129.

$$f(x) = a_1 \sin x + a_2 \sin 2x + \cdots + a_n \sin nx$$
$$f'(x) = a_1 \cos x + 2a_2 \cos 2x + \cdots + na_n \cos nx$$
$$f'(0) = a_1 + 2a_2 + \cdots + na_n$$
$$|a_1 + 2a_2 + \cdots + na_n| = |f'(0)| = \lim_{x\to 0}\left|\frac{f(x) - f(0)}{x - 0}\right| = \lim_{x\to 0}\left|\frac{f(x)}{\sin x}\right|\cdot\left|\frac{\sin x}{x}\right| = \lim_{x\to 0}\left|\frac{f(x)}{\sin x}\right| \le 1$$

130. $$\frac{d}{dx}\left[\frac{P_n(x)}{(x^k - 1)^{n+1}}\right] = \frac{(x^k - 1)^{n+1}P_n'(x) - P_n(x)(n + 1)(x^k - 1)^n kx^{k-1}}{(x^k - 1)^{2n+2}} = \frac{(x^k - 1)P_n'(x) - (n + 1)kx^{k-1}P_n(x)}{(x^k - 1)^{n+2}}$$

$$P_n(x) = (x^k - 1)^{n+1}\frac{d^n}{dx^n}\left[\frac{1}{x^k - 1}\right] \Rightarrow$$

$$P_{n+1}(x) = (x^k - 1)^{n+2}\frac{d}{dx}\left[\frac{d^n}{dx^n}\left[\frac{1}{x^k - 1}\right]\right] = (x^k - 1)P_n'(x) - (n + 1)kx^{k-1}P_n(x)$$

$$P_{n+1}(1) = -(n + 1)kP_n(1)$$

For $n = 1$, $\frac{d}{dx}\left[\frac{1}{x^k - 1}\right] = \frac{-kx^{k-1}}{(x^k - 1)^2} = \frac{P_1(x)}{(x^k - 1)^2} \Rightarrow P_1(1) = -k$. Also, $P_0(1) = 1$.

You now use mathematical induction to verify that $P_n(1) = (-k)^n n!$ for $n \ge 0$. Assume true for n. Then

$P_{n+1}(1) = -(n + 1)k\,P_n(1) = -(n + 1)k(-k)^n n! = (-k)^{n+1}(n + 1)!$.

Section 2.5 Implicit Differentiation

1.
$$x^2 + y^2 = 9$$
$$2x + 2yy' = 0$$
$$y' = -\frac{x}{y}$$

2.
$$x^2 - y^2 = 25$$
$$2x - 2yy' = 0$$
$$y' = \frac{x}{y}$$

3.
$$x^{1/2} + y^{1/2} = 16$$
$$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}y' = 0$$
$$y' = -\frac{x^{-1/2}}{y^{-1/2}} = -\sqrt{\frac{y}{x}}$$

4.
$$2x^3 + 3y^3 = 64$$
$$6x^2 + 9y^2y' = 0$$
$$9y^2y' = -6x^2$$
$$y' = \frac{-6x^2}{9y^2} = -\frac{2x^2}{3y^2}$$

5.
$$x^3 - xy + y^2 = 7$$
$$3x^2 - xy' - y + 2yy' = 0$$
$$(2y - x)y' = y - 3x^2$$
$$y' = \frac{y - 3x^2}{2y - x}$$

6.
$$x^2y + y^2x = -2$$
$$x^2y' + 2xy + y^2 + 2yxy' = 0$$
$$(x^2 + 2xy)y' = -(y^2 + 2xy)$$
$$y' = \frac{-y(y + 2x)}{x(x + 2y)}$$

7.
$$x^3y^3 - y - x = 0$$
$$3x^3y^2y' + 3x^2y^3 - y' - 1 = 0$$
$$(3x^3y^2 - 1)y' = 1 - 3x^2y^3$$
$$y' = \frac{1 - 3x^2y^3}{3x^3y^2 - 1}$$

8.
$$\sqrt{xy} = x^2y + 1$$
$$\frac{1}{2}(xy)^{-1/2}(xy' + y) = 2xy + x^2y'$$
$$\frac{x}{2\sqrt{xy}}y' + \frac{y}{2\sqrt{xy}} = 2xy + x^2y'$$
$$\left(\frac{x}{2\sqrt{xy}} - x^2\right)y' = 2xy - \frac{y}{2\sqrt{xy}}$$
$$y' = \frac{2xy - \dfrac{y}{2\sqrt{xy}}}{\dfrac{x}{2\sqrt{xy}} - x^2}$$
$$y' = \frac{4xy\sqrt{xy} - y}{x - 2x^2\sqrt{xy}}$$

9.
$$x^3 - 3x^2y + 2xy^2 = 12$$
$$3x^2 - 3x^2y' - 6xy + 4xyy' + 2y^2 = 0$$
$$(4xy - 3x^2)y' = 6xy - 3x^2 - 2y^2$$
$$y' = \frac{6xy - 3x^2 - 2y^2}{4xy - 3x^2}$$

10.
$$4\cos x \sin y = 1$$
$$4[-\sin x \sin y + \cos x \cos y\, y'] = 0$$
$$\cos x \cos y\, y' = \sin x \sin y$$
$$y' = \frac{\sin x \sin y}{\cos x \cos y}$$
$$= \tan x \tan y$$

11.
$$\sin x + 2\cos 2y = 1$$
$$\cos x - 4(\sin 2y)y' = 0$$
$$y' = \frac{\cos x}{4\sin 2y}$$

12.
$$(\sin \pi x + \cos \pi y)^2 = 2$$
$$2(\sin \pi x + \cos \pi y)[\pi \cos \pi x - \pi(\sin \pi y)y'] = 0$$
$$\pi \cos \pi x - \pi(\sin \pi y)y' = 0$$
$$y' = \frac{\cos \pi x}{\sin \pi y}$$

13.
$$\sin x = x(1 + \tan y)$$
$$\cos x = x(\sec^2 y)y' + (1 + \tan y)(1)$$
$$y' = \frac{\cos x - \tan y - 1}{x \sec^2 y}$$

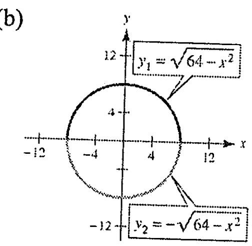
14.
$$\cot y = x - y$$
$$(-\csc^2 y)y' = 1 - y'$$
$$y' = \frac{1}{1 - \csc^2 y} = \frac{1}{-\cot^2 y} = -\tan^2 y$$

15.
$$y = \sin xy$$
$$y' = [xy' + y]\cos(xy)$$
$$y' - x\cos(xy)y' = y\cos(xy)$$
$$y' = \frac{y\cos(xy)}{1 - x\cos(xy)}$$

16.
$$x = \sec\frac{1}{y}$$
$$1 = -\frac{y'}{y^2}\sec\frac{1}{y}\tan\frac{1}{y}$$
$$y' = \frac{-y^2}{\sec(1/y)\tan(1/y)} = -y^2\cos\left(\frac{1}{y}\right)\cot\left(\frac{1}{y}\right)$$

17. (a)
$$x^2 + y^2 = 64$$
$$y^2 = 64 - x^2$$
$$y = \pm\sqrt{64 - x^2}$$

(b)



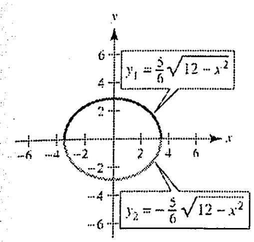
(c) Explicitly:
$$\frac{dy}{dx} = \pm\frac{1}{2}(64 - x^2)^{-1/2}(-2x) = \frac{\mp x}{\sqrt{64 - x^2}} = \frac{-x}{\pm\sqrt{64 - x^2}} = -\frac{x}{y}$$

(d) Implicitly: $2x + 2yy' = 0$
$$y' = -\frac{x}{y}$$

18. (a) $25x^2 + 36y^2 = 300$

$$36y^2 = 300 - 25x^2 = 25(12 - x^2)$$
$$y^2 = \frac{25}{36}(12 - x^2)$$
$$y = \pm\frac{5}{6}\sqrt{12 - x^2}$$

(b)



(c) Explicitly:

$$\frac{dy}{dx} = \pm\frac{5}{6}\left(\frac{1}{2}\right)(12 - x^2)^{-1/2}(-2x)$$
$$= \mp\frac{5x}{6\sqrt{12 - x^2}}$$
$$= -\frac{25x}{36y}$$

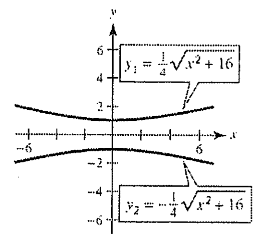
(d) Implicitly: $50x + 72y \cdot y' = 0$

$$y' = \frac{-50x}{72y} = -\frac{25x}{36y}$$

19. (a) $16y^2 - x^2 = 16$

$$16y^2 = x^2 + 16$$
$$y^2 = \frac{x^2}{16} + 1 = \frac{x^2 + 16}{16}$$
$$y = \frac{\pm\sqrt{x^2 + 16}}{4}$$

(b)



(c) Explicitly:

$$\frac{dy}{dx} = \frac{\pm\frac{1}{2}(x^2 + 16)^{-1/2}(-2x)}{4}$$
$$= \frac{\pm x}{4\sqrt{x^2 + 16}} = \frac{\pm x}{4(\pm 4y)} = -\frac{x}{16y}$$

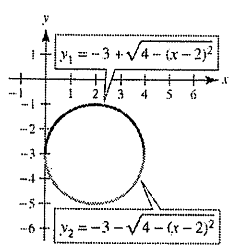
(d) Implicitly: $16y^2 - x^2 = 16$

$$32yy' - 2x = 0$$
$$32yy' = 2x$$
$$y' = \frac{2x}{32y} = \frac{x}{16y}$$

20. (a) $x^2 + y^2 - 4x + 6y + 9 = 0$

$$(x^2 - 4x + 4) + (y^2 + 6y + 9) = -9 + 4 + 9$$
$$(x - 2)^2 + (y + 3)^2 = 4$$
$$(y + 3)^2 = 4 - (x - 2)^2$$
$$y + 3 = \pm\sqrt{4 - (x - 2)^2}$$
$$y = -3 \pm \sqrt{4 - (x - 2)^2}$$

(b)



(c) Explicitly:

$$\frac{dy}{dx} = \pm\frac{1}{2}\left[4 - (x - 2)^2\right]^{-1/2}\left[-2(x - 2)\right]$$
$$= \mp\frac{x - 2}{\sqrt{4 - (x - 2)^2}}$$
$$= -\frac{x - 2}{y + 3}$$

(d) Implicitly:

$$2x + 2yy' - 4 + 6y' = 0$$
$$2yy' + 6y' = -2x + 4$$
$$y'(2y + 6) = -2(x - 2)$$
$$y' = \frac{-2(x - 2)}{2(y + 3)} = -\frac{x - 2}{y + 3}$$

21.

$$\begin{aligned} xy &= 6 \\ xy' + y(1) &= 0 \\ xy' &= -y \\ y' &= -\frac{y}{x} \end{aligned}$$

At $(-6, -1)$: $y' = -\dfrac{1}{6}$

22.

$$\begin{aligned} y^3 - x^2 &= 4 \\ 3y^2y' - 2x &= 0 \\ y' &= \frac{2x}{3y^2} \end{aligned}$$

At $(2, 2)$: $y' = \dfrac{2(2)}{3(2^2)} = \dfrac{1}{3}$

23.

$$\begin{aligned} y^2 &= \frac{x^2 - 49}{x^2 + 49} \\ 2yy' &= \frac{(x^2 + 49)(2x) - (x^2 - 49)(2x)}{(x^2 + 49)^2} \\ 2yy' &= \frac{196x}{(x^2 + 49)^2} \\ y' &= \frac{98x}{y(x^2 + 49)^2} \end{aligned}$$

At $(7, 0)$: y' is undefined.

25 ~~**24.**~~

$$\begin{aligned} (x + y)^3 &= x^3 + y^3 \\ x^3 + 3x^2y + 3xy^2 + y^3 &= x^3 + y^3 \\ 3x^2y + 3xy^2 &= 0 \\ x^2y + xy^2 &= 0 \\ x^2y' + 2xy + 2xyy' + y^2 &= 0 \\ (x^2 + 2xy)y' &= -(y^2 + 2xy) \\ y' &= -\frac{y(y + 2x)}{x(x + 2y)} \end{aligned}$$

At $(-1, 1)$: $y' = -1$

24 ~~**25.**~~

$$\begin{aligned} x^{2/3} + y^{2/3} &= 5 \\ \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' &= 0 \\ y' &= \frac{-x^{-1/3}}{y^{-1/3}} = -\sqrt[3]{\frac{y}{x}} \end{aligned}$$

At $(8, 1)$: $y' = -\dfrac{1}{2}$

26.

$$\begin{aligned} x^3 + y^3 &= 6xy - 1 \\ 3x^2 + 3y^2y' &= 6xy' + 6y \\ (3y^2 - 6x)y' &= 6y - 3x^2 \\ y' &= \frac{6y - 3x^2}{3y^2 - 6x} \end{aligned}$$

At $(2, 3)$: $y' = \dfrac{18 - 12}{27 - 12} = \dfrac{6}{15} = \dfrac{2}{5}$

27.

$$\begin{aligned} \tan(x + y) &= x \\ (1 + y')\sec^2(x + y) &= 1 \\ y' &= \frac{1 - \sec^2(x + y)}{\sec^2(x + y)} \\ &= \frac{-\tan^2(x + y)}{\tan^2(x + y) + 1} \\ &= -\sin^2(x + y) \\ &= -\frac{x^2}{x^2 + 1} \end{aligned}$$

At $(0, 0)$: $y' = 0$

28.

$$\begin{aligned} x\cos y &= 1 \\ x[-y'\sin y] + \cos y &= 0 \\ y' &= \frac{\cos y}{x\sin y} \\ &= \frac{1}{x}\cot y \\ &= \frac{\cot y}{x} \end{aligned}$$

At $\left(2, \dfrac{\pi}{3}\right)$: $y' = \dfrac{1}{2\sqrt{3}}$

29.

$$\begin{aligned} (x^2 + 4)y &= 8 \\ (x^2 + 4)y' + y(2x) &= 0 \\ y' &= \frac{-2xy}{x^2 + 4} \\ &= \frac{-2x[8/(x^2 + 4)]}{x^2 + 4} \\ &= \frac{-16x}{(x^2 + 4)^2} \end{aligned}$$

At $(2, 1)$: $y' = \dfrac{-32}{64} = -\dfrac{1}{2}$

$\left(\text{Or, you could just solve for } y\text{: } y = \dfrac{8}{x^2 + 4}\right)$

30. $$(4 - x)y^2 = x^3$$
$$(4 - x)(2yy') + y^2(-1) = 3x^2$$
$$y' = \frac{3x^2 + y^2}{2y(4 - x)}$$
At $(2, 2)$: $y' = 2$

31. $$\left(x^2 + y^2\right)^2 = 4x^2y$$
$$2\left(x^2 + y^2\right)(2x + 2yy') = 4x^2y' + y(8x)$$
$$4x^3 + 4x^2yy' + 4xy^2 + 4y^3y' = 4x^2y' + 8xy$$
$$4x^2yy' + 4y^3y' - 4x^2y' = 8xy - 4x^3 - 4xy^2$$
$$4y'\left(x^2y + y^3 - x^2\right) = 4\left(2xy - x^3 - xy^2\right)$$
$$y' = \frac{2xy - x^3 - xy^2}{x^2y + y^3 - x^2}$$
At $(1, 1)$: $y' = 0$

32. $$x^3 + y^3 - 6xy = 0$$
$$3x^2 + 3y^2y' - 6xy' - 6y = 0$$
$$y'\left(3y^2 - 6x\right) = 6y - 3x^2$$
$$y' = \frac{6y - 3x^2}{3y^2 - 6x} = \frac{2y - x^2}{y^2 - 2x}$$
At $\left(\frac{4}{3}, \frac{8}{3}\right)$: $y' = \frac{(16/3) - (16/9)}{(64/9) - (8/3)} = \frac{32}{40} = \frac{4}{5}$

33. $(y - 3)^2 = 4(x - 5)$, $(6, 1)$
$$2(y - 3)y' = 4$$
$$y' = \frac{2}{y - 3}$$
At $(6, 1)$: $y' = \frac{2}{1 - 3} = -1$

Tangent line: $y - 1 = -1(x - 6)$
$$y = -x + 7$$

34. $(x + 2)^2 + (y - 3)^2 = 37$, $(4, 4)$
$$2(x + 2) + 2(y - 3)y' = 0$$
$$(y - 3)y' = -(x + 2)$$
$$y' = -\frac{(x + 2)}{y - 3}$$
At $(4, 4)$: $y' = -\frac{6}{1} = -6$

Tangent line: $y - 4 = -6(x - 4)$
$$y = -6x + 28$$

35. $xy = 1$, $(1, 1)$
$$xy' + y = 0$$
$$y' = \frac{-y}{x}$$
At $(1, 1)$: $y' = -1$

Tangent line: $y - 1 = -1(x - 1)$
$$y = -x + 2$$

36. $7x^2 - 6\sqrt{3}xy + 13y^2 - 16 = 0$, $\left(\sqrt{3}, 1\right)$
$$14x - 6\sqrt{3}xy' - 6\sqrt{3}y + 26yy' = 0$$
$$y' = \frac{6\sqrt{3}y - 14x}{26y - 6\sqrt{3}x}$$
At $\left(\sqrt{3}, 1\right)$: $y' = \frac{6\sqrt{3} - 14\sqrt{3}}{26 - 6\sqrt{3}\sqrt{3}} = \frac{-8\sqrt{3}}{8} = -\sqrt{3}$

Tangent line: $y - 1 = -\sqrt{3}\left(x - \sqrt{3}\right)$
$$y = -\sqrt{3}x + 4$$

37. $x^2y^2 - 9x^2 - 4y^2 = 0$, $\left(-4, 2\sqrt{3}\right)$
$$x^2 2yy' + 2xy^2 - 18x - 8yy' = 0$$
$$y' = \frac{18x - 2xy^2}{2x^2y - 8y}$$
At $\left(-4, 2\sqrt{3}\right)$: $y' = \frac{18(-4) - 2(-4)(12)}{2(16)\left(2\sqrt{3}\right) - 16\sqrt{3}}$
$$= \frac{24}{48\sqrt{3}} = \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}$$

Tangent line: $y - 2\sqrt{3} = \frac{\sqrt{3}}{6}(x + 4)$
$$y = \frac{\sqrt{3}}{6}x + \frac{8}{3}\sqrt{3}$$

38. $x^{2/3} + y^{2/3} = 5$, $(8, 1)$
$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$$
$$y' = \frac{-x^{-1/3}}{y^{-1/3}} = -\left(\frac{y}{x}\right)^{1/3}$$
At $(8, 1)$: $y' = -\frac{1}{2}$

Tangent line: $y - 1 = -\frac{1}{2}(x - 8)$
$$y = -\frac{1}{2}x + 5$$

39. $3(x^2 + y^2)^2 = 100(x^2 - y^2), \quad (4, 2)$

$$6(x^2 + y^2)(2x + 2yy') = 100(2x - 2yy')$$

At $(4, 2)$:

$$6(16 + 4)(8 + 4y') = 100(8 - 4y')$$
$$960 + 480y' = 800 - 400y'$$
$$880y' = -160$$
$$y' = -\tfrac{2}{11}$$

Tangent line: $y - 2 = -\tfrac{2}{11}(x - 4)$

$$11y + 2x - 30 = 0$$
$$y = -\tfrac{2}{11}x + \tfrac{30}{11}$$

40. $y^2(x^2 + y^2) = 2x^2, \quad (1, 1)$

$$y^2x^2 + y^4 = 2x^2$$
$$2yy'x^2 + 2xy^2 + 4y^3y' = 4x$$

At $(1, 1)$:

$$2y' + 2 + 4y' = 4$$
$$6y' = 2$$
$$y' = \tfrac{1}{3}$$

Tangent line: $y - 1 = \tfrac{1}{3}(x - 1)$

$$y = \tfrac{1}{3}x + \tfrac{2}{3}$$

41. (a) $\dfrac{x^2}{2} + \dfrac{y^2}{8} = 1, \quad (1, 2)$

$$x + \frac{yy'}{4} = 0$$
$$y' = -\frac{4x}{y}$$

At $(1, 2)$: $y' = -2$

Tangent line: $y - 2 = -2(x - 1)$

$$y = -2x + 4$$

(b) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \Rightarrow \dfrac{2x}{a^2} + \dfrac{2yy'}{b^2} = 0 \Rightarrow y' = \dfrac{-b^2x}{a^2y}$

$y - y_0 = \dfrac{-b^2x_0}{a^2y_0}(x - x_0)$, Tangent line at (x_0, y_0)

$$\frac{y_0y}{b^2} - \frac{y_0^2}{b^2} = \frac{-x_0x}{a^2} + \frac{x_0^2}{a^2}$$

Because $\dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2} = 1$, you have $\dfrac{y_0y}{b^2} + \dfrac{x_0x}{a^2} = 1$.

Note: From part (a),

$\dfrac{1(x)}{2} + \dfrac{2(y)}{8} = 1 \Rightarrow \dfrac{1}{4}y = -\dfrac{1}{2}x + 1 \Rightarrow y = -2x + 4$,

Tangent line.

42. (a) $\dfrac{x^2}{6} - \dfrac{y^2}{8} = 1, \quad (3, -2)$

$$\frac{x}{3} - \frac{y}{4}y' = 0$$
$$\frac{y}{4}y' = \frac{x}{3}$$
$$y' = \frac{4x}{3y}$$

At $(3, -2)$: $y' = \dfrac{4(3)}{3(-2)} = -2$

Tangent line: $y + 2 = -2(x - 3)$

$$y = -2x + 4$$

(b) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \Rightarrow \dfrac{2x}{a^2} - \dfrac{2yy'}{b^2} = 0 \Rightarrow y' = \dfrac{xb^2}{ya^2}$

$y - y_0 = \dfrac{x_0b^2}{y_0a^2}(x - x_0)$, Tangent line at (x_0, y_0)

$$\frac{yy_0}{b^2} - \frac{y_0^2}{b^2} = \frac{x_0x}{a^2} - \frac{x_0^2}{a^2}$$

Because $\dfrac{x_0^2}{a^2} - \dfrac{y_0^2}{b^2} = 1$, you have $\dfrac{x_0x}{a^2} - \dfrac{yy_0}{b^2} = 1$.

Note: From part (a),

$\dfrac{3x}{6} - \dfrac{(-2)y}{8} = 1 \Rightarrow \dfrac{1}{2}x + \dfrac{y}{4} = 1 \Rightarrow y = -2x + 4$,

Tangent line.

43. $\tan y = x$

$$y' \sec^2 y = 1$$
$$y' = \frac{1}{\sec^2 y} = \cos^2 y, \; -\frac{\pi}{2} < y < \frac{\pi}{2}$$
$$\sec^2 y = 1 + \tan^2 y = 1 + x^2$$
$$y' = \frac{1}{1 + x^2}$$

44. $\cos y = x$

$$-\sin y \cdot y' = 1$$
$$y' = \frac{-1}{\sin y}, \quad 0 < y < \pi$$
$$\sin^2 y + \cos^2 y = 1$$
$$\sin^2 y = 1 - \cos^2 y$$
$$\sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - x^2}$$
$$y' = \frac{-1}{\sqrt{1 - x^2}}, \quad -1 < x < 1$$

45.
$$\begin{aligned} x^2 + y^2 &= 4 \\ 2x + 2yy' &= 0 \\ y' &= \frac{-x}{y} \\ y'' &= \frac{y(-1) + xy'}{y^2} \\ &= \frac{-y + x(-x/y)}{y^2} \\ &= \frac{-y^2 - x^2}{y^3} \\ &= -\frac{4}{y^3} \end{aligned}$$

46.
$$\begin{aligned} x^2y - 4x &= 5 \\ x^2y' + 2xy - 4 &= 0 \\ y' &= \frac{4 - 2xy}{x^2} \\ x^2y'' + 2xy' + 2xy' + 2y &= 0 \\ x^2y'' + 4x\left[\frac{4 - 2xy}{x^2}\right] + 2y &= 0 \\ x^4y'' + 4x(4 - 2xy) + 2x^2y &= 0 \\ x^4y'' + 16x - 8x^2y + 2x^2y &= 0 \\ x^4y'' &= 6x^2y - 16x \\ y'' &= \frac{6xy - 16}{x^3} \end{aligned}$$

47.
$$\begin{aligned} x^2 - y^2 &= 36 \\ 2x - 2yy' &= 0 \\ y' &= \frac{x}{y} \\ x - yy' &= 0 \\ 1 - yy'' - (y')^2 &= 0 \\ 1 - yy'' - \left(\frac{x}{y}\right)^2 &= 0 \\ y^2 - y^3y'' &= x^2 \\ y'' &= \frac{y^2 - x^2}{y^3} = -\frac{36}{y^3} \end{aligned}$$

48.
$$\begin{aligned} xy - 1 &= 2x + y^2 \\ xy' + y &= 2 + 2yy' \\ xy' - 2yy' &= 2 - y \\ (x - 2y)y' &= 2 - y \\ y' &= \frac{2 - y}{x - 2y} \\ xy'' + y' + y' &= 2yy'' + 2(y')^2 \\ xy'' - 2yy'' &= 2(y')^2 - 2y' \\ (x - 2y)y'' &= 2(y')^2 - 2y' = 2\left(\frac{2 - y}{x - 2y}\right)^2 - 2\left(\frac{2 - y}{x - 2y}\right) \\ y'' &= \frac{2(2 - y)[(2 - y) - (x - 2y)]}{(x - 2y)^3} \\ &= \frac{2(2 - y)(2 - x + y)}{(x - 2y)^3} \\ &= \frac{2(4 - 2x + 2y - 2y + xy - y^2)}{(x - 2y)^3} \\ &= \frac{2(y^2 - xy + 2x - 4)}{(2y - x)^3} = \frac{2(-5)}{(2y - x)^3} = \frac{10}{(x - 2y)^3} \end{aligned}$$

49. $y^2 = x^3$

$2yy' = 3x^2$

$y' = \frac{3x^2}{2y} = \frac{3x^2}{2y} \cdot \frac{xy}{xy} = \frac{3y}{2x} \cdot \frac{x^3}{y^2} = \frac{3y}{2x}$

$y'' = \frac{2x(3y') - 3y(2)}{4x^2}$

$= \frac{2x[3 \cdot (3y/2x)] - 6y}{4x^2} = \frac{3y}{4x^2} = \frac{3x}{4y}$

50. $y^3 = 4x$

$3y^2y' = 4$

$y' = \frac{4}{3y^2}$

$3y^2y'' + 6y(y')^2 = 0$

$yy'' + 2(y')^2 = 0$

$y'' = \frac{-2(y')^2}{y} = \frac{-2}{y}\left(\frac{4}{3y^2}\right)^2$

$y'' = -\frac{32}{9y^5}$

Note: $y = (4x)^{1/3}$

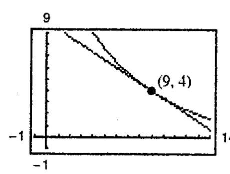
$y' = \frac{4}{3}(4x)^{-2/3}$

$y'' = -\frac{8}{9}(4)(4x)^{-5/3} = -\frac{32}{9(4x)^{5/3}} = -\frac{32}{9y^5}$

51. $\sqrt{x} + \sqrt{y} = 5$

$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}y' = 0$

$y' = \frac{-\sqrt{y}}{\sqrt{x}}$



At $(9, 4)$: $y' = -\frac{2}{3}$

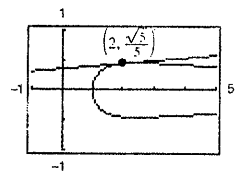
Tangent line: $y - 4 = -\frac{2}{3}(x - 9)$

$2x + 3y - 30 = 0$

52. $y^2 = \frac{x - 1}{x^2 + 1}$

$2yy' = \frac{(x^2 + 1)(1) - (x - 1)(2x)}{(x^2 + 1)^2} = \frac{x^2 + 1 - 2x^2 + 2x}{(x^2 + 1)^2}$

$y' = \frac{1 + 2x - x^2}{2y(x^2 + 1)^2}$



At $\left(2, \frac{\sqrt{5}}{5}\right)$: $y' = \frac{1 + 4 - 4}{\left[\frac{(2\sqrt{5})}{5}\right](4 + 1)^2} = \frac{1}{10\sqrt{5}}$

Tangent line: $y - \frac{\sqrt{5}}{5} = \frac{1}{10\sqrt{5}}(x - 2)$

$10\sqrt{5}y - 10 = x - 2$

$x - 10\sqrt{5}y + 8 = 0$

53. $x^2 + y^2 = 25$

$2x + 2yy' = 0$

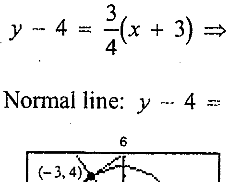
$y' = \frac{-x}{y}$

At $(4, 3)$:

Tangent line:

$y - 3 = \frac{-4}{3}(x - 4) \Rightarrow 4x + 3y - 25 = 0$

Normal line: $y - 3 = \frac{3}{4}(x - 4) \Rightarrow 3x - 4y = 0$

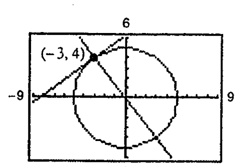


At $(-3, 4)$:

Tangent line:

$y - 4 = \frac{3}{4}(x + 3) \Rightarrow 3x - 4y + 25 = 0$

Normal line: $y - 4 = \frac{-4}{3}(x + 3) \Rightarrow 4x + 3y = 0$



54. $x^2 + y^2 = 36$

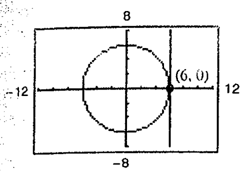
$2x + 2yy' = 0$

$y' = -\frac{x}{y}$

At $(6, 0)$; slope is undefined.

Tangent line: $x = 6$

Normal line: $y = 0$



At $(5, \sqrt{11})$, slope is $\frac{-5}{\sqrt{11}}$

Tangent line: $y - \sqrt{11} = \frac{-5}{\sqrt{11}}(x - 5)$

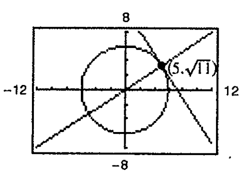
$\sqrt{11}y - 11 = -5x + 25$

$5x + \sqrt{11}y - 36 = 0$

Normal line: $y - \sqrt{11} = \frac{\sqrt{11}}{5}(x - 5)$

$5y - 5\sqrt{11} = \sqrt{11}x - 5\sqrt{11}$

$5y - \sqrt{11}x = 0$



55. $x^2 + y^2 = r^2$

$2x + 2yy' = 0$

$y' = \frac{-x}{y}$ = slope of tangent line

$\frac{y}{x}$ = slope of normal line

Let (x_0, y_0) be a point on the circle. If $x_0 = 0$, then the tangent line is horizontal, the normal line is vertical and, hence, passes through the origin. If $x_0 \neq 0$, then the equation of the normal line is

$y - y_0 = \frac{y_0}{x_0}(x - x_0)$

$y = \frac{y_0}{x_0}x$

which passes through the origin.

56. $y^2 = 4x$

$2yy' = 4$

$y' = \frac{2}{y} = 1$ at $(1, 2)$

Equation of normal line at $(1, 2)$ is

$y - 2 = -1(x - 1)$, $y = 3 - x$. The centers of the circles must be on the normal line and at a distance of 4 units from $(1, 2)$. Therefore,

$(x - 1)^2 + [(3 - x) - 2]^2 = 16$

$2(x - 1)^2 = 16$

$x = 1 \pm 2\sqrt{2}.$

Centers of the circles: $(1 + 2\sqrt{2}, 2 - 2\sqrt{2})$ and $(1 - 2\sqrt{2}, 2 + 2\sqrt{2})$

Equations: $(x - 1 - 2\sqrt{2})^2 + (y - 2 + 2\sqrt{2})^2 = 16$

$(x - 1 + 2\sqrt{2})^2 + (y - 2 - 2\sqrt{2})^2 = 16$

57. $25x^2 + 16y^2 + 200x - 160y + 400 = 0$

$50x + 32yy' + 200 - 160y' = 0$

$y' = \frac{200 + 50x}{160 - 32y}$

Horizontal tangents occur when $x = -4$:

$25(16) + 16y^2 + 200(-4) - 160y + 400 = 0$

$y(y - 10) = 0 \Rightarrow y = 0, 10$

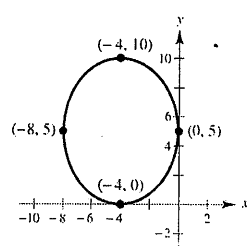
Horizontal tangents: $(-4, 0)$, $(-4, 10)$

Vertical tangents occur when $y = 5$:

$25x^2 + 400 + 200x - 800 + 400 = 0$

$25x(x + 8) = 0 \Rightarrow x = 0, -8$

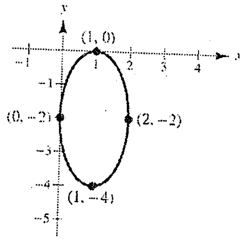
Vertical tangents: $(0, 5)$, $(-8, 5)$



58. $4x^2 + y^2 - 8x + 4y + 4 = 0$

$$8x + 2yy' - 8 + 4y' = 0$$

$$y' = \frac{8 - 8x}{2y + 4} = \frac{4 - 4x}{y + 2}$$



Horizontal tangents occur when $x = 1$:

$$4(1)^2 + y^2 - 8(1) + 4y + 4 = 0$$

$$y^2 + 4y = y(y + 4) = 0 \Rightarrow y = 0, -4$$

Horizontal tangents: $(1, 0), (1, -4)$

Vertical tangents occur when $y = -2$:

$$4x^2 + (-2)^2 - 8x + 4(-2) + 4 = 0$$

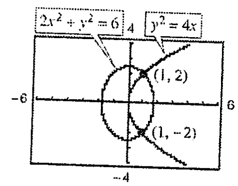
$$4x^2 - 8x = 4x(x - 2) = 0 \Rightarrow x = 0, 2$$

Vertical tangents: $(0, -2), (2, -2)$

59. Find the points of intersection by letting $y^2 = 4x$ in the equation $2x^2 + y^2 = 6$.

$2x^2 + 4x = 6$ and $(x + 3)(x - 1) = 0$

The curves intersect at $(1, \pm 2)$.



Ellipse:	*Parabola*:
$4x + 2yy' = 0$	$2yy' = 4$
$y' = -\frac{2x}{y}$	$y' = \frac{2}{y}$

At $(1, 2)$, the slopes are:

$y' = -1$ $\qquad$ $y' = 1$

At $(1, -2)$, the slopes are:

$y' = 1$ $\qquad$ $y' = -1$

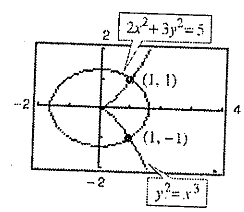
Tangents are perpendicular.

60. Find the points of intersection by letting $y^2 = x^3$ in the equation $2x^2 + 3y^2 = 5$.

$2x^2 + 3x^3 = 5$ and $3x^3 + 2x^2 - 5 = 0$

Intersect when $x = 1$.

Points of intersection: $(1, \pm 1)$



$y^2 = x^3$:	$2x^2 + 3y^2 = 5$:
$2yy' = 3x^2$	$4x + 6yy' = 0$
$y' = \frac{3x^2}{2y}$	$y' = -\frac{2x}{3y}$

At $(1, 1)$, the slopes are:

$y' = \frac{3}{2}$ $\qquad$ $y' = -\frac{2}{3}$

At $(1, -1)$, the slopes are:

$y' = -\frac{3}{2}$ $\qquad$ $y' = \frac{2}{3}$

Tangents are perpendicular.

61. $y = -x$ and $x = \sin y$

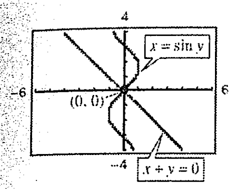
Point of intersection: $(0, 0)$

$y = -x$:	$x = \sin y$:
$y' = -1$	$1 = y'\cos y$
	$y' = \sec y$

At $(0, 0)$, the slopes are:

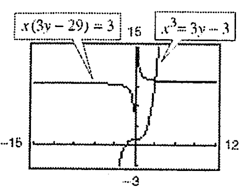
$y' = -1 \qquad y' = 1$

Tangents are perpendicular.



62. Rewriting each equation and differentiating:

$x^3 = 3(y - 1)$	$x(3y - 29) = 3$
$y = \dfrac{x^3}{3} + 1$	$y = \dfrac{1}{3}\left(\dfrac{3}{x} + 29\right)$
$y' = x^2$	$y' = -\dfrac{1}{x^2}$

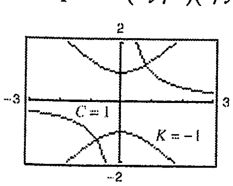


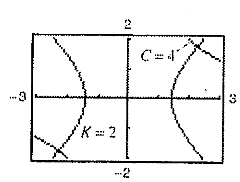
For each value of x, the derivatives are negative reciprocals of each other. So, the tangent lines are orthogonal at both points of intersection.

63.

$xy = C$	$x^2 - y^2 = K$
$xy' + y = 0$	$2x - 2yy' = 0$
$y' = -\dfrac{y}{x}$	$y' = \dfrac{x}{y}$

At any point of intersection (x, y) the product of the slopes is $(-y/x)(x/y) = -1$. The curves are orthogonal.

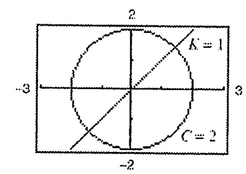


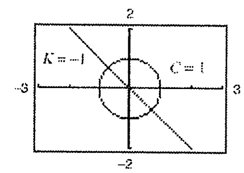


64.

$x^2 + y^2 = C^2$	$y = Kx$
$2x + 2yy' = 0$	$y' = K$
$y' = -\dfrac{x}{y}$	

At the point of intersection (x, y), the product of the slopes is $(-x/y)(K) = (-x/Kx)(K) = -1$. The curves are orthogonal.

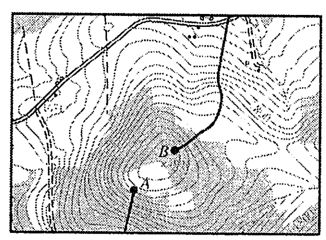




65. Answers will vary. *Sample answer:* In the explicit form of a function, the variable is explicitly written as a function of x. In an implicit equation, the function is only implied by an equation. An example of an implicit function is $x^2 + xy = 5$. In explicit form it would be $y = (5 - x^2)/x$.

66. Answers will vary. *Sample answer:* Given an implicit equation, first differentiate both sides with respect to x. Collect all terms involving y' on the left, and all other terms to the right. Factor out y' on the left side. Finally, divide both sides by the left-hand factor that does not contain y'.

67.



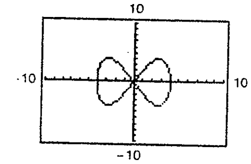
Use starting point B.

68. (a) The slope is greater at $x = -3$.

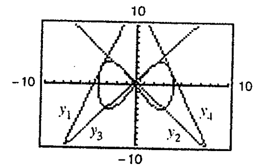
(b) The graph has vertical tangent lines at about $(-2, 3)$ and $(2, 3)$.

(c) The graph has a horizontal tangent line at about $(0, 6)$.

69. (a)
$$x^4 = 4(4x^2 - y^2)$$
$$4y^2 = 16x^2 - x^4$$
$$y^2 = 4x^2 - \frac{1}{4}x^4$$
$$y = \pm\sqrt{4x^2 - \frac{1}{4}x^4}$$



(b)
$$y = 3 \Rightarrow 9 = 4x^2 - \frac{1}{4}x^4$$
$$36 = 16x^2 - x^4$$
$$x^4 - 16x^2 + 36 = 0$$
$$x^2 = \frac{16 \pm \sqrt{256 - 144}}{2} = 8 \pm \sqrt{28}$$



Note that $x^2 = 8 \pm \sqrt{28} = 8 \pm 2\sqrt{7} = \left(1 \pm \sqrt{7}\right)^2$. So, there are four values of x:
$-1 - \sqrt{7}, 1 - \sqrt{7}, -1 + \sqrt{7}, 1 + \sqrt{7}$

To find the slope, $2yy' = 8x - x^3 \Rightarrow y' = \dfrac{x(8 - x^2)}{2(3)}$.

For $x = -1 - \sqrt{7}$, $y' = \frac{1}{3}\left(\sqrt{7} + 7\right)$, and the line is

$$y_1 = \frac{1}{3}\left(\sqrt{7} + 7\right)\left(x + 1 + \sqrt{7}\right) + 3 = \frac{1}{3}\left[\left(\sqrt{7} + 7\right)x + 8\sqrt{7} + 23\right].$$

For $x = 1 - \sqrt{7}$, $y' = \frac{1}{3}\left(\sqrt{7} - 7\right)$, and the line is

$$y_2 = \frac{1}{3}\left(\sqrt{7} - 7\right)\left(x - 1 + \sqrt{7}\right) + 3 = \frac{1}{3}\left[\left(\sqrt{7} - 7\right)x + 23 - 8\sqrt{7}\right].$$

For $x = -1 + \sqrt{7}$, $y' = -\frac{1}{3}\left(\sqrt{7} - 7\right)$, and the line is

$$y_3 = -\frac{1}{3}\left(\sqrt{7} - 7\right)\left(x + 1 - \sqrt{7}\right) + 3 = -\frac{1}{3}\left[\left(\sqrt{7} - 7\right)x - \left(23 - 8\sqrt{7}\right)\right].$$

For $x = 1 + \sqrt{7}$, $y' = -\frac{1}{3}\left(\sqrt{7} + 7\right)$, and the line is

$$y_4 = -\frac{1}{3}\left(\sqrt{7} + 7\right)\left(x - 1 - \sqrt{7}\right) + 3 = -\frac{1}{3}\left[\left(\sqrt{7} + 7\right)x - \left(8\sqrt{7} + 23\right)\right].$$

(c) Equating y_3 and y_4:

$$-\frac{1}{3}\left(\sqrt{7} - 7\right)\left(x + 1 - \sqrt{7}\right) + 3 = -\frac{1}{3}\left(\sqrt{7} + 7\right)\left(x - 1 - \sqrt{7}\right) + 3$$
$$\left(\sqrt{7} - 7\right)\left(x + 1 - \sqrt{7}\right) = \left(\sqrt{7} + 7\right)\left(x - 1 - \sqrt{7}\right)$$
$$\sqrt{7}x + \sqrt{7} - 7 - 7x - 7 + 7\sqrt{7} = \sqrt{7}x - \sqrt{7} - 7 + 7x - 7 - 7\sqrt{7}$$
$$16\sqrt{7} = 14x$$
$$x = \frac{8\sqrt{7}}{7}$$

If $x = \dfrac{8\sqrt{7}}{7}$, then $y = 5$ and the lines intersect at $\left(\dfrac{8\sqrt{7}}{7}, 5\right)$.

70. $\sqrt{x} + \sqrt{y} = \sqrt{c}$

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

Tangent line at (x_0, y_0): $y - y_0 = -\frac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0)$

x-intercept: $\left(x_0 + \sqrt{x_0}\sqrt{y_0}, 0\right)$

y-intercept: $\left(0, y_0 + \sqrt{x_0}\sqrt{y_0}\right)$

Sum of intercepts:

$$\left(x_0 + \sqrt{x_0}\sqrt{y_0}\right) + \left(y_0 + \sqrt{x_0}\sqrt{y_0}\right) = x_0 + 2\sqrt{x_0}\sqrt{y_0} + y_0 = \left(\sqrt{x_0} + \sqrt{y_0}\right)^2 = \left(\sqrt{c}\right)^2 = c$$

71. $y = x^{p/q}$; p, q integers and $q > 0$

$$y^q = x^p$$

$$qy^{q-1}y' = px^{p-1}$$

$$y' = \frac{p}{q} \cdot \frac{x^{p-1}}{y^{q-1}} = \frac{p}{q} \cdot \frac{x^{p-1}y}{y^q}$$

$$= \frac{p}{q} \cdot \frac{x^{p-1}}{x^p}x^{p/q} = \frac{p}{q}x^{p/q-1}$$

So, if $y = x^n$, $n = p/q$, then $y' = nx^{n-1}$.

72. $x^2 + y^2 = 100$, slope $= \frac{3}{4}$

$$2x + 2yy' = 0$$

$$y' = -\frac{x}{y} = \frac{3}{4} \Rightarrow y = -\frac{4}{3}x$$

$$x^2 + \left(\frac{16}{9}x^2\right) = 100$$

$$\frac{25}{9}x^2 = 100$$

$$x = \pm 6$$

Points: $(6, -8)$ and $(-6, 8)$

73. $\frac{x^2}{4} + \frac{y^2}{9} = 1, \quad (4, 0)$

$$\frac{2x}{4} + \frac{2yy'}{9} = 0$$

$$y' = \frac{-9x}{4y}$$

$$\frac{-9x}{4y} = \frac{y - 0}{x - 4}$$

$$-9x(x - 4) = 4y^2$$

But, $9x^2 + 4y^2 = 36 \Rightarrow 4y^2 = 36 - 9x^2$. So, $-9x^2 + 36x = 4y^2 = 36 - 9x^2 \Rightarrow x = 1$.

Points on ellipse: $\left(1, \pm\frac{3}{2}\sqrt{3}\right)$

At $\left(1, \frac{3}{2}\sqrt{3}\right)$: $y' = \frac{-9x}{4y} = \frac{-9}{4\left[(3/2)\sqrt{3}\right]} = -\frac{\sqrt{3}}{2}$

At $\left(1, -\frac{3}{2}\sqrt{3}\right)$: $y' = \frac{\sqrt{3}}{2}$

Tangent lines: $y = -\frac{\sqrt{3}}{2}(x - 4) = -\frac{\sqrt{3}}{2}x + 2\sqrt{3}$

$$y = \frac{\sqrt{3}}{2}(x - 4) = \frac{\sqrt{3}}{2}x - 2\sqrt{3}$$

74. $x = y^2$

$1 = 2yy'$

$y' = \frac{1}{2y}$, slope of tangent line

Consider the slope of the normal line joining $(x_0, 0)$ and $(x, y) = (y^2, y)$ on the parabola.

$$-2y = \frac{y - 0}{y^2 - x_0}$$

$$y^2 - x_0 = -\frac{1}{2}$$

$$y^2 = x_0 - \frac{1}{2}$$

(a) If $x_0 = \frac{1}{4}$, then $y^2 = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$, which is impossible. So, the only normal line is the x-axis $(y = 0)$.

(b) If $x_0 = \frac{1}{2}$, then $y^2 = 0 \Rightarrow y = 0$. Same as part (a).

(c) If $x_0 = 1$, then $y^2 = \frac{1}{2} = x$ and there are three normal lines.

The x-axis, the line joining $(x_0, 0)$ and $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$, and the line joining $(x_0, 0)$ and $\left(\frac{1}{2}, -\frac{1}{\sqrt{2}}\right)$

If two normals are perpendicular, then their slopes are -1 and 1. So,

$$-2y = -1 = \frac{y - 0}{y^2 - x_0} \Rightarrow y = \frac{1}{2}$$

and

$$\frac{1/2}{(1/4) - x_0} = -1 \Rightarrow \frac{1}{4} - x_0 = -\frac{1}{2} \Rightarrow x_0 = \frac{3}{4}.$$

The perpendicular normal lines are $y = -x + \frac{3}{4}$ and $y = x - \frac{3}{4}$.

75. (a) $\frac{x^2}{32} + \frac{y^2}{8} = 1$

$$\frac{2x}{32} + \frac{2yy'}{8} = 0 \Rightarrow y' = \frac{-x}{4y}$$

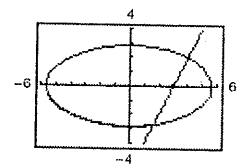
At $(4, 2)$: $y' = \frac{-4}{4(2)} = -\frac{1}{2}$

Slope of normal line is 2.

$$y - 2 = 2(x - 4)$$

$$y = 2x - 6$$

(b)



(c) $\frac{x^2}{32} + \frac{(2x - 6)^2}{8} = 1$

$$x^2 + 4(4x^2 - 24x + 36) = 32$$

$$17x^2 - 96x + 112 = 0$$

$$(17x - 28)(x - 4) = 0 \Rightarrow x = 4, \frac{28}{17}$$

Second point: $\left(\frac{28}{17}, -\frac{46}{17}\right)$

Section 2.6 Related Rates

1. $y = \sqrt{x}$

$$\frac{dy}{dt} = \left(\frac{1}{2\sqrt{x}}\right)\frac{dx}{dt}$$

$$\frac{dx}{dt} = 2\sqrt{x}\,\frac{dy}{dt}$$

(a) When $x = 4$ and $dx/dt = 3$:

$$\frac{dy}{dt} = \frac{1}{2\sqrt{4}}(3) = \frac{3}{4}$$

(b) When $x = 25$ and $dy/dt = 2$:

$$\frac{dx}{dt} = 2\sqrt{25}(2) = 20$$