The velocity v of the falling object is

$$v=\frac{ds}{dt}=\frac{d}{dt}(-ct^2)=-2ct$$

and its acceleration a is

$$a=\frac{dv}{dt}=\frac{d^2s}{dt^2}=-2c$$

which is a constant. Usually, we denote the constant $2c$ by g so $c=\frac{1}{2}g$. Then

$$a=-g \qquad v=-gt \qquad s=-\frac{1}{2}gt^2$$

The number g is called the **acceleration due to gravity**. For our planet, g is approximately 32 ft/s^2, or 9.8 m/s^2. On the planet Jupiter, $g\approx 26.0$ m/s^2, and our moon, $g\approx 1.60$ m/s^2.

2.4 Assess Your Understanding

Concepts and Vocabulary

1. *True or False* The derivative of a product is the product of the derivatives.
2. If $F(x)=f(x)g(x)$, then $F'(x)=$ ______.
3. *True or False* $\frac{d}{dx}x^n=nx^{n+1}$, for any integer n.
4. If f and $g\neq 0$ are two differentiable functions, then $\frac{d}{dx}\frac{f(x)}{g(x)}=$ ______.
5. *True or False* $f(x)=\frac{e^x}{x^2}$ can be differentiated using the Quotient Rule or by writing $f(x)=\frac{e^x}{x^2}=x^{-2}e^x$ and using the Product Rule.
6. If $g\neq 0$ is a differentiable function, then $\frac{d}{dx}\frac{1}{g(x)}=$ ______.
7. If $f(x)=x$, then $f''(x)=$ ______.
8. When an object in rectilinear motion is modeled by the position function $s=s(t)$, then the acceleration a of the object at time t is given by $a=a(t)=$ ______.

Skill Building

In Problems 9–40, find the derivative of each function.

PAGE 195 9. $f(x)=xe^x$
10. $f(x)=x^2e^x$
11. $f(x)=x^2(x^3-1)$
12. $f(x)=x^4(x+5)$
PAGE 196 13. $f(x)=(3x^2-5)(2x+1)$
14. $f(x)=(3x-2)(4x+5)$
15. $s(t)=(2t^5-t)(t^3-2t+1)$
16. $F(u)=(u^4-3u^2+1)(u^2-u+2)$
17. $f(x)=(x^3+1)(e^x+1)$
18. $f(x)=(x^2+1)(e^x+x)$
19. $g(s)=\frac{2s}{s+1}$
20. $F(z)=\frac{z+1}{2z}$
21. $G(u)=\frac{1-2u}{1+2u}$
22. $f(w)=\frac{1-w^2}{1+w^2}$
PAGE 197 23. $f(x)=\frac{4x^2-2}{3x+4}$
24. $f(x)=\frac{-3x^3-1}{2x^2+1}$
PAGE 197 25. $f(w)=\frac{1}{w^3-1}$
26. $g(v)=\frac{1}{v^2+5v-1}$
27. $s(t)=t^{-3}$
28. $G(u)=u^{-4}$
29. $f(x)=-\frac{4}{e^x}$
30. $f(x)=\frac{3}{4e^x}$
PAGE 198 31. $f(x)=\frac{10}{x^4}+\frac{3}{x^2}$
32. $f(x)=\frac{2}{x^5}-\frac{3}{x^3}$
33. $f(x)=3x^3-\frac{1}{3x^2}$
34. $f(x)=x^5-\frac{5}{x^5}$
35. $s(t)=\frac{1}{t}-\frac{1}{t^2}+\frac{1}{t^3}$
36. $s(t)=\frac{1}{t}+\frac{1}{t^2}+\frac{1}{t^3}$
37. $f(x)=\frac{e^x}{x^2}$
38. $f(x)=\frac{x^2}{e^x}$
39. $f(x)=\frac{x^2+1}{xe^x}$
40. $f(x)=\frac{xe^x}{x^2-x}$

In Problems 41–54, find f' and f'' for each function.

PAGE 199 41. $f(x)=3x^2+x-2$
42. $f(x)=-5x^2-3x$
43. $f(x)=e^x-3$
44. $f(x)=x-e^x$
PAGE 200 45. $f(x)=(x+5)e^x$
46. $f(x)=3x^4e^x$
47. $f(x)=(2x+1)(x^3+5)$
48. $f(x)=(3x-5)(x^2-2)$
49. $f(x)=x+\frac{1}{x}$
50. $f(x)=x-\frac{1}{x}$
51. $f(t)=\frac{t^2-1}{t}$
52. $f(u)=\frac{u+1}{u}$
53. $f(x)=\frac{e^x+x}{x}$
54. $f(x)=\frac{e^x}{x}$
55. Find y' and y'' for **(a)** $y=\frac{1}{x}$ and **(b)** $y=\frac{2x-5}{x}$.
56. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for **(a)** $y=\frac{5}{x^2}$ and **(b)** $y=\frac{2-3x}{x}$.

Rectilinear Motion *In Problems 57–60, find the velocity $v = v(t)$ and acceleration $a = a(t)$ of an object in rectilinear motion whose signed distance s from the origin at time t is modeled by the position function $s = s(t)$.*

57. $s(t) = 16t^2 + 20t$

58. $s(t) = 16t^2 + 10t + 1$

59. $s(t) = 4.9t^2 + 4t + 4$

60. $s(t) = 4.9t^2 + 5t$

In Problems 61–68, find the indicated derivative.

61. $f^{(4)}(x)$ if $f(x) = x^3 - 3x^2 + 2x - 5$

62. $f^{(5)}(x)$ if $f(x) = 4x^3 + x^2 - 1$

63. $\dfrac{d^8}{dt^8}\left(\dfrac{1}{8}t^8 - \dfrac{1}{7}t^7 + t^5 - t^3\right)$

64. $\dfrac{d^6}{dt^6}(t^6 + 5t^5 - 2t + 4)$

65. $\dfrac{d^7}{du^7}(e^u + u^2)$

66. $\dfrac{d^{10}}{du^{10}}(2e^u)$

67. $\dfrac{d^5}{dx^5}(-e^x)$

68. $\dfrac{d^8}{dx^8}(12x - e^x)$

In Problems 69–72:

(a) Find the slope of the tangent line for each function f at the given point.

(b) Find an equation of the tangent line to the graph of each function f at the given point.

(c) Find the points, if any, where the graph of the function has a horizontal tangent line.

(d) Graph each function, the tangent line found in (b), and any tangent lines found in (c) on the same set of axes.

69. $f(x) = \dfrac{x^2}{x-1}$ at $\left(-1, -\dfrac{1}{2}\right)$

70. $f(x) = \dfrac{x}{x+1}$ at $(0, 0)$

71. $f(x) = \dfrac{x^3}{x+1}$ at $\left(1, \dfrac{1}{2}\right)$

72. $f(x) = \dfrac{x^2+1}{x}$ at $\left(2, \dfrac{5}{2}\right)$

In Problems 73–80:

(a) Find the points, if any, at which the graph of each function f has a horizontal tangent line.

(b) Find an equation for each horizontal tangent line.

(c) Solve the inequality $f'(x) > 0$.

(d) Solve the inequality $f'(x) < 0$.

(e) Graph f and any horizontal lines found in (b) on the same set of axes.

(f) Describe the graph of f for the results obtained in (c) and (d).

73. $f(x) = (x+1)(x^2 - x - 11)$

74. $f(x) = (3x^2 - 2)(2x + 1)$

75. $f(x) = \dfrac{x^2}{x+1}$

76. $f(x) = \dfrac{x^2+1}{x}$

77. $f(x) = xe^x$

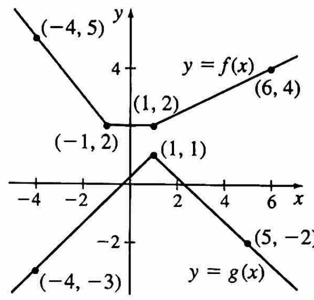
78. $f(x) = x^2e^x$

79. $f(x) = \dfrac{x^2-3}{e^x}$

80. $f(x) = \dfrac{e^x}{x^2+1}$

In Problems 81 and 82, use the graphs to determine each derivative.

81. Let $u(x) = f(x) \cdot g(x)$ and $v(x) = \dfrac{g(x)}{f(x)}$.



(a) $u'(0)$

(b) $u'(4)$

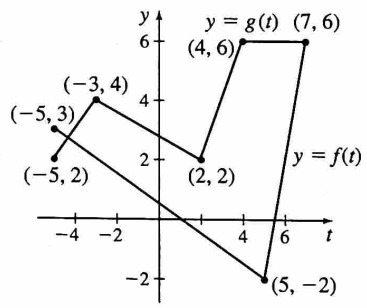
(c) $v'(-2)$

(d) $v'(6)$

(e) $\dfrac{d}{dx}\dfrac{1}{f(x)}$ at $x = -2$

(f) $\dfrac{d}{dx}\dfrac{1}{g(x)}$ at $x = 4$

82. Let $F(t) = f(t) \cdot g(t)$ and $G(t) = \dfrac{f(t)}{g(t)}$.



(a) $F'(0)$

(b) $F'(3)$

(c) $F'(-4)$

(d) $G'(-2)$

(e) $G'(-1)$

(f) $\dfrac{d}{dt}\dfrac{1}{f(t)}$ at $t = 3$

Applications and Extensions

PAGE 201

83. Vertical Motion An object is propelled vertically upward from the ground with an initial velocity of 39.2 m/s. The distance s (in meters) of the object from the ground after t seconds is given by the position function $s = s(t) = -4.9t^2 + 39.2t$.

(a) What is the velocity of the object at time t?

(b) When will the object reach its maximum height?

(c) What is the maximum height?

(d) What is the acceleration of the object at any time t?

(e) How long is the object in the air?

(f) What is the velocity of the object upon impact with the ground? What is its speed?

(g) What is the total distance traveled by the object?

84. Vertical Motion A ball is thrown vertically upward from a height of 6 ft with an initial velocity of 80 ft/s. The distance s (in feet) of the ball from the ground after t seconds is given by the position function $s = s(t) = 6 + 80t - 16t^2$.

(a) What is the velocity of the ball after 2 s?
(b) When will the ball reach its maximum height?
(c) What is the maximum height the ball reaches?
(d) What is the acceleration of the ball at any time t?
(e) How long is the ball in the air?
(f) What is the velocity of the ball upon impact with the ground? What is its speed?
(g) What is the total distance traveled by the ball?

85. Environmental Cost The cost C, in thousands of dollars, for the removal of a pollutant from a certain lake is given by the function $C(x) = \frac{5x}{110 - x}$, where x is the percent of pollutant removed.

(a) What is the domain of C?
(b) Graph C.
(c) What is the cost to remove 80% of the pollutant?
(d) Find $C'(x)$, the rate of change of the cost C with respect to the amount of pollutant removed.
(e) Find the rate of change of the cost for removing 40%, 60%, 80%, and 90% of the pollutant.
(f) Interpret the answers found in (e).

86. Investing in Fine Art The value V of a painting t years after it is purchased is modeled by the function

$$V(t) = \frac{100t^2 + 50}{t} + 400 \quad 1 \le t \le 5$$

(a) Find the rate of change in the value V with respect to time.
(b) What is the rate of change in value after 2 years?
(c) What is the rate of change in value after 3 years?
(d) Interpret the answers in (b) and (c).

87. Drug Concentration The concentration of a drug in a patient's blood t hours after injection is given by the function $f(t) = \frac{0.4t}{2t^2 + 1}$ (in milligrams per liter).

(a) Find the rate of change of the concentration with respect to time.
(b) What is the rate of change of the concentration after 10 min? After 30 min? After 1 hour?
(c) Interpret the answers found in (b).
(d) Graph f for the first 5 hours after administering the drug.
(e) From the graph, approximate the time (in minutes) at which the concentration of the drug is highest. What is the highest concentration of the drug in the patient's blood?

88. Population Growth A population of 1000 bacteria is introduced into a culture and grows in number according to the formula

$$P(t) = 1000\left(1 + \frac{4t}{100 + t^2}\right), \text{ where } t \text{ is measured in hours.}$$

(a) Find the rate of change in population with respect to time.
(b) What is the rate of change in population at $t = 1$, $t = 2$, $t = 3$, and $t = 4$?
(c) Interpret the answers found in (b).
(d) Graph $P = P(t)$, $0 \le t \le 20$.
(e) From the graph, approximate the time (in hours) when the population is the greatest. What is the maximum population of the bacteria in the culture?

89. Economics The price-demand function for a popular e-book is given by $D(p) = \frac{100,000}{p^2 + 10p + 50}$, $4 \le p \le 20$, where $D = D(p)$ is the quantity demanded at the price p dollars.

(a) Find $D'(p)$, the rate of change of demand with respect to price.
(b) Find $D'(5)$, $D'(10)$, and $D'(15)$.
(c) Interpret the results found in (b).

90. Intensity of Light The intensity of illumination I on a surface is inversely proportional to the square of the distance r from the surface to the source of light. If the intensity is 1000 units when the distance is 1 m from the light, find the rate of change of the intensity with respect to the distance when the source is 10 meters from the surface.

PAGE 198 **91. Ideal Gas Law** The Ideal Gas Law, used in chemistry and thermodynamics, relates the pressure p, the volume V, and the absolute temperature T (in Kelvin) of a gas, using the equation $pV = nRT$, where n is the amount of gas (in moles) and $R = 8.31$ is the ideal gas constant. In an experiment, a spherical gas container of radius r meters is placed in a pressure chamber and is slowly compressed while keeping its temperature at 273 K.

(a) Find the rate of change of the pressure p with respect to the radius r of the chamber.
Hint: The volume V of a sphere is $V = \frac{4}{3}\pi r^3$.
(b) Interpret the sign of the answer found in (a).
(c) If the sphere contains 1.0 mol of gas, find the rate of change of the pressure when $r = \frac{1}{4}$ m.
Note: The metric unit of pressure is the pascal, Pa.

92. Body Density The density ρ of an object is its mass m divided by its volume V; that is, $\rho = \frac{m}{V}$. If a person dives below the surface of the ocean, the water pressure on the diver will steadily increase, compressing the diver and therefore increasing body density. Suppose the diver is modeled as a sphere of radius r.

(a) Find the rate of change of the diver's body density with respect to the radius r of the sphere.
Hint: The volume V of a sphere is $V = \frac{4}{3}\pi r^3$.
(b) Interpret the sign of the answer found in (a).
(c) Find the rate of change of the diver's body density when the radius is 45 cm and the mass is 80,000 g (80 kg).

Jerk and Snap *Problems 93–96 use the following discussion:* Suppose that an object is moving in rectilinear motion so that its signed distance s from the origin at time t is given by the position function $s = s(t)$. The velocity $v = v(t)$ of the object at time t is the rate of change of s with respect to time, namely, $v = v(t) = \frac{ds}{dt}$. The acceleration $a = a(t)$

of the object at time t is the rate of change of the velocity with respect to time,

$$a = a(t) = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}$$

There are also physical interpretations of the third derivative and the fourth derivative of $s = s(t)$. The **jerk** $J = J(t)$ of the object at time t is the rate of change of the acceleration a with respect to time; that is,

$$J = J(t) = \frac{da}{dt} = \frac{d}{dt}\left(\frac{dv}{dt}\right) = \frac{d^2v}{dt^2} = \frac{d^3s}{dt^3}$$

The **snap** $S = S(t)$ of the object at time t is the rate of change of the jerk J with respect to time; that is,

$$S = S(t) = \frac{dJ}{dt} = \frac{d^2a}{dt^2} = \frac{d^3v}{dt^3} = \frac{d^4s}{dt^4}$$

Engineers take jerk into consideration when designing elevators, aircraft, and cars. In these cases, they try to minimize jerk, making for a smooth ride. But when designing thrill rides, such as roller coasters, the jerk is increased, making for an exciting experience.

93. **Rectilinear Motion** As an object in rectilinear motion moves, its signed distance s from the origin at time t is given by the position function $s = s(t) = t^3 - t + 1$, where s is in meters and t is in seconds.

(a) Find the velocity v, acceleration a, jerk J, and snap S of the object at time t.
(b) When is the velocity of the object 0 m/s?
(c) Find the acceleration of the object at $t = 2$ and at $t = 5$.
(d) Does the jerk of the object ever equal 0 m/s^3?
(e) How would you interpret the snap for this object in rectilinear motion?

94. **Rectilinear Motion** As an object in rectilinear motion moves, its signed distance s from the origin at time t is given by the position function $s = s(t) = \frac{1}{6}t^4 - t^2 + \frac{1}{2}t + 4$, where s is in meters and t is in seconds.

(a) Find the velocity v, acceleration a, jerk J, and snap S of the object at any time t.
(b) Find the velocity of the object at $t = 0$ and at $t = 3$.
(c) Find the acceleration of the object at $t = 0$. Interpret your answer.
(d) Is the jerk of the object constant? In your own words, explain what the jerk says about the acceleration of the object.
(e) How would you interpret the snap for this object in rectilinear motion?

95. **Elevator Ride Quality** The ride quality of an elevator depends on several factors, two of which are acceleration and jerk. In a study of 367 persons riding in a 1600-kg elevator that moves at an average speed of 4 m/s, the majority of riders were comfortable in an elevator with vertical motion given by

$$s(t) = 4t + 0.8t^2 + 0.333t^3$$

(a) Find the acceleration that the riders found acceptable.
(b) Find the jerk that the riders found acceptable.

Source: Elevator Ride Quality, January 2007, http://www.lift-report.de/index.php/news/176/368/Elevator-Ride-Quality

96. **Elevator Ride Quality** In a hospital, the effects of high acceleration or jerk may be harmful to patients, so the acceleration and jerk need to be lower than in standard elevators. It has been determined that a 1600-kg elevator that is installed in a hospital and that moves at an average speed of 4 m/s should have vertical motion

$$s(t) = 4t + 0.55t^2 + 0.1167t^3$$

(a) Find the acceleration of a hospital elevator.
(b) Find the jerk of a hospital elevator.

Source: Elevator Ride Quality, January 2007, http://www.lift-report.de/index.php/news/176/368/Elevator-Ride-Quality

97. **Current Density in a Wire** The current density J in a wire is a measure of how much an electrical current is compressed as it flows through a wire and is modeled by the function $J(A) = \frac{I}{A}$, where I is the current (in amperes) and A is the cross-sectional area of the wire. In practice, current density, rather than merely current, is often important. For example, superconductors lose their superconductivity if the current density is too high.

(a) As current flows through a wire, it heats the wire, causing it to expand in area A. If a constant current is maintained in a cylindrical wire, find the rate of change of the current density J with respect to the radius r of the wire.
(b) Interpret the sign of the answer found in (a).
(c) Find the rate of change of current density with respect to the radius r when a current of 2.5 amps flows through a wire of radius $r = 0.50$ mm.

98. **Derivative of a Reciprocal, Function** Prove that if a function g is differentiable, then $\frac{d}{dx}\frac{1}{g(x)} = -\frac{g'(x)}{[g(x)]^2}$, provided $g(x) \neq 0$.

99. **Extended Product Rule** Show that if f, g, and h are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)h(x)] = f(x)g(x)h'(x) + f(x)g'(x)h(x) + f'(x)g(x)h(x)$$

From this, deduce that

$$\frac{d}{dx}[f(x)]^3 = 3[f(x)]^2 f'(x)$$

In Problems 100–105, use the Extended Product Rule (Problem 99) to find y'.

100. $y = (x^2 + 1)(x - 1)(x + 5)$

101. $y = (x - 1)(x^2 + 5)(x^3 - 1)$

102. $y = (x^4 + 1)^3$

103. $y = (x^3 + 1)^3$

104. $y = (3x + 1)\left(1 + \frac{1}{x}\right)(x^{-5} + 1)$

105. $y = \left(1 - \frac{1}{x}\right)\left(1 - \frac{1}{x^2}\right)\left(1 - \frac{1}{x^3}\right)$

106. **(Further) Extended Product Rule** Write a formula for the derivative of the product of four differentiable functions. That is, find a formula for $\frac{d}{dx}[f_1(x)f_2(x)f_3(x)f_4(x)]$. Also find a formula for $\frac{d}{dx}[f(x)]^4$.

107. If f and g are differentiable functions, show that if $F(x)=\dfrac{1}{f(x)g(x)}$, then

$$F'(x)=-F(x)\left[\frac{f'(x)}{f(x)}+\frac{g'(x)}{g(x)}\right]$$

provided $f(x)\neq 0$, $g(x)\neq 0$.

108. Higher-Order Derivatives If $f(x)=\dfrac{1}{1-x}$, find a formula for the nth derivative of f. That is, find $f^{(n)}(x)$.

109. Let $f(x)=\dfrac{x^6-x^4+x^2}{x^4+1}$. Rewrite f in the form $(x^4+1)f(x)=x^6-x^4+x^2$. Now find $f'(x)$ without using the quotient rule.

110. If f and g are differentiable functions with $f\neq -g$, find the derivative of $\dfrac{fg}{f+g}$.

CAS **111.** $f(x)=\dfrac{2x}{x+1}$.

(a) Use technology to find $f'(x)$.

(b) Simplify f' to a single fraction using either algebra or a CAS.

(c) Use technology to find $f^{(5)}(x)$.
Hint: Your CAS may have a method for finding higher-order derivatives without finding other derivatives first.

Challenge Problems

112. Suppose f and g have derivatives up to the fourth order. Find the first four derivatives of the product fg and simplify the answers. In particular, show that the fourth derivative is

$$\frac{d^4}{dx^4}(fg)=f^{(4)}g+4f^{(3)}g^{(1)}+6f^{(2)}g^{(2)}+4f^{(1)}g^{(3)}+fg^{(4)}$$

Identify a pattern for the higher-order derivatives of fg.

113. Suppose $f_1(x),\ldots,f_n(x)$ are differentiable functions.

(a) Find $\dfrac{d}{dx}[f_1(x)\cdot\ldots\cdot f_n(x)]$.

(b) Find $\dfrac{d}{dx}\dfrac{1}{f_1(x)\cdot\ldots\cdot f_n(x)}$.

114. Let a, b, c, and d be real numbers. Define

$$\begin{vmatrix} a & b\\ c & d\end{vmatrix}=ad-bc$$

This is called a 2×2 **determinant** and it arises in the study of linear equations. Let $f_1(x)$, $f_2(x)$, $f_3(x)$, and $f_4(x)$ be differentiable and let

$$D(x)=\begin{vmatrix} f_1(x) & f_2(x)\\ f_3(x) & f_4(x)\end{vmatrix}$$

Show that

$$D'(x)=\begin{vmatrix} f_1'(x) & f_2'(x)\\ f_3(x) & f_4(x)\end{vmatrix}+\begin{vmatrix} f_1(x) & f_2(x)\\ f_3'(x) & f_4'(x)\end{vmatrix}$$

115. Let $f_0(x)=x-1$

$$f_1(x)=1+\frac{1}{x-1}$$

$$f_2(x)=1+\cfrac{1}{1+\cfrac{1}{x-1}}$$

$$f_3(x)=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{x-1}}}$$

(a) Write f_1, f_2, f_3, f_4, and f_5 in the form $\dfrac{ax+b}{cx+d}$.

(b) Using the results from (a), write the sequence of numbers representing the coefficients of x in the numerator, beginning with $f_0(x)=x-1$.

(c) Write the sequence in (b) as a recursive sequence.
Hint: Look at the sum of consecutive terms.

(d) Find f_0', f_1', f_2', f_3', f_4', and f_5'.

AP® Practice Problems

Preparing for the AP® Exam

PAGE 197 **1.** What is the instantaneous rate of change at $x=-2$ of the function $f(x)=\dfrac{x-1}{x^2+2}$?

(A) $-\dfrac{1}{6}$ (B) $\dfrac{1}{9}$ (C) $\dfrac{1}{2}$ (D) -1

PAGE 197 **2.** An equation of the tangent line to the graph of $f(x)=\dfrac{5x-3}{3x-6}$ at the point $(3, 4)$ is

(A) $7x+3y=37$ (B) $7x+3y=33$
(C) $7x-3y=9$ (D) $13x+3y=51$

PAGE 197 **3.** If f, g, and h are nonzero differentiable functions of x, then $\dfrac{d}{dx}\left(\dfrac{gh}{f}\right)=$

(A) $\dfrac{fgh'+fg'h-f'gh}{f^2}$ (B) $\dfrac{g'h'-ghf'}{f^2}$

(C) $\dfrac{gh'+g'h}{f'}$ (D) $\dfrac{fgh'+fg'h+f'gh}{f^2}$

PAGE 195 **4.** If $y=x^3e^x$, then $\dfrac{dy}{dx}=$

(A) $3x^2e^x$ (B) $3x^2+e^x$
(C) $3x^2e^x(x+1)$ (D) $x^2e^x(x+3)$

PAGE 198 **5.** $\dfrac{d}{dt}\left(t^2-\dfrac{1}{t^2}+\dfrac{1}{t}\right)$ at $t=2$ is

(A) $\dfrac{7}{2}$ (B) $\dfrac{9}{2}$ (C) $\dfrac{9}{4}$ (D) 4

PAGE 201 6
PAGE 197 7

PAGE 201 6. The position of an object moving along a straight line at time t, in seconds, is given by $s(t) = 16t^2 - 5t + 20$ meters. What is the acceleration of the object when $t = 2$?

(A) 32 m/s (B) 0 m/s^2 (C) 32 m/s^2 (D) 64 m/s^2

PAGE 197 7. If $y = \frac{x-3}{x+3}$, $x \neq -3$, the instantaneous rate of change of y with respect to x at $x = 3$ is

(A) $-\frac{1}{6}$ (B) $\frac{1}{6}$ (C) $\frac{1}{36}$ (D) 1

PAGE 197 8. Find an equation of the normal line to the graph of the function $f(x) = \frac{x^2}{x+1}$ at $x = 1$.

(A) $8x + 6y = 11$ (B) $-8x + 6y = -5$
(C) $-3x + 4y = -1$ (D) $3x + 4y = 5$

PAGE 200 9. If $y = xe^x$, then the nth derivative of y is

(A) e^x (B) $(x+n)e^x$ (C) ne^x (D) $x^n e^x$

2.5 The Derivative of the Trigonometric Functions

OBJECTIVE ***When you finish this section, you should be able to:***

1 Differentiate trigonometric functions (p. 207)

1 Differentiate Trigonometric Functions

To find the derivatives of $y = \sin x$ and $y = \cos x$, we use the limits

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \quad \text{and} \quad \lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$$

that were established in Section 1.4.

THEOREM Derivative of $y = \sin x$

The derivative of $y = \sin x$ is $y' = \cos x$. That is,

$$\boxed{y' = \frac{d}{dx}\sin x = \cos x}$$

Proof

$$y' = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$ **The definition of a derivative**

$$= \lim_{h \to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}$$ **$\sin(A+B) = \sin A \cos B + \sin B \cos A$**

$$= \lim_{h \to 0} \left[\frac{\sin x \cos h - \sin x}{h} + \frac{\sin h \cos x}{h}\right]$$ **Rearrange terms.**

$$= \lim_{h \to 0} \left[\sin x \cdot \frac{\cos h - 1}{h} + \frac{\sin h}{h} \cdot \cos x\right]$$ **Factor.**

$$= \left[\lim_{h \to 0} \sin x\right]\left[\lim_{h \to 0} \frac{\cos h - 1}{h}\right] + \left[\lim_{h \to 0} \cos x\right]\left[\lim_{h \to 0} \frac{\sin h}{h}\right]$$ **Use properties of limits.**

$$= \sin x \cdot 0 + \cos x \cdot 1 = \cos x$$ **$\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0;\ \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$** ■

NEED TO REVIEW? The trigonometric functions are discussed in Section P.6, pp. 52–58. Trigonometric identities are discussed in Appendix A.4, pp. A33 to A36.

The geometry of the derivative $\frac{d}{dx}\sin x = \cos x$ is shown in Figure 30. On the graph of $f(x) = \sin x$, the horizontal tangents are marked as well as the tangent lines that have slopes of 1 and -1. The derivative function is plotted on the second graph, and those points are connected with a smooth curve.