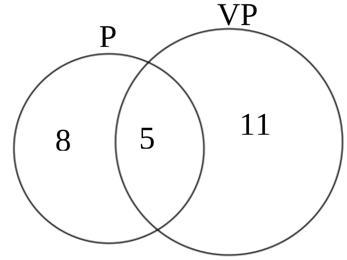
THE 2015–2016 KENNESAW STATE UNIVERSITY
HIGH SCHOOL MATHEMATICS COMPETITION

Solutions

1. D Make a Venn diagram. Answer is 8.



The eight were: Franklin Roosevelt, Dwight Eisenhower, John Kennedy, Jimmy Carter, Ronald Reagan, Bill Clinton, George W. Bush, Barack Obama.

2. C Clearly, R = 0. The number F I V E will be smallest if F = 1. This means O must be 2, 3, or 4, and 2 makes the I smallest. Thus, I = 4. This leaves 3 and 5 for U and N (in either order), making V = 8. This leaves only 6, 7, or 9 for E. Therefore, the smallest value for the number F I V E is 1486.

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   F O U R        1 2 3 0
+    O N E     +    2 5 6
   -------        -------
   F I V E        1 4 8 6
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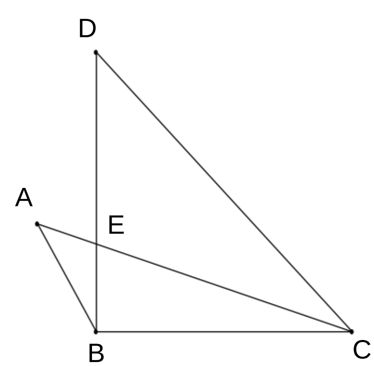
3. E Let n = number of boys in the club at the start. Then $(n)(n) + 4(n + 4) = 301$ From which $n^2 + 4n - 285 = 0$. Factoring, $(n - 15)(n + 19) = 0$, and $n = 15$. Thus, there are 15 + 4 = 19 boys now in the club.

4. C Method 1: The probability that the second number is the same as the first is 1/6. Therefore, 5/6 of the time, one die has a higher number than the other. By symmetry, the probability that the second die has the higher number is (1/2)(5/6) = 5/12.

Method 2: There is a 1/6 probability that the first die is a one, and in that case, a 5/6 probability that the second die is larger. Therefore, the probability that both occur is (1/6)(5/6). Similarly, if the first die is a 2, 3, 4, or 5, the required probabilities are, respectively, (1/6)(4/6), (1/6)(3/6), (1/6)(2/6), (1/6)(1/6). Therefore, the probability that the second die is larger than the first is

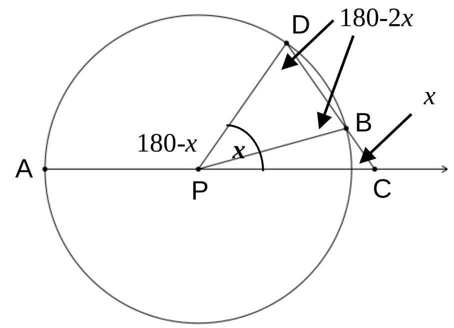
$$(1/6)(5/6) + (1/6)(4/6) + (1/6)(3/6) + (1/6)(2/6) + (1/6)(1/6) = 15/36 = 5/12.$$

5. C In ΔDBC, $m\angle D = 40$. Since $\angle ABD \cong \angle DCA$ and vertical angles DEC and AEB are congruent, the third angles of triangles DEC and AEB must also be congruent. Therefore, $\angle A \cong \angle D$, making $m\angle A = 40$.



6. B Since $\underline{A}\,\underline{B}$ is prime, B must be 1, 3, 7, or 9. Since $\underline{B}\,\underline{C}$ is a square and there are no squares in the 70's or 90's, $B = 1$ or 3, and this means $C = 6$. A quick check of the integers from 61 to 69 shows that 63 = (9)(7) and 68 = (4)(17) satisfy the third condition of the problem. Thus A = 3 or 8. Remembering the first condition, the only 4-digit numbers that work are 3163 and 8368. The required sum is 11,531

7. B Since PD = DC, ΔPDC is isosceles and ∠CPD ≅ ∠PCD. Since ΔDPB is also isosceles, ∠PDB ≅ ∠PBD. Let m∠PCD = x. Representing angle measures as shown in the diagram, m∠BPD = 180 – 2(180 – 2x) = 4x – 180. Therefore, m∠APB = (180 – x) + (4x – 180) = 3x. Hence, the required ratio is 1:3.



8. C It is easy enough to list all 10 possibilities: {0,1,2,8,9,10}, {0,1,3,7,9,10}, {0,1,4,6,9,10} {0,2,3,7,8,10}, {0, 2,4,6,8,10}, {0,3,4,6,7,10}, {1,2,3,7,8,9}, {1,2,4,6,8,9}, {1,3,4,6,7,9}, and {2,3,4,6,7,8}.

9. A Let 3k and k denote the number of students in each subgroup and let M denote the class mean. Then $4kM - 3k(M-5) = kM + 15k$ represents the total points scored by the smaller subgroup of k students. Therefore, the mean of this group is

$$\frac{kM + 15k}{k} = M + 15.$$

10. E Let S = the sum of the 28 consecutive odd integers. Then

$$S = a + (a+2) + (a+4) + \ldots + (a+54) = \frac{28}{2}(2a+54) = 28(a+27).$$

Since $28 = 7(2^2)$, this last expression will be a perfect cube if $(a + 27) = 2(7^2) = 98$. Therefore, $a = 71$.

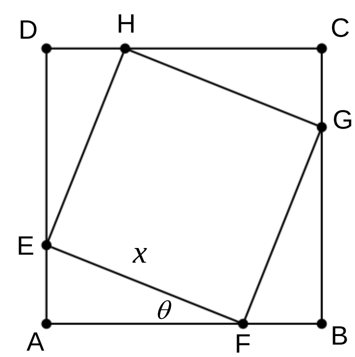
11. C Let P, N, and Q represent the total number of pennies, nickels, and quarters, respectively. Then $P + 5N + 25Q = P + 35P + 25Q = 607$. Therefore, $Q = \frac{607 - 36P}{25}$. Since Q is a positive integer, the numerator of this fraction must be divisible by 25 and so must end in a 5 (it cannot end in 0 since 607 is odd and 36P is even). This can only happen if P ends in 2 or 7. Trying P = 2, 7, and 12, we find only 12 works. If $P \geq 17$, the numerator is negative. Therefore, $P = 12$, $N = 84$, and $Q = 7$ and the total number of coins is 103.

12. B Let EF = x. Then AE = $x \sin\theta$, and DE = AF = $x \cos\theta$. Therefore,

$$x\sin\theta + x\cos\theta = x(\sin\theta + \cos\theta) = 1.$$

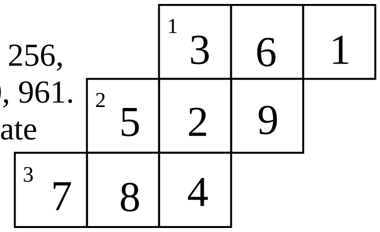
Squaring both sides and solving for x^2, we obtain

$$x^2 = \frac{1}{\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta} = \frac{1}{1 + 2\sin\theta\cos\theta} = \frac{1}{1+\sin 2\theta}$$



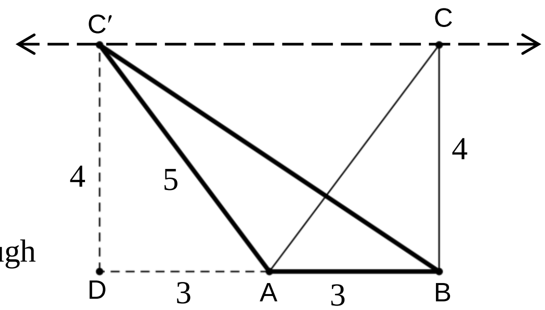
13. B List all the three digit perfect squares: 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400, 441, 484, 529, 576, 625, 676, 729, 784, 841, 900, 961. Since each digit from 1 to 9 is to be represented exactly once, eliminate all those with duplicate digits or a 0. We are left with

169, 196, 256, 289, 324, 361, 529, 576, 625, 729, 784, 841, 961.



Only two of the numbers contain the digit 3, namely 324 and 361. If we choose 324 as one of the "across" entries, then the other two cannot contain the digits 3, 2, and 4. This leaves only 169, 196, 576, and 961. But then it is impossible to have the digit 8. Therefore we must choose 361 instead. Eliminating the other numbers which contain a 3, 6, or 1, we are left with 289, 529, 729, and 784. From these we must choose 529 and 784 because they are the only occurrences of 5 and 4, respectively. Only the arrangement shown gives us a perfect square 324 for 1-down. Therefore, 2-down is 58.

14. B Represent the solutions with n, $n+1$, and $n+2$. Then $a = -[n + (n+1) + (n+2)]$ and $a^2 = (3n+3)^2 = 9(n+1)^2$. Also, $b = n(n+1) + n(n+2) + (n+1)(n+2) = 3n^2 + 6n + 2$, so that $b + 1 = 3n^2 + 6n + 3 = 3(n+1)^2$. Therefore, $\frac{a^2}{b+1} = \frac{9(n+1)^2}{3(n+1)^2} = 3$.



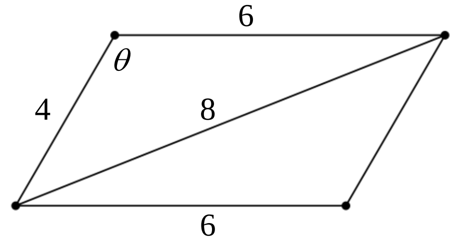
15. C Since the area of the triangle is 6, the altitude to the 3″ side must have length 4″. In the diagram, ΔABC illustrates the 3-4-5 right triangle that fulfills the requirement. To find the other triangle, extend $\overline{BA}$ through point A to a point D that is 3″ from A. Let C′ be the point on the parallel to $\overline{BA}$ through point C that is 4″ from D, as shown. ΔAC′B has the required properties. Using the Pythagorean Theorem on ΔBC′D, C′B = $\sqrt{52}$.

16. D The ratio of the geometric progression is $\frac{i+1}{i} = 1 - i$. Therefore, the third term is $(i+1)(1-i) = 2$. The fourth term is $2(1-i) = 2 - 2i$, the fifth term is $(1-i)(2-2i) = -4i$, and the sixth term is $(1-i)(-4i) = -4i - 4 = -4(1+i)$. Therefore, the sixth term is –4 times the second term. Thus the tenth term is $16(1+i)$ and the fourteenth term is $-64(1+i)$. Multiplying this by $(1-i)$ gives –128 as the fifteenth term.

17. E Adding the given equations gives $a + b + ab = 4030$. Adding one to both sides, we get $a + b + ab + 1 = (a+1)(b+1) = 4031$. Since 4031 = (29)(139), and both 29 and 139 are prime, $a = 28$ and $b = 138$ (or vice versa, since a and b are interchangeable in this situation). Therefore $a + b + c = 28 + 138 + c = 2015$, and $c = 1849$.

18. C The answer is c, b, a. Clearly, $k! > 2^k$ for all integers $k > 3$, which implies that $c >$ b. To see that $b > a$, we need to show that $2^n > (\sqrt{n})!$. Let $n = m^2$. Then we wish to show that $2^{m^2} > m!$. This is certainly true because $2^{m^2} = (2^m)^m$ and $2^m > m$ for $m > 2$.

19. E Using the Law of Cosines on the triangle with sides 4, 6, 8,



$8^2 = 4^2 + 6^2 - 2(4)(6)\cos\theta \quad \Rightarrow \quad \cos\theta = -\frac{1}{4}$

Since the consecutive angles of a parallelogram are supplementary, the other angle of the parallelogram is $(180 - \theta)°$. Let d represent the length of the other diagonal. Now using the Law of Cosines on the triangle with sides 4, 6, d,

$d^2 = 4^2 + 6^2 - 2(4)(6)[\cos(180-\theta)] = 52 - 48(-\cos\theta) = 52 - 48\left(\frac{1}{4}\right) = 40$ and $d = \sqrt{40}$.

20. A $f(11) = 11$, $f(14) = \frac{f(11)-1}{f(11)+1} = \frac{5}{6}$, $f(17) = \frac{f(14)-1}{f(14)+1} = -\frac{1}{11}$,

$f(20) = \frac{f(17)-1}{f(17)+1} = -\frac{6}{5}$ and $f(23) = \frac{f(20)-1}{f(20)+1} = 11$.

Therefore, for all positive integers n, $f(11) = f(23) = f(35) = \ldots = f(11+12n)$.
Since $f(2015) = f(11 + 12\cdot 167)$, then $f(2015) = 11$.

21. E Let $A = 10x + y$ and $B = 10y + x$, where $x = 1, 2, 3, \ldots, 9$, and $y = 0, 1, 2, 3, \ldots, 9$. Then

$$A^2 - B^2 = (10x+y)^2 - (10y+x)^2 = 99x^2 - 99y^2 = (9)(11)(x+y)(x-y)$$

Since this must be a perfect square and $x - y < 10$, 11 must be a factor of $x + y$. But $x + y \le 17$, so $x + y = 11$. Therefore, $x - y$ is a perfect square. Hence, $x - y = 1$, 4, or 9. Since x +y and x – y are either both even or both odd, the possible combinations are:

(i) $x + y = 11$ and $x - y = 1$ and (ii) $x + y = 11$ and $x - y = 9$.

(i) yields $x = 6$, $y = 5$ from which $A = 65$. (ii) yields $x = 10$, $y = 1$ which is not acceptable. Therefore, $A = 65$ and $A^2 + B^2 = 65^2 + 56^2 = 7{,}361$.

22. C The measure of ∠C is $\frac{180(n-2)}{n}$. The measure of ∠CBD is $\frac{180 - \frac{180(n-2)}{n}}{2} = \frac{180}{n}$.

The measure of ∠DBA is $\frac{180(n-2)}{n} - \frac{180}{n} = \frac{180n - 540}{n}$

For the second polygon to be a regular polygon with m sides, $\frac{180(m-2)}{m} = \frac{180n - 540}{n}$.

Simplifying this last equation gives $m = \frac{2}{3}n$. Since m is an integer, any value of $n > 4$ which is a multiple of 3 will suffice. The only choice that is a multiple of 3 is 57.

23. E Expand 101^{20} using the binomial theorem:

$$(100+1)^{20} = 1 + (20)(100) + \binom{20}{2}(100^2) + \binom{20}{3}(100^3) + \binom{20}{4}(100^4) + \ldots$$

The first 3 terms will not affect the ninth digit from the right since each has fewer than 8 digits. None of the omitted terms at the end will affect the ninth digit from right since each has more than 9 terminal zeros. The fourth and fifth terms are 1,140,000,000 and 484,500,000,000, so that the ninth digit from the right in their sum is 5 + 1 = 6.

24. D $\log_{\sin x}(\tan x) = \log_{\tan x}(\sin x) = \dfrac{1}{\log_{\sin x}(\tan x)}$ using the change of base formula,

where $\sin x > 0$ and $\tan x > 0$.

Therefore, $[\log_{\sin x}(\tan x)]^2 = 1$, which implies $\log_{\sin x}(\tan x) = \pm 1$.

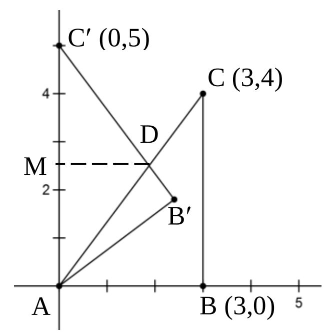
$\log_{\sin x}(\tan x) = 1 \Rightarrow \tan x = \sin x \Rightarrow \cos x = 1$. However, this would make $\sin x = 0$.

$\log_{\sin x}(\tan x) = -1 \Rightarrow \cos x = \sin^2 x = 1 - \cos^2 x$

Therefore, $\cos^2 x + \cos x - 1 = 0 \Rightarrow \cos x = \dfrac{-1 \pm \sqrt{5}}{2}$.

Since $\dfrac{-1-\sqrt{5}}{2} < -1$, the only possible value of $\cos x$ is $\dfrac{-1+\sqrt{5}}{2}$.

25. A Since ABC is a right triangle, AC = AC′ = 5. Let D be the point of intersection of $\overline{AC}$ and $\overline{B'C'}$, as shown in the diagram. Since BC is parallel to the y-axis, ∠C ≅ ∠CAC′. Therefore, ∠B′C′A ≅ ∠CA C′, so that ΔADC′ is isosceles. Thus, the altitude from D meets $\overline{AC'}$ at its midpoint, M, and C′M = 2.5.



Since ΔMDC′ and ΔBAC are both right triangles with ∠C ≅ ∠C′, the triangles are similar. Thus, $\dfrac{MC'}{BC} = \dfrac{C'D}{AC}$. Substituting, $\dfrac{2.5}{4} = \dfrac{C'D}{5}$, from which $C'D = \dfrac{25}{8}$.

Hence, $DB' = 4 - \dfrac{25}{8} = \dfrac{7}{8}$, and the area of right triangle DAB′ is

$$\frac{1}{2}(AB')(B'D) = \frac{1}{2}(3)\left(\frac{7}{8}\right) = \frac{21}{16}$$