -4.9(5.6)^2 + S_0$

$S_0 = 4.9(5.6)^2$

$\boxed{S_0 \approx 153.7 \text{m}}$