MATHEMATICAL ASSOCIATION OF AMERICA
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# Solutions Pamphlet

American Mathematics Competitions

67th Annual

# AMC 12A

**American Mathematics Contest 12A**

**Tuesday, February 2, 2016**

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic versus geometric, computational versus conceptual, elementary versus advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions for this contest during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, internet, or media of any type is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 12 and orders for any publications should be addressed to:


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The problems and solutions for this AMC 12 were prepared by
MAA's Subcommittee on the AMC10/AMC12 Exams, under the direction of the
co-chairs Jerrold W. Grossman and Silvia Fernandez.

1. **Answer (B):**

$$\frac{11! - 10!}{9!} = \frac{10! \cdot (11-1)}{9!} = \frac{10 \cdot 9! \cdot 10}{9!} = 100$$

2. **Answer (C):** The equation can be written $10^x \cdot (10^2)^{2x} = (10^3)^5$ or $10^x \cdot 10^{4x} = 10^{15}$. Thus $10^{5x} = 10^{15}$, so $5x = 15$ and $x = 3$.

3. **Answer (B):**

$$\frac{3}{8} - \left(-\frac{2}{5}\right)\left\lfloor \frac{\frac{3}{8}}{-\frac{2}{5}} \right\rfloor = \frac{3}{8} + \frac{2}{5}\left\lfloor -\frac{15}{16} \right\rfloor = \frac{3}{8} + \frac{2}{5}(-1) = -\frac{1}{40}$$

4. **Answer (D):** The mean of the data values is

$$\frac{60 + 100 + x + 40 + 50 + 200 + 90}{7} = \frac{x + 540}{7} = x.$$

Solving this equation for $x$ gives $x = 90$. Thus the data in nondecreasing order are $40, 50, 60, 90, 90, 100, 200$, so the median is 90 and the mode is 90, as required.

5. **Answer (E):** A counterexample must satisfy the hypothesis of being an even integer greater than 2 but fail to satisfy the conclusion that it can be written as the sum of two prime numbers.

6. **Answer (D):** There are

$$1 + 2 + \cdots + N = \frac{N(N+1)}{2}$$

coins in the array. Therefore $N(N+1) = 2 \cdot 2016 = 4032$. Because $N(N+1) \approx N^2$, it follows that $N \approx \sqrt{4032} \approx \sqrt{2^{12}} = 2^6 = 64$. Indeed, $63 \cdot 64 = 4032$, so $N = 63$ and the sum of the digits of $N$ is 9.

7. **Answer (D):** The given equation is equivalent to $(x^2 - y^2)(x + y + 1) = 0$, which is in turn equivalent to $(x + y)(x - y)(x + y + 1) = 0$. A product is 0 if and only if one of the factors is 0, so the graph is the union of the graphs of $x + y = 0$, $x - y = 0$, and $x + y + 1 = 0$. These are three straight lines, two of which intersect at the origin and the third of which does not pass through the origin. Therefore the graph consists of three lines that do not all pass through a common point.

8. **Answer (D):** The diagonal of the rectangle from upper left to lower right divides the shaded region into four triangles. Two of them have a 1-unit horizontal base and altitude $\frac{1}{2} \cdot 5 = 2\frac{1}{2}$, and the other two have a 1-unit vertical base and altitude $\frac{1}{2} \cdot 8 = 4$. Therefore the total area is $2 \cdot \frac{1}{2} \cdot 1 \cdot 2\frac{1}{2} + 2 \cdot \frac{1}{2} \cdot 1 \cdot 4 = 6\frac{1}{2}$.

9. **Answer (E):** Let $x$ be the common side length. Draw a diagonal between opposite corners of the unit square. The length of this diagonal is $\sqrt{2}$. The diagonal consists of two small-square diagonals and one small-square side length. Combining the previous two observations yields

$$2x\sqrt{2} + x = \sqrt{2}.$$

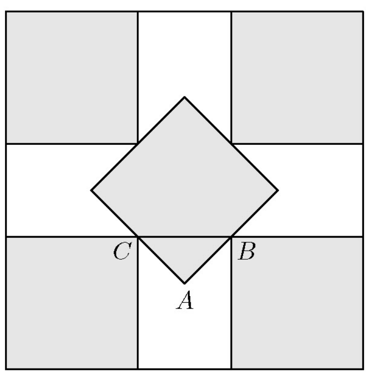
Solving this equation for $x$ gives $x = \frac{4-\sqrt{2}}{7}$. The requested sum is $4 + 7 = 11$.

**OR**

Again let $x$ be the common side length. Triangle $ABC$ in the figure shown is a right triangle with sides $\frac{x}{2}$, $\frac{x}{2}$, and $1 - 2x$. By the Pythagorean Theorem,

$$\left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^2 = (1 - 2x)^2.$$

Solving this equation and noting that $x < \frac{1}{2}$ yields $x = \frac{4-\sqrt{2}}{7}$, as above.




10. **Answer (B):** The total number of seats moved to the right among the five friends must equal the total number of seats moved to the left. One of Dee and Edie moved some number of seats to the right, and the other moved the same number of seats to the left. Because Bea moved two seats to the right and Ceci moved one seat to the left, Ada must also move one seat to the left upon her

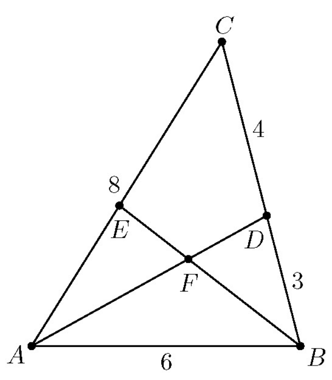
return. Because her new seat is an end seat and its number cannot be 5, it must be seat 1. Therefore Ada occupied seat 2 before she got up. The order before moving was Bea-Ada-Ceci-Dee-Edie (or Bea-Ada-Ceci-Edie-Dee), and the order after moving was Ada-Ceci-Bea-Edie-Dee (or Ada-Ceci-Bea-Dee-Edie).

11. **Answer (E):** Because 42 students cannot sing, $100 - 42 = 58$ can sing. Similarly, $100 - 65 = 35$ can dance, and $100 - 29 = 71$ can act. This gives a total of $58 + 35 + 71 = 164$. However, the students with two talents have been counted twice in this sum. Because there are 100 students in all, $164 - 100 = 64$ students must have been counted twice.

**OR**

Consider the three sets referred to in the problem: those who cannot sing, those who cannot dance, and those who cannot act. Students with one talent are in two of those sets, whereas students with two talents are in only one. Thus the total $42 + 65 + 29 = 136$ counts all students twice except those with two talents. The number of students with two talents is therefore $2 \cdot 100 - 136 = 64$.

12. **Answer (C):** Applying the Angle Bisector Theorem to $\triangle BAC$ gives $BD : DC = 6 : 8$, so $BD = \frac{6}{6+8} \cdot 7 = 3$. Then applying the Angle Bisector Theorem to $\triangle ABD$ gives $AF : FD = 6 : 3 = 2 : 1$.




**Note:** More generally the ratio $AF : FD$ is $(AB + CA) : BC$, which equals $2 : 1$ whenever $AB, BC, CA$ forms an arithmetic progression.

13. **Answer (A):** Let $N = 5k$, where $k$ is a positive integer. There are $5k + 1$ equally likely possible positions for the red ball in the line of balls. Number

these $0, 1, 2, 3, \ldots, 5k-1, 5k$ from one end. The red ball will *not* divide the green balls so that at least $\frac{3}{5}$ of them are on the same side if it is in position $2k+1, 2k+2, \ldots, 3k-1$. This includes $(3k-1)-2k = k-1$ positions. The probability that $\frac{3}{5}$ or more of the green balls will be on the same side is therefore $1 - \frac{k-1}{5k+1} = \frac{4k+2}{5k+1}$.

Solving the inequality $\frac{4k+2}{5k+1} < \frac{321}{400}$ for $k$ yields $k > \frac{479}{5} = 95\frac{4}{5}$. The value of $k$ corresponding to the required least value of $N$ is therefore 96, so $N = 480$. The sum of the digits of $N$ is 12.

14. **Answer (C):** The sum of the four numbers on the vertices of each face must be $\frac{1}{6} \cdot 3 \cdot (1+2+\cdots+8) = 18$. The only sets of four of the numbers that include 1 and have a sum of 18 are $\{1, 2, 7, 8\}$, $\{1, 3, 6, 8\}$, $\{1, 4, 5, 8\}$, and $\{1, 4, 6, 7\}$. Three of these sets contain both 1 and 8. Because two specific vertices can belong to at most two faces, the vertices of one face must be labeled with the numbers $1, 4, 6, 7$, and two of the faces must include vertices labeled 1 and 8. Thus 1 and 8 must mark two adjacent vertices. The cube can be rotated so that the vertex labeled 1 is at the lower left front, and the vertex labeled 8 is at the lower right front. The numbers 4, 6, and 7 must label vertices on the left face. There are $3! = 6$ ways to assign these three labels to the three remaining vertices of the left face. Then the numbers 5, 3, and 2 must label the vertices of the right face adjacent to the vertices labeled 4, 6, and 7, respectively. Hence there are 6 possible arrangements.

15. **Answer (D):** Let $X$ be the foot of the perpendicular from $P$ to $\overline{QQ'}$, and let $Y$ be the foot of the perpendicular from $Q$ to $\overline{RR'}$. By the Pythagorean Theorem,

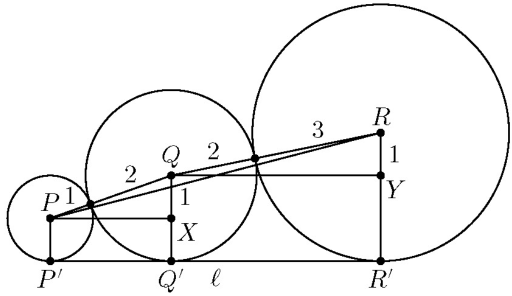
$$P'Q' = PX = \sqrt{(2+1)^2 - (2-1)^2} = \sqrt{8}$$

and

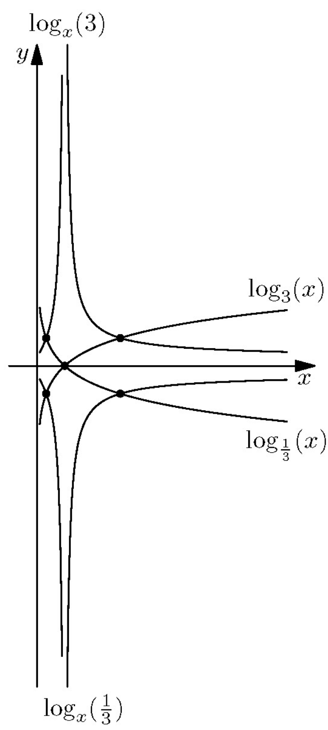
$$Q'R' = QY = \sqrt{(3+2)^2 - (3-2)^2} = \sqrt{24}.$$

The required area can be computed as the sum of the areas of the two smaller trapezoids, $PQQ'P'$ and $QRR'Q'$, minus the area of the large trapezoid, $PRR'P'$:

$$\frac{1+2}{2}\sqrt{8} + \frac{2+3}{2}\sqrt{24} - \frac{1+3}{2}\left(\sqrt{8} + \sqrt{24}\right) = \sqrt{6} - \sqrt{2}.$$




16. **Answer (D):** Let $u = \log_3 x$. Then $\log_x 3 = \frac{1}{u}$, $\log_{\frac{1}{3}} x = -u$, and $\log_x \frac{1}{3} = -\frac{1}{u}$. Thus each point at which two of the graphs of the given functions intersect in the $(x, y)$-plane corresponds to a point at which two of the graphs of $y = u$, $y = \frac{1}{u}$, $y = -u$, and $y = -\frac{1}{u}$ intersect in the $(u, y)$-plane. There are 5 such points $(u, y)$, namely $(0, 0)$, $(1, 1)$, $(-1, 1)$, $(1, -1)$, and $(-1, -1)$. The corresponding points of intersection on the graphs of the given functions are $(1, 0)$, $(3, 1)$, $(\frac{1}{3}, 1)$, $(3, -1)$, and $(\frac{1}{3}, -1)$.



17. **Answer (B):** Without loss of generality, let the square and equilateral triangles have side length 6. Then the height of the equilateral triangles is $3\sqrt{3}$, and the distance of each of the triangle centers, $E$, $F$, $G$, and $H$, to the square $ABCD$ is $\sqrt{3}$. It follows that the diagonal of square $ABCD$ has length $6\sqrt{2}$, and the diagonal of square $EFGH$ has length equal to the side length of square $ABCD$ plus twice the distance from the center of an equilateral triangle to square $ABCD$ or $6+2\sqrt{3}$. The required ratio of the areas of the two squares is equal to the square of the ratio of the lengths of the diagonals of the two squares, or

$$\left(\frac{6+2\sqrt{3}}{6\sqrt{2}}\right)^2 = \left(\frac{3+\sqrt{3}}{3\sqrt{2}}\right)^2 = \frac{12+6\sqrt{3}}{18} = \frac{2+\sqrt{3}}{3}.$$

**OR**

Without loss of generality, place the square in the Cartesian plane with coordinates $A(-6,0)$, $B(0,0)$, $C(0,-6)$, and $D(-6,-6)$. The center of each equilateral triangle is the point at which the medians intersect, and this point is one third of the way from the midpoint of a side of the triangle to the opposite vertex. The height of an equilateral triangle with side 6 is $3\sqrt{3}$, so the centers are $\sqrt{3}$ units from the sides of the square. Therefore the coordinates are $E(-3,\sqrt{3})$, $F(\sqrt{3},-3)$, $G(-3,-6-\sqrt{3})$, and $H(-6-\sqrt{3},-3)$. The area of square $EFGH$ is half the product of the lengths of its diagonals, or $\frac{1}{2}(6+2\sqrt{3})^2 = 24+12\sqrt{3}$. Square $ABCD$ has area 36, so the desired ratio is $\frac{2+\sqrt{3}}{3}$.

18. **Answer (D):** Let $110n^3 = p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$, where the $p_j$ are distinct primes and the $r_j$ are positive integers. Then $\tau(110n^3)$, the number of positive integer divisors of $110n^3$, is given by

$$\tau(110n^3) = (r_1+1)(r_2+1)\cdots(r_k+1) = 110.$$

Because $110 = 2\cdot 5\cdot 11$, it follows that $k=3$, $\{p_1, p_2, p_3\} = \{2, 5, 11\}$, and, without loss of generality, $r_1 = 1$, $r_2 = 4$, and $r_3 = 10$. Therefore

$$n^3 = \frac{p_1\cdot p_2^4\cdot p_3^{10}}{110} = p_2^3\cdot p_3^9, \quad \text{so} \quad n = p_2\cdot p_3^3.$$

It follows that $81n^4 = 3^4\cdot p_2^4\cdot p_3^{12}$, and because 3, $p_2$, and $p_3$ are distinct primes, $\tau(81n^4) = 5\cdot 5\cdot 13 = 325$.

19. **Answer (B):** Jerry arrives at 4 for the first time after an even number of tosses. Because Jerry tosses 8 coins, he arrives at 4 for the first time after either

4, 6, or 8 tosses. If Jerry arrives at 4 for the first time after 4 tosses, then he must have tossed HHHH. The probability of this occurring is $\frac{1}{16}$. If Jerry arrives at 4 for the first time after 6 tosses, he must have tossed 5 heads and 1 tail among the 6 tosses, and the 1 tail must have come among the first 4 tosses. Thus, there are 4 possible sequences of valid tosses, each with probability $\frac{1}{64}$, for a total of $\frac{1}{16}$. If Jerry arrives at 4 for the first time after 8 tosses, then he must have tossed 6 heads and 2 tails among the 8 tosses. Both tails must occur among the first 6 tosses; otherwise Jerry would have already reached 4 before the $8^{\text{th}}$ toss. Further, at least 1 tail must occur in the first 4 tosses; otherwise Jerry would have already reached 4 after the $4^{\text{th}}$ toss. Therefore there are $\binom{6}{2} - 1 = 14$ sequences for which Jerry first arrives at 4 after 8 tosses, each with probability $\frac{1}{256}$, for a total of $\frac{14}{256} = \frac{7}{128}$. Thus the probability that Jerry reaches 4 at some time during the process is $\frac{1}{16} + \frac{1}{16} + \frac{7}{128} = \frac{23}{128}$. The requested sum is $23 + 128 = 151$.

**OR**

Count the sequences of 8 heads or tails that result in Jerry arriving at 4. Any sequence with T appearing fewer than 3 times results in Jerry reaching 4. There are $\binom{8}{0} + \binom{8}{1} + \binom{8}{2} = 1 + 8 + 28 = 37$ such sequences. If Jerry's sequence contains exactly 3 Ts, then he reaches 4 only if he does so before getting his second T. As a result, Jerry can get at most one T in his first 5 tosses. This happens if the first 4 tosses are H and there is exactly one H in the last 4 tosses, or there is one T within the first 4 tosses followed by the remaining 5 Hs, accounting for $4 + 4 = 8$ ways for Jerry to get to 4 with exactly 3 Ts. Finally, the only way for Jerry to get to 4 by tossing exactly 4 Ts is HHHHTTTT. Jerry cannot get to 4 by tossing fewer than 4 Hs. Thus there are $37 + 8 + 1 = 46$ sequences where he reaches 4, out of $2^8 = 256$ equally likely ways to toss the coin 8 times. The required probability is $\frac{46}{256} = \frac{23}{128}$, and the requested sum is $23 + 128 = 151$.

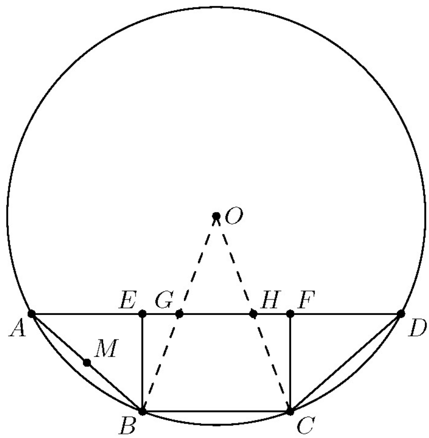
20. **Answer (A):** From the given properties, $a \diamondsuit 1 = a \diamondsuit (a \diamondsuit a) = (a \diamondsuit a) \cdot a = 1 \cdot a = a$ for all nonzero $a$. Then for nonzero $a$ and $b$, $a = a \diamondsuit 1 = a \diamondsuit (b \diamondsuit b) = (a \diamondsuit b) \cdot b$. It follows that $a \diamondsuit b = \frac{a}{b}$. Thus

$$100 = 2016 \diamondsuit (6 \diamondsuit x) = 2016 \diamondsuit \frac{6}{x} = \frac{2016}{\frac{6}{x}} = 336x,$$

so $x = \frac{100}{336} = \frac{25}{84}$. The requested sum is $25 + 84 = 109$.

21. **Answer (E):** Let $ABCD$ be the given cyclic quadrilateral with $AB = BC = CD = 200$, and let $E$ and $F$ be the feet of the perpendicular segments from $B$ and $C$, respectively, to $\overline{AD}$, as shown in the figure. Let the center of the circle be $O$, and let $\angle AOB = \angle BOC = \angle COD = \theta$. Because inscribed $\angle BAD$ is half the

size of central $\angle BOD = 2\theta$, it follows that $\angle BAD = \theta$. Let $M$ be the midpoint of $\overline{AB}$. Then $\sin\left(\frac{\theta}{2}\right) = \frac{AM}{AO} = \frac{100}{200\sqrt{2}} = \frac{1}{2\sqrt{2}}$. Then $\cos\theta = 1 - 2\sin^2\left(\frac{\theta}{2}\right) = \frac{3}{4}$. Hence $AE = AB\cos\theta = 200 \cdot \frac{3}{4} = 150$, and $FD = 150$ as well. Because $EF = BC = 200$, the remaining side $AD = AE + EF + FD = 150 + 200 + 150 = 500$.




**OR**

Label the quadrilateral $ABCD$ and the center of the circle as in the first solution. Because the chords $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$ are shorter than the radius, each of $\angle AOB$, $\angle BOC$, and $\angle COD$ is less than $60^\circ$, so $O$ is outside the quadrilateral $ABCD$. Let $G$ and $H$ be the intersections of $\overline{AD}$ with $\overline{OB}$ and $\overline{OC}$, respectively. Because $\overline{AD}$ and $\overline{BC}$ are parallel, and $\triangle OAB$ and $\triangle OBC$ are congruent and isosceles, it follows that $\angle ABO = \angle OBC = \angle OGH = \angle AGB$. Thus $\triangle ABG$, $\triangle OGH$, and $\triangle OBC$ are similar and isosceles with $\frac{AB}{BG} = \frac{OG}{GH} = \frac{OB}{BC} = \frac{200\sqrt{2}}{200} = \sqrt{2}$. Then $AG = AB = 200$, $BG = \frac{AB}{\sqrt{2}} = \frac{200}{\sqrt{2}} = 100\sqrt{2}$, and $GH = \frac{OG}{\sqrt{2}} = \frac{BO-BG}{\sqrt{2}} = \frac{200\sqrt{2}-100\sqrt{2}}{\sqrt{2}} = 100$. Therefore $AD = AG + GH + HD = 200 + 100 + 200 = 500$.

**OR**

Let $\theta$ be the central angle that subtends the side of length 200. Then by the Law of Cosines, $\left(200\sqrt{2}\right)^2 + \left(200\sqrt{2}\right)^2 - 2\left(200\sqrt{2}\right)^2\cos\theta = 200^2$, which gives $\cos\theta = \frac{3}{4}$. The Law of Cosines also gives the square of the fourth side of the quadrilateral as

$$\left(200\sqrt{2}\right)^2 + \left(200\sqrt{2}\right)^2 - 2\left(200\sqrt{2}\right)^2\cos(3\theta)$$

$$= 160{,}000 - 160{,}000(4\cos^3\theta - 3\cos\theta) = 250{,}000.$$

Thus the fourth side has length $\sqrt{250{,}000} = 500$.

22. **Answer (A):** Because $\text{lcm}(x, y) = 2^3 \cdot 3^2$ and $\text{lcm}(x, z) = 2^3 \cdot 3 \cdot 5^2$, it follows that $5^2$ divides $z$, but neither $x$ nor $y$ is divisible by 5. Furthermore, $y$ is divisible by $3^2$, and neither $x$ nor $z$ is divisible by $3^2$, but at least one of $x$ or $z$ is divisible by 3. Finally, because $\text{lcm}(y, z) = 2^2 \cdot 3^2 \cdot 5^2$, at least one of $y$ or $z$ is divisible by $2^2$, but neither is divisible by $2^3$. However, $x$ must be divisible by $2^3$. Thus $x = 2^3 \cdot 3^j$, $y = 2^k \cdot 3^2$, and $z = 2^m \cdot 3^n \cdot 5^2$, where $\max(j, n) = 1$ and $\max(k, m) = 2$. There are 3 choices for $(j, n)$ and 5 choices for $(k, m)$, so there are 15 possible ordered triples $(x, y, z)$.

23. **Answer (C):** Let the chosen numbers be $x$, $y$, and $z$. The set of possible ordered triples $(x, y, z)$ forms a solid unit cube, two of whose vertices are $(0, 0, 0)$ and $(1, 1, 1)$. The numbers fail to be the side lengths of a triangle with positive area if and only if one of the numbers is at least as great as the sum of the other two. The ordered triples that satisfy $z \geq x + y$ lie in the region on and above the plane $z = x + y$. The intersection of this region with the solid cube is a solid tetrahedron with vertices $(0, 0, 0)$, $(0, 0, 1)$, $(0, 1, 1)$, and $(1, 0, 1)$. The volume of this tetrahedron is $\frac{1}{6}$. The intersections of the solid cube with the regions defined by the inequalities $y \geq x + z$ and $x \geq y + z$ are solid tetrahedra with the same volume. Because at most one of the inequalities $z > x + y$, $y > x + z$, and $x > y + z$ can be true for any choice of $x$, $y$, and $z$, the three tetrahedra have disjoint interiors. Thus the required probability is $1 - 3 \cdot \frac{1}{6} = \frac{1}{2}$.

**OR**

As in the first solution, the set of possible ordered triples $(x, y, z)$ forms a solid unit cube. First consider only the points for which $x > y$ and $x > z$. These points form a square pyramid whose vertex is $(0, 0, 0)$ and whose base has vertices at $(0, 0, 1)$, $(1, 0, 1)$, $(1, 1, 1)$, and $(0, 1, 1)$. Such an ordered triple corresponds to the side lengths of a triangle if and only if $z < x + y$. The plane $z = x + y$ passes through the vertex of the pyramid and bisects its base, so it bisects the volume of the pyramid. The probability of forming a triangle is the same as the probability of not forming a triangle. The same argument applies when $y$ or $z$ is the largest element in the triple. The probability of any two of $x$, $y$, and $z$ being equal is 0, so this case can be ignored. Thus this event and its complement are equally likely; the probability is $\frac{1}{2}$.

24. **Answer (B):** Because $a$ and $b$ are positive, all the roots must be positive. Let the roots be $r$, $s$, and $t$. Then

$$x^3 - ax^2 + bx - a = (x-r)(x-s)(x-t) = x^3 - (r+s+t)x^2 + (rs+st+tr)x - rst.$$

Therefore $r+s+t = a = rst$. The Arithmetic Mean–Geometric Mean Inequality implies that $27rst \le (r+s+t)^3 = (rst)^3$, from which $a = rst \ge 3\sqrt{3}$. Furthermore, equality is achieved if and only if $r = s = t = \sqrt{3}$. In this case $b = rs + st + tr = 9$.

25. **Answer (E):** Assume that $k = 2j \ge 2$ is even. The smallest perfect square with $k+1$ digits is $10^k = (10^j)^2$. Thus the sequence of numbers written on the board after Silvia erases the last $k$ digits of each number is the sequence

$$1 = \left\lfloor \frac{\left(10^j\right)^2}{10^k} \right\rfloor, \left\lfloor \frac{\left(10^j+1\right)^2}{10^k} \right\rfloor, \ldots, \left\lfloor \frac{n^2}{10^k} \right\rfloor, \ldots.$$

The sequence ends the first time that

$$\left\lfloor \frac{(n+1)^2}{10^k} \right\rfloor - \left\lfloor \frac{n^2}{10^k} \right\rfloor \ge 2;$$

before that, every two consecutive terms are either equal or they differ by 1. Suppose that

$$\left\lfloor \frac{n^2}{10^k} \right\rfloor = a \quad \text{and} \quad \left\lfloor \frac{(n+1)^2}{10^k} \right\rfloor \ge a+2.$$

Then $n^2 < (a+1)10^k$ and $(a+2)10^k \le (n+1)^2$. Thus

$$10^k = (a+2)10^k - (a+1)10^k < (n+1)^2 - n^2 = 2n+1.$$

It follows that $n = \frac{10^k}{2} + m$ for some positive integer $m$. Note that

$$\frac{n^2}{10^k} = \frac{1}{10^k}\left(\frac{10^k}{2} + m\right)^2 = \frac{1}{10^k}\left(\frac{10^{2k}}{4} + m \cdot 10^k + m^2\right) = \frac{10^k}{4} + m + \frac{m^2}{10^k}.$$

Because $k \ge 2$, it follows that $10^k$ is divisible by 4, and so

$$\left\lfloor \frac{n^2}{10^k} \right\rfloor = \frac{10^k}{4} + m + \left\lfloor \frac{m^2}{10^k} \right\rfloor \quad \text{and} \quad \left\lfloor \frac{(n+1)^2}{10^k} \right\rfloor = \frac{10^k}{4} + m + 1 + \left\lfloor \frac{(m+1)^2}{10^k} \right\rfloor.$$

The difference will be at least 2 for the first time when

$$\left\lfloor \frac{m^2}{10^k} \right\rfloor = 0 \quad \text{and} \quad \left\lfloor \frac{(m+1)^2}{10^k} \right\rfloor \ge 1,$$

that is, for $m$ such that $m^2 < 10^k \le (m+1)^2$, equivalently, $m < 10^j \le m+1$. Thus $m = 10^j - 1$ and then

$$f(k) = f(2j) = a + 1 = \left\lfloor \frac{n^2}{10^k} \right\rfloor + 1 = \frac{10^k}{4} + m + 1 = \frac{10^{2j}}{4} + 10^j.$$

Therefore

$$\begin{aligned}\sum_{j=1}^{1008} f(2j) &= \sum_{j=1}^{1008} \left( \frac{10^{2j}}{4} + 10^j \right) = 25 \sum_{j=0}^{1007} 10^{2j} + 10 \sum_{j=0}^{1007} 10^j \\ &= \underbrace{252525\ldots25}_{2016 \text{ digits}} + \underbrace{111\ldots10}_{1009 \text{ digits}}.\end{aligned}$$

Because there are no carries in the sum, the required sum of digits equals $1008 \cdot (2+5) + 1008 \cdot 1 = 1008 \cdot 8 = 8064$.

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