Ch. 2.4 Notes: The Chain Rule

Chain Rule: Method of computing the derivative of the composition of 2 or more functions (function within a function)

$\ast$ Rule: $\frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$

Steps:

1) Take the derivative of the outside while keeping the inside portion unchanged
2) Then multiply by the derivative of the inside function.

$\boxed{\text{Ex. 1}}$ $f(x) = (3x^2+2)^5$

$$f'(x) = \underbrace{5(3x^2+2)^4}_{f'(g(x))} \cdot \underbrace{(6x)}_{g'(x)} \qquad \boxed{f'(x) = 30x(3x^2+2)^4}$$

$\boxed{\text{Ex. 2}}$ Find all values of x of $f(x) = \sqrt[3]{(x^2-1)^2}$ for which $f'(x) = 0$ and where $f'(x)$ does not exist

$f(x) = (x^2-1)^{2/3}$

$f'(x) = \frac{2}{3}(x^2-1)^{-1/3} \cdot (2x)$

$f'(x) = \dfrac{4x}{3(x^2-1)^{1/3}}$

To find where $f'(x)=0$, set numerator $=0$

$4x = 0 \rightarrow x = 0$ $\quad$ $\underline{f'(x) = 0 \text{ at } x = 0}$

$f'(x) = \text{DNE}$, set denominator $=0$

$3(x^2-1)^{1/3} = 0 \quad x^2-1=0 \quad x = \pm 1$

$\underline{f'(x) = \text{undefined at } x=1,\ x=-1}$

*choose to use Rule that affects larger portion of the problem first.

$3(x^2-1)^{1/3} = 0$ | $x^2-1=0$

$\left(\sqrt[3]{x^2-1}\right)^3 = (0)^3$ | $x = \pm 1$

Ex.3 $y=\dfrac{4}{(x+2)^2}$ find equation of tangent line to y at $x=-3$

$y=4(x+2)^{-2}$

$y'=-2\cdot4(x+2)^{-3}(1)$

$y'=\dfrac{-8}{(x+2)^3}$

$y(-3)=\dfrac{4}{(-3+2)^2}=4$

$y'(-3)=\dfrac{-8}{(-3+2)^3}=8$

point: $(-3,4)$

slope: $m=8$

$y-y_1=m(x-x_1)$

$\boxed{y-4=8(x+3)}$

Ex.4 $y=\left(\dfrac{x-1}{x^2-4}\right)^3$

quotient and chain

* Apply the Rule that affects the larger portion of the expression first

$f(x)=(\ )^3 \quad g(x)=\dfrac{x-1}{x^2-4} \quad \dfrac{f'g-fg'}{g^2}$

$$y'=3\left(\frac{x-1}{x^2-4}\right)^2\left[\frac{\overbrace{(1)(x^2-4)-(x-1)(2x)}^{x^2-4-2x^2+2x}}{(x^2-4)^2}\right]$$

$$=\frac{3(x-1)^2(-x^2+2x-4)}{(x^2-4)^2(x^2-4)^2}=\boxed{\frac{3(x-1)^2(-x^2+2x-4)}{(x^2-4)^4}}$$

Ex.5 $y=\dfrac{x}{\sqrt{x^2-1}}=\dfrac{x}{(x^2-1)^{1/2}}$ quotient first, then chain

$\dfrac{f'g-fg'}{g^2}$

$$y'=\frac{1(x^2-1)^{1/2}-x\cdot\frac{1}{2}(x^2-1)^{-1/2}(2x)}{\left[(x^2-1)^{1/2}\right]^2}$$

$$y'=\frac{\left[(x^2-1)^{1/2}-\frac{x^2}{(x^2-1)^{1/2}}\right]}{\left[\ (x^2-1)^1\ \right]}\cdot\frac{(x^2-1)^{1/2}}{(x^2-1)^{1/2}}$$

$$y'=\frac{x^2-1-x^2}{(x^2-1)^{3/2}}=\boxed{\frac{-1}{(x^2-1)^{3/2}}}$$

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11) $f(t)=\sqrt{1-t} = (1-t)^{1/2}$ $f'(t)=\frac{1}{2}(1-t)^{-1/2}(-1)$

$= \frac{-1}{2\sqrt{1-t}}$ or $\frac{-1}{2(1-t)^{1/2}}$

13) $y=2\sqrt[4]{4-x^2} = 2(4-x^2)^{1/4}$

$y'=\frac{1}{4}\cdot 2(4-x^2)^{-3/4}(-2x)$ $y'=\frac{-x}{(4-x^2)^{3/4}}$

17) $y=\frac{1}{x-2}=(x-2)^{-1}$ $y'=-1(x-2)^{-2}(1) = \boxed{\frac{-1}{(x-2)^2}}$

21) $y=\frac{1}{\sqrt{x+2}}=(x+2)^{-1/2}$ $y'=-\frac{1}{2}(x+2)^{-3/2}(1)=\frac{-1}{2(x+2)^{3/2}}$

23) $f(x)=\underbrace{x^2}_{f}\underbrace{(x-2)^4}_{g}$ ∗ use product rule, chain rule

$f'g+fg'$

$f'(x)=\overbrace{2x}^{f'}\overbrace{(x-2)^4}^{g}+\overbrace{x^2}^{f}\cdot\overbrace{4(x-2)^3(1)}^{g'} = 2x(x-2)^4+4x^2(x-2)^3$

$=2x(x-2)^3[x-2+2x] = \boxed{2x(x-2)^3(3x-2)}$

25) $y=x\sqrt{1-x^2}=x(1-x^2)^{1/2}$ ∗ use product rule, chain rule

$y'=\underbrace{1}_{f'}\underbrace{(1-x^2)^{1/2}}_{g}+\underbrace{x}_{f}\cdot\underbrace{\frac{1}{2}(1-x^2)^{-1/2}(-2x)}_{g'}$

$=\boxed{\sqrt{1-x^2}-\frac{x^2}{\sqrt{1-x^2}}}$

27) $y = \dfrac{x}{\sqrt{x^2+1}} = \dfrac{x}{(x^2+1)^{1/2}}$ * use quotient rule, chain rule

$$y' = \frac{1(x^2+1)^{1/2} - x \cdot \frac{1}{2}(x^2+1)^{-1/2}(2x)}{\left[(x^2+1)^{1/2}\right]^2} = \frac{\left(\sqrt{x^2+1} - \frac{x^2}{\sqrt{x^2+1}}\right)}{\left(x^2+1\right)} \cdot \frac{\sqrt{x^2+1}}{\sqrt{x^2+1}}$$

$$= \frac{x^2+1-x^2}{(x^2+1)(x^2+1)^{1/2}} = \boxed{\frac{1}{(x^2+1)^{3/2}}}$$

29) $g(x) = \left(\dfrac{x+5}{x^2+2}\right)^2$ * Chain rule, quotient rule

$$g'(x) = 2\left(\frac{x+5}{x^2+2}\right)^1 \cdot \left[\frac{1(x^2+2)-(x+5)(2x)}{(x^2+2)^2}\right] = \frac{2(x+5)(x^2+2-2x^2-10x)}{(x^2+2)(x^2+2)^2}$$

$$g'(x) = \frac{2(x+5)(-x^2-10x+2)}{(x^2+2)^3}$$

31) $f(v) = \left(\dfrac{1-2v}{1+v}\right)^3$ * Chain rule, quotient rule

$$f'(v) = 3\left(\frac{1-2v}{1+v}\right)^2\left[\frac{(1+v)(-2)-(1-2v)}{(1+v)^2}\right] = \boxed{\frac{-9(1-2v)^2}{(1+v)^4}}$$

59) $s(t) = \sqrt{t^2+2t+8}$ at $(2,4)$ Evaluate derivative at given point

$s(t) = (t^2+2t+8)^{1/2}$

$s'(t) = \frac{1}{2}(t^2+2t+8)^{-1/2}(2t+2)$

$s'(2) = \frac{1}{2}(2^2+2(2)+8)^{-1/2}(4+2)$

$s'(2) = \frac{1}{2}(16)^{-1/2}(6) = \frac{1}{2}\left(\frac{1}{\sqrt{16}}\right)(6)$

$s'(2) = \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)(6) = \frac{3}{4}$

61) $f(x) = \dfrac{3}{x^3-4}$ at $(-1, -3/5)$

$f(x) = 3(x^3-4)^{-1}$

$f'(x) = -3(x^3-4)^{-2}(3x^2) = \dfrac{-9x^2}{(x^3-4)^2}$

$f'(-1) = \dfrac{-9(-1)^2}{(-1-4)^2} = \boxed{\dfrac{-9}{25}}$

63) $f(t) = \dfrac{3t+2}{t-1}$ $(0,-2)$

$f'(t) = \dfrac{3(t-1)-(3t+2)(1)}{(t-1)^2} = \dfrac{3t-3-3t-2}{(t-1)^2} = \dfrac{-5}{(t-1)^2}$

$f'(0) = \dfrac{-5}{(0-1)^2} = \boxed{-5}$

67) $f(x) = \sqrt{3x^2-2}$ at $(3,5)$ Find equation of tangent line

$f(x) = (3x^2-2)^{1/2}$

$f'(x) = \frac{1}{2}(3x^2-2)^{-1/2}(6x)$

$f'(x) = \dfrac{3x}{\sqrt{3x^2-2}}$ $f'(3) = \dfrac{9}{\sqrt{25}} = \dfrac{9}{5}$

$y - y_1 = m(x - x_1)$

$\boxed{y-5 = \frac{9}{5}(x-3)}$

69) $y = (2x^3+1)^2$ at $(-1,1)$

$y' = 2(2x^3+1)^1(6x^2)$

$y' = 12x^2(2x^3+1)$

$y'(-1) = 12(-1) = -12$

$\boxed{y-1 = -12(x+1)}$

97) Given: $g(5)=-3$ $\quad h(5)=3$ $\quad$ Find $f'(5)$

$g'(5)=6$ $\quad h'(5)=-2$

a) $f(x)=g(x)h(x)$ $\quad$ * product rule

$f'(x)=g'(x)h(x)+g(x)h'(x)$

$f'(5)=g'(5)h(5)+g(5)h'(5)$

$=(6)(3)+(-3)(-2) = \boxed{24}$

b) $f(x)=g(h(x))$ $\quad$ * chain rule

$f'(x)=g'(h(x))\cdot h'(x)$