2.4 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Decomposition of a Composite Function In Exercises 1–6, complete the table.

$y = f(g(x))$	$u = g(x)$	$y = f(u)$
1. $y = (5x - 8)^4$		
2. $y = \dfrac{1}{\sqrt{x+1}}$		
3. $y = \sqrt{x^3 - 7}$		
4. $y = 3\tan(\pi x^2)$		
5. $y = \csc^3 x$		
6. $y = \sin \dfrac{5x}{2}$		

Finding a Derivative In Exercises 7–34, find the derivative of the function.

7. $y = (4x - 1)^3$

8. $y = 5(2 - x^3)^4$

9. $g(x) = 3(4 - 9x)^4$

10. $f(t) = (9t + 2)^{2/3}$

11. $f(t) = \sqrt{5 - t}$

12. $g(x) = \sqrt{4 - 3x^2}$

13. $y = \sqrt[3]{6x^2 + 1}$

14. $f(x) = \sqrt{x^2 - 4x + 2}$

15. $y = 2\sqrt[4]{9 - x^2}$

16. $f(x) = \sqrt[3]{12x - 5}$

17. $y = \dfrac{1}{x - 2}$

18. $s(t) = \dfrac{1}{4 - 5t - t^2}$

19. $f(t) = \left(\dfrac{1}{t - 3}\right)^2$

20. $y = -\dfrac{3}{(t - 2)^4}$

21. $y = \dfrac{1}{\sqrt{3x + 5}}$

22. $g(t) = \dfrac{1}{\sqrt{t^2 - 2}}$

23. $f(x) = x^2(x - 2)^4$

24. $f(x) = x(2x - 5)^3$

25. $y = x\sqrt{1 - x^2}$

26. $y = \frac{1}{2}x^2\sqrt{16 - x^2}$

27. $y = \dfrac{x}{\sqrt{x^2 + 1}}$

28. $y = \dfrac{x}{\sqrt{x^4 + 4}}$

29. $g(x) = \left(\dfrac{x + 5}{x^2 + 2}\right)^2$

30. $h(t) = \left(\dfrac{t^2}{t^3 + 2}\right)^2$

31. $f(v) = \left(\dfrac{1 - 2v}{1 + v}\right)^3$

32. $g(x) = \left(\dfrac{3x^2 - 2}{2x + 3}\right)^3$

33. $f(x) = ((x^2 + 3)^5 + x)^2$

34. $g(x) = (2 + (x^2 + 1)^4)^3$

Finding a Derivative Using Technology In Exercises 35–40, use a computer algebra system to find the derivative of the function. Then use the utility to graph the function and its derivative on the same set of coordinate axes. Describe the behavior of the function that corresponds to any zeros of the graph of the derivative.

35. $y = \dfrac{\sqrt{x} + 1}{x^2 + 1}$

36. $y = \sqrt{\dfrac{2x}{x + 1}}$

37. $y = \sqrt{\dfrac{x + 1}{x}}$

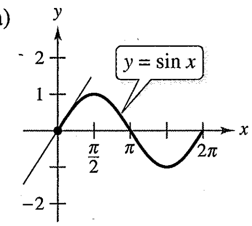
38. $g(x) = \sqrt{x - 1} + \sqrt{x + 1}$

39. $y = \dfrac{\cos \pi x + 1}{x}$

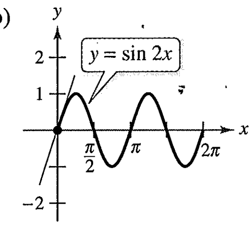
40. $y = x^2 \tan \dfrac{1}{x}$

Slope of a Tangent Line In Exercises 41 and 42, find the slope of the tangent line to the sine function at the origin. Compare this value with the number of complete cycles in the interval $[0, 2\pi]$. What can you conclude about the slope of the sine function $\sin ax$ at the origin?

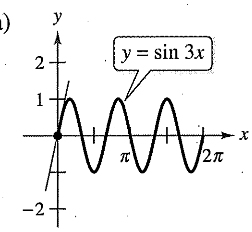
41. (a)



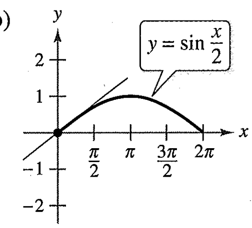
(b)



42. (a)



(b)



Finding a Derivative In Exercises 43–64, find the derivative of the function.

43. $y = \cos 4x$

44. $y = \sin \pi x$

45. $g(x) = 5\tan 3x$

46. $h(x) = \sec x^2$

47. $y = \sin(\pi x)^2$

48. $y = \cos(1 - 2x)^2$

49. $h(x) = \sin 2x \cos 2x$

50. $g(\theta) = \sec\left(\frac{1}{2}\theta\right)\tan\left(\frac{1}{2}\theta\right)$

51. $f(x) = \dfrac{\cot x}{\sin x}$

52. $g(v) = \dfrac{\cos v}{\csc v}$

53. $y = 4\sec^2 x$

54. $g(t) = 5\cos^2 \pi t$

55. $f(\theta) = \tan^2 5\theta$

56. $g(\theta) = \cos^2 8\theta$

57. $f(\theta) = \frac{1}{4}\sin^2 2\theta$

58. $h(t) = 2\cot^2(\pi t + 2)$

59. $f(t) = 3\sec^2(\pi t - 1)$

60. $y = 3x - 5\cos(\pi x)^2$

61. $y = \sqrt{x} + \frac{1}{4}\sin(2x)^2$

62. $y = \sin \sqrt[3]{x} + \sqrt[3]{\sin x}$

63. $y = \sin(\tan 2x)$

64. $y = \cos\sqrt{\sin(\tan \pi x)}$

Evaluating a Derivative In Exercises 65–72, find and evaluate the derivative of the function at the given point. Use a graphing utility to verify your result.

65. $y = \sqrt{x^2 + 8x}$, $(1, 3)$

66. $y = \sqrt[5]{3x^3 + 4x}$, $(2, 2)$

67. $f(x) = \dfrac{5}{x^3 - 2}$, $\left(-2, -\dfrac{1}{2}\right)$

68. $f(x) = \dfrac{1}{(x^2 - 3x)^2}$, $\left(4, \dfrac{1}{16}\right)$

69. $f(t) = \dfrac{3t + 2}{t - 1}$, $(0, -2)$

70. $f(x) = \dfrac{x + 4}{2x - 5}$, $(9, 1)$

71. $y = 26 - \sec^3 4x$, $(0, 25)$

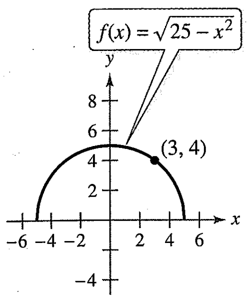
72. $y = \dfrac{1}{x} + \sqrt{\cos x}$, $\left(\dfrac{\pi}{2}, \dfrac{2}{\pi}\right)$

Finding an Equation of a Tangent Line In Exercises 73–80, (a) find an equation of the tangent line to the graph of f at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of the graphing utility to confirm your results.

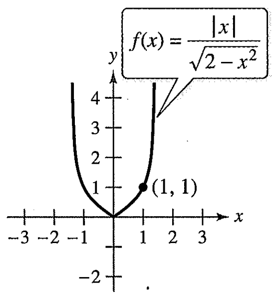
73. $f(x) = \sqrt{2x^2 - 7},\ (4, 5)$
74. $f(x) = \frac{1}{3}x\sqrt{x^2 + 5},\ (2, 2)$
75. $y = (4x^3 + 3)^2,\ (-1, 1)$
76. $f(x) = (9 - x^2)^{2/3},\ (1, 4)$
77. $f(x) = \sin 2x,\ (\pi, 0)$
78. $y = \cos 3x,\ \left(\frac{\pi}{4}, -\frac{\sqrt{2}}{2}\right)$
79. $f(x) = \tan^2 x,\ \left(\frac{\pi}{4}, 1\right)$
80. $y = 2\tan^3 x,\ \left(\frac{\pi}{4}, 2\right)$

Famous Curves In Exercises 81 and 82, find an equation of the tangent line to the graph at the given point. Then use a graphing utility to graph the function and its tangent line in the same viewing window.

81. Top half of circle



82. Bullet-nose curve



83. **Horizontal Tangent Line** Determine the point(s) in the interval $(0, 2\pi)$ at which the graph of

$$f(x) = 2\cos x + \sin 2x$$

has a horizontal tangent.

84. **Horizontal Tangent Line** Determine the point(s) at which the graph of

$$f(x) = \frac{x}{\sqrt{2x - 1}}$$

has a horizontal tangent.

Finding a Second Derivative In Exercises 85–90, find the second derivative of the function.

85. $f(x) = 5(2 - 7x)^4$
86. $f(x) = 6(x^3 + 4)^3$
87. $f(x) = \frac{1}{x - 6}$
88. $f(x) = \frac{8}{(x - 2)^2}$
89. $f(x) = \sin x^2$
90. $f(x) = \sec^2 \pi x$

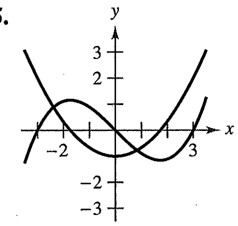
Evaluating a Second Derivative In Exercises 91–94, evaluate the second derivative of the function at the given point. Use a computer algebra system to verify your result.

91. $h(x) = \frac{1}{9}(3x + 1)^3,\ \left(1, \frac{64}{9}\right)$
92. $f(x) = \frac{1}{\sqrt{x + 4}},\ \left(0, \frac{1}{2}\right)$
93. $f(x) = \cos x^2,\ (0, 1)$
94. $g(t) = \tan 2t,\ \left(\frac{\pi}{6}, \sqrt{3}\right)$

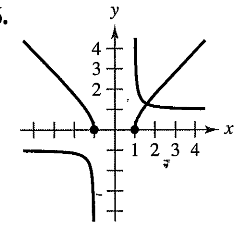
WRITING ABOUT CONCEPTS

Identifying Graphs In Exercises 95–98, the graphs of a function f and its derivative f' are shown. Label the graphs as f or f' and write a short paragraph stating the criteria you used in making your selection. To print an enlarged copy of the graph, go to *MathGraphs.com*.

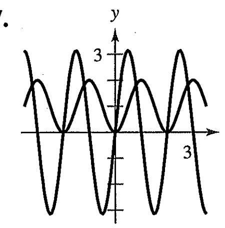
95.



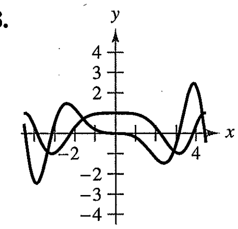
96.



97.



98.



Describing a Relationship In Exercises 99 and 100, the relationship between f and g is given. Explain the relationship between f' and g'.

99. $g(x) = f(3x)$
100. $g(x) = f(x^2)$

101. **Think About It** The table shows some values of the derivative of an unknown function f. Complete the table by finding the derivative of each transformation of f, if possible.

(a) $g(x) = f(x) - 2$
(b) $h(x) = 2f(x)$
(c) $r(x) = f(-3x)$
(d) $s(x) = f(x + 2)$

x	-2	-1	0	1	2	3
$f'(x)$	4	$\frac{2}{3}$	$-\frac{1}{3}$	-1	-2	-4
$g'(x)$						
$h'(x)$						
$r'(x)$						
$s'(x)$						

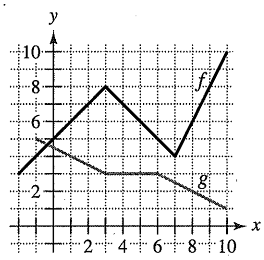
102. **Using Relationships** Given that $g(5) = -3$, $g'(5) = 6$, $h(5) = 3$, and $h'(5) = -2$, find $f'(5)$ for each of the following, if possible. If it is not possible, state what additional information is required.

(a) $f(x) = g(x)h(x)$
(b) $f(x) = g(h(x))$
(c) $f(x) = \frac{g(x)}{h(x)}$
(d) $f(x) = [g(x)]^3$

Finding Derivatives **In Exercises 103 and 104, the graphs of f and g are shown. Let $h(x) = f(g(x))$ and $s(x) = g(f(x))$. Find each derivative, if it exists. If the derivative does not exist, explain why.**

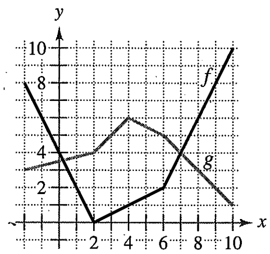
103. (a) Find $h'(1)$.

(b) Find $s'(5)$.



104. (a) Find $h'(3)$.

(b) Find $s'(9)$.



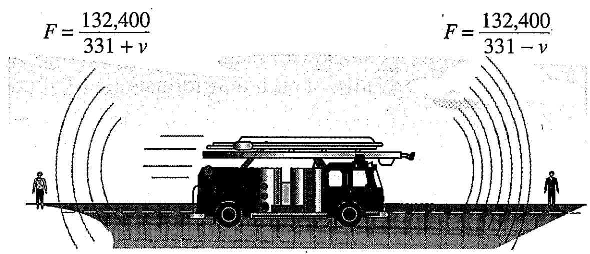
105. Doppler Effect The frequency F of a fire truck siren heard by a stationary observer is

$$F = \frac{132{,}400}{331 \pm v}$$

where $\pm v$ represents the velocity of the accelerating fire truck in meters per second (see figure). Find the rate of change of F with respect to v when

(a) the fire truck is approaching at a velocity of 30 meters per second (use $-v$).

(b) the fire truck is moving away at a velocity of 30 meters per second (use $+v$).



106. Harmonic Motion The displacement from equilibrium of an object in harmonic motion on the end of a spring is

$$y = \frac{1}{3}\cos 12t - \frac{1}{4}\sin 12t$$

where y is measured in feet and t is the time in seconds. Determine the position and velocity of the object when $t = \pi/8$.

107. Pendulum A 15-centimeter pendulum moves according to the equation $\theta = 0.2 \cos 8t$, where θ is the angular displacement from the vertical in radians and t is the time in seconds. Determine the maximum angular displacement and the rate of change of θ when $t = 3$ seconds.

108. Wave Motion A buoy oscillates in simple harmonic motion $y = A \cos \omega t$ as waves move past it. The buoy moves a total of 3.5 feet (vertically) from its low point to its high point. It returns to its high point every 10 seconds.

(a) Write an equation describing the motion of the buoy if it is at its high point at $t = 0$.

(b) Determine the velocity of the buoy as a function of t.

109. Modeling Data The normal daily maximum temperatures T (in degrees Fahrenheit) for Chicago, Illinois, are shown in the table. *(Source: National Oceanic and Atmospheric Administration)*

Month	Jan	Feb	Mar	Apr
Temperature	29.6	34.7	46.1	58.0

Month	May	Jun	Jul	Aug
Temperature	69.9	79.2	83.5	81.2

Month	Sep	Oct	Nov	Dec
Temperature	73.9	62.1	47.1	34.4

(a) Use a graphing utility to plot the data and find a model for the data of the form

$$T(t) = a + b \sin(ct - d)$$

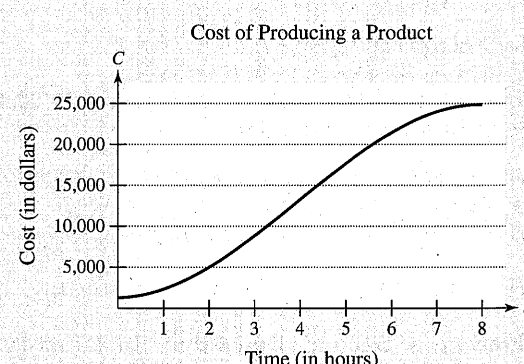
where T is the temperature and t is the time in months, with $t = 1$ corresponding to January.

(b) Use a graphing utility to graph the model. How well does the model fit the data?

(c) Find T' and use a graphing utility to graph the derivative.

(d) Based on the graph of the derivative, during what times does the temperature change most rapidly? Most slowly? Do your answers agree with your observations of the temperature changes? Explain.

110. HOW DO YOU SEE IT? The cost C (in dollars) of producing x units of a product is $C = 60x + 1350$. For one week, management determined that the number of units produced x at the end of t hours can be modeled by $x = -1.6t^3 + 19t^2 - 0.5t - 1$. The graph shows the cost C in terms of the time t.



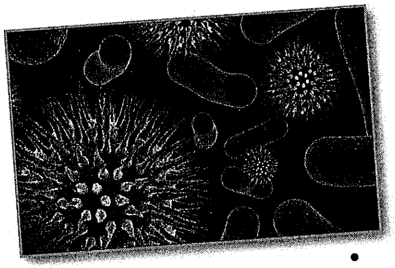
(a) Using the graph, which is greater, the rate of change of the cost after 1 hour or the rate of change of the cost after 4 hours?

(b) Explain why the cost function is not increasing at a constant rate during the eight-hour shift.

111. Biology The number N of bacteria in a culture after t days is modeled by

$$N = 400\left[1 - \frac{3}{(t^2 + 2)^2}\right].$$

Find the rate of change of N with respect to t when (a) $t = 0$, (b) $t = 1$, (c) $t = 2$, (d) $t = 3$, and (e) $t = 4$. (f) What can you conclude?



112. Depreciation The value V of a machine t years after it is purchased is inversely proportional to the square root of $t + 1$. The initial value of the machine is $10,000.

(a) Write V as a function of t.

(b) Find the rate of depreciation when $t = 1$.

(c) Find the rate of depreciation when $t = 3$.

113. Finding a Pattern Consider the function $f(x) = \sin \beta x$, where β is a constant.

(a) Find the first-, second-, third-, and fourth-order derivatives of the function.

(b) Verify that the function and its second derivative satisfy the equation $f''(x) + \beta^2 f(x) = 0$.

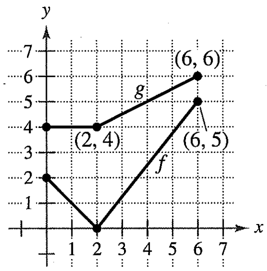
(c) Use the results of part (a) to write general rules for the even- and odd-order derivatives $f^{(2k)}(x)$ and $f^{(2k-1)}(x)$. [*Hint:* $(-1)^k$ is positive if k is even and negative if k is odd.]

114. Conjecture Let f be a differentiable function of period p.

(a) Is the function f' periodic? Verify your answer.

(b) Consider the function $g(x) = f(2x)$. Is the function $g'(x)$ periodic? Verify your answer.

115. Think About It Let $r(x) = f(g(x))$ and $s(x) = g(f(x))$, where f and g are shown in the figure. Find (a) $r'(1)$ and (b) $s'(4)$.



116. Using Trigonometric Functions

(a) Find the derivative of the function $g(x) = \sin^2 x + \cos^2 x$ in two ways.

(b) For $f(x) = \sec^2 x$ and $g(x) = \tan^2 x$, show that

$$f'(x) = g'(x).$$

117. Even and Odd Functions

(a) Show that the derivative of an odd function is even. That is, if $f(-x) = -f(x)$, then $f'(-x) = f'(x)$.

(b) Show that the derivative of an even function is odd. That is, if $f(-x) = f(x)$, then $f'(-x) = -f'(x)$.

118. Proof Let u be a differentiable function of x. Use the fact that $|u| = \sqrt{u^2}$ to prove that

$$\frac{d}{dx}[|u|] = u'\frac{u}{|u|}, \quad u \neq 0.$$

Using Absolute Value **In Exercises 119–122, use the result of Exercise 118 to find the derivative of the function.**

119. $g(x) = |3x - 5|$

120. $f(x) = |x^2 - 9|$

121. $h(x) = |x| \cos x$

122. $f(x) = |\sin x|$

Linear and Quadratic Approximations **The linear and quadratic approximations of a function f at $x = a$ are**

$$P_1(x) = f'(a)(x - a) + f(a) \quad \text{and}$$

$$P_2(x) = \tfrac{1}{2}f''(a)(x - a)^2 + f'(a)(x - a) + f(a).$$

In Exercises 123 and 124, (a) find the specified linear and quadratic approximations of f, (b) use a graphing utility to graph f and the approximations, (c) determine whether P_1 or P_2 is the better approximation, and (d) state how the accuracy changes as you move farther from $x = a$.

123. $f(x) = \tan x; \quad a = \frac{\pi}{4}$

124. $f(x) = \sec x; \quad a = \frac{\pi}{6}$

True or False? **In Exercises 125–128, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

125. If $y = (1 - x)^{1/2}$, then $y' = \frac{1}{2}(1 - x)^{-1/2}$.

126. If $f(x) = \sin^2(2x)$, then $f'(x) = 2(\sin 2x)(\cos 2x)$.

127. If y is a differentiable function of u, and u is a differentiable function of x, then y is a differentiable function of x.

128. If y is a differentiable function of u, u is a differentiable function of v, and v is a differentiable function of x, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}.$$

PUTNAM EXAM CHALLENGE

129. Let $f(x) = a_1 \sin x + a_2 \sin 2x + \cdot \cdot \cdot + a_n \sin nx$, where $a_1, a_2, \ldots, a_n$ are real numbers and where n is a positive integer. Given that $|f(x)| \leq |\sin x|$ for all real x, prove that $|a_1 + 2a_2 + \cdot \cdot \cdot + na_n| \leq 1$.

130. Let k be a fixed positive integer. The nth derivative of $\frac{1}{x^k - 1}$ has the form $\frac{P_n(x)}{(x^k - 1)^{n+1}}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$.

These problems were composed by the Committee on the Putnam Prize Competition.