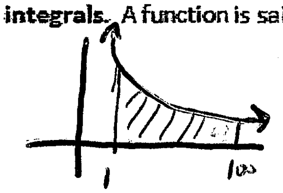
BC Calculus   Ch. 8.8 Notes   Improper Integrals

Key

The definition of a definite integral $\int_a^b f(x)dx$ requires that the interval [a, b] be finite. Furthermore, the Fundamental Theorem of Calculus requires that f be continuous on [a, b]. In this section, we will learn a way to evaluate integrals that do not meet these requirements. We'll see problems where one or both of the limits of integration are infinity and integrate functions that have infinite discontinuities in the interval [a, b]. Integrals that possess either property are called **improper integrals.** A function is said to have an **infinite discontinuity** at c if $\lim_{x\to c} f(x) = \infty$ or $\boxed{\lim_{x\to c}} f(x) = -\infty$.

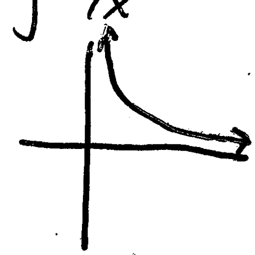


Example 1:
Evaluate by using your calculator.

(a) $\int_1^{100} \frac{1}{x}dx =$ 4.605   (b) $\int_1^{1000} \frac{1}{x}dx =$ 6.908   (c) $\int_1^{1{,}000{,}000} \frac{1}{x}dx =$ 13.816

Based on your values above, what do you think $\int_1^{\infty} \frac{1}{x}dx$ equals? ∞ divergent

$y = 1/x^2$



Fact:

If $a > 0$, then $\int_a^{\infty} \frac{1}{x^p}dx$ is convergent if $p > 1$ and divergent if $p \le 1$. These are called *p-series integrals*.

If $a = 1$ and $p > 1$, then $\int_1^{\infty} \frac{1}{x^p}dx$ converges to $\boxed{\frac{1}{p-1}}$ sum $= \frac{1}{p-1}$ $\int_1^{\infty} \frac{1}{x^2}dx = \frac{1}{2-1} = \boxed{1}$

Example 2:

(a) $\int_1^{\infty} \frac{1}{x^{2/3}}dx = \infty$

(b) $\int_1^{\infty} \frac{1}{x^{1.1}}dx = \frac{1}{1.1-1}$ $\frac{1}{0.1} = \boxed{10}$

(c) $\int_1^{\infty} x^{-7}dx = \frac{1}{x^7}$ $S = \frac{1}{7-1} = \boxed{\frac{1}{6}}$

(d) $\int_1^{\infty} 3 \cdot x^{-3/2}dx$ $\int \frac{3}{x^{3/2}}dx$ $3 \cdot \frac{1}{3/2 - 1} = \boxed{6}$

Integrals such as $\int_a^{\infty} f(x)dx$, $\int_{-\infty}^{a} f(x)dx$, and $\int_{-\infty}^{\infty} f(x)dx$ are called **improper integrals,**

1. They have an infinite interval of integration.
2. They have a discontinuity on the interior of the interval of integration
3. Both 1) and 2).

(a) $\int_1^{\infty} e^{-x}dx = \lim_{b\to\infty} \int_1^b e^{-x}dx$

$u = -x$ $\frac{du}{dx} = -1$ $dx = -du$

$-\int_1^b e^u du = -e^{-x}\Big]_1^b$

$\lim_{b\to\infty} -e^{-b} - -e^{-1}$

$\lim_{b\to\infty} -\frac{1}{e^b} + \frac{1}{e} = \boxed{\frac{1}{e}}$

(b) $\int_{-\infty}^{0} e^{x/4}dx =$ $\lim_{b\to-\infty} \int_b^0 e^{\frac{1}{4}x}dx$ $u = \frac{1}{4}x$ $\frac{du}{dx} = \frac{1}{4}$ $dx = 4du$

$\int_b^0 e^u \cdot 4du = 4e^{x/4}\Big]_b^0$

$\lim_{b\to-\infty} 4e^0 - 4e^{b/4}$

$4 - \frac{4}{e^{\infty}} = \boxed{4}$

(c) $\int_1^{\infty} \frac{1}{x}dx = \boxed{\infty}$

Example 4:

(a) $\int_1^\infty (1-x)e^{-x}\,dx =$

	u	dv
$+$	$1-x$	e^{-x}
$-$	-1	$-e^{-x}$
$+$	0	e^{-x}

$$-e^{-x}(1-x) + e^{-x}$$

$$-\cancel{e^{-x}} + xe^{-x} + \cancel{e^{-x}} = \frac{x}{e^x}\Big]_1^b$$

$$\lim_{b\to\infty} \frac{b}{e^b} - \frac{1}{e^1} = \boxed{\frac{-1}{e}}$$

(b) $\int_0^\infty \frac{2\,dx}{x^2+4x+3} =$

$$\frac{2}{(x+3)(x+1)} = \frac{\overset{-1}{A}}{x+3} + \frac{\overset{1}{B}}{x+1}$$

$x=-3 \qquad x=-1$

$$\int_0^b \frac{-1}{x+3} + \frac{1}{x+1}\,dx$$

$$\lim_{b\to\infty} -\ln|x+3| + \ln|x+1|\Big]_0^b$$

$$\lim_{b\to\infty} -\ln|b+3| + \ln|b+1| - (-\ln 3 + \overset{0}{\ln 1})$$

$$\lim_{b\to\infty} \ln\left|\frac{b+1}{b+3}\right| + \ln 3 + 0$$

$$0 + \ln 3 = \boxed{\ln 3}$$

Example 5:

Determine if the following converge or diverge. If they converge, find the value to which they converge.
This might be helpful:

Convergent plus Convergent = Convergent

Divergent plus Convergent = Divergent

Divergent plus Divergent = Divergent
$(\infty + \infty$ or $-\infty - \infty)$

Divergent – Divergent . . . . . = $\boxed{\text{Indeterminate}}$
$(\infty - \infty$ or $-\infty + \infty)$

(a) $\int_1^\infty \frac{2+x}{x^2}\,dx \qquad \int_1^\infty \frac{2}{x^2} + \frac{x}{x^2}\,dx$

$$\lim_{b\to\infty} \int_1^b 2x^{-2} + \frac{1}{\cancel{x}}\,dx = \frac{2x^{-1}}{-1} + \ln x\Big]_1^b$$

$$\lim_{b\to\infty} \frac{-2}{b} + \ln b - \left(\frac{-2}{1} + \ln 1\right)$$

$$0 + \infty + 2 - 0$$

$\boxed{\text{Diverges}}$

(b) $\int_1^\infty \left[\frac{1}{x} - \frac{3x}{1+5x^2}\right]dx$ — u-sub $\quad u = 1+5x^2 \quad \frac{du}{dx} = 10x \quad dx = \frac{du}{10x}$

$$\int \frac{1}{x} - \frac{3x}{u}\cdot\frac{du}{10x} = \ln x - \frac{3}{10}\ln|1+5x^2|\Big]_1^b$$

$$\lim_{b\to\infty} \underset{\infty}{\ln b} - \underset{\infty}{\frac{3}{10}\ln|1+5b^2|} - \left[\ln 1 - \frac{3}{10}\ln|1+5|\right]$$

$$\lim_{b\to\infty} \ln\left|\frac{b}{(1+5b^2)^{3/10}}\right| = \infty \quad \boxed{\text{Divergent}}$$

Sometimes, an integral can be doubly improper.

If $\int_{-\infty}^{c} f(x)dx$ and $\int_{c}^{\infty} f(x)dx$ are both convergent, then $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx$, where c is any number. Symmetry can also be used to circumvent the doubleness of the impropriety. Note as well that this requires BOTH of the integrals to be convergent in order for this integral to also be convergent. If either of the two integrals is divergent then so is this integral.

Example 6:

*split into 2 integrals, choose a convenient internal bound value

(a) $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}$

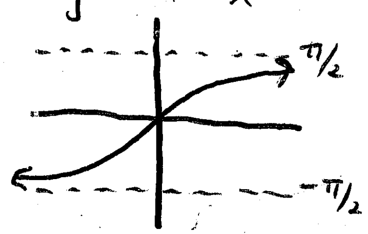
$\lim_{b \to -\infty} \arctan(x)\Big]_b^0 + \lim_{b \to \infty} \arctan x\Big]_0^b$

$\cancel{\arctan 0} - \arctan(b) + \arctan(b) - \cancel{\arctan 0}$

$-(-\pi/2) + \pi/2$

$= \boxed{\pi}$

$y = \arctan x$



(b) $\int_{-\infty}^{\infty} xe^{-x^2} dx = \int_{-\infty}^{0} xe^{-x^2} + \int_{0}^{\infty} xe^{-x^2}$

$u = -x^2$

$\frac{du}{dx} = -2x$

$dx = \frac{du}{-2x}$

$\int xe^{u} \cdot \frac{du}{-2x}$

$= -\frac{1}{2}e^{-x^2}$

$\lim_{b \to -\infty} -\frac{1}{2}e^{-x^2}\Big]_b^0 = -\frac{1}{2} - \frac{1}{2e^{\infty}}$

$= -\frac{1}{2} - 0$

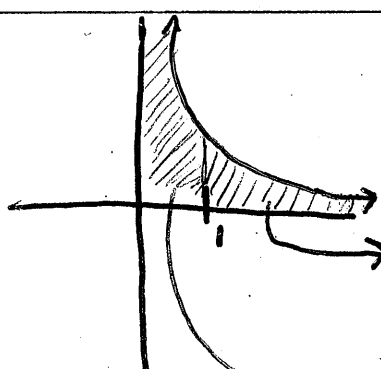
$\lim_{b \to \infty} -\frac{1}{2}e^{-x^2}\Big]_0^b = \frac{-1}{2e^b} + \frac{1}{2}e^0$

$0 + \frac{1}{2}$

$= \boxed{0}$

When an integral is improper has a finite interval of integration, it is improper because its interval spans an infinite discontinuity (vertical asymptote). These are harder to spot, so be vigilant!!

As you can see, if a function of the form $\frac{1}{x^p}$ tends to converge as $x \to \infty$ will tend to diverge as $x \to 0^+$ at the vertical asymptote. The exception, of course is $\frac{1}{x}$, which diverges in both directions.



$y = \frac{1}{x^2}$

$\int_1^{\infty} \frac{1}{x^2}dx$ converges

$\int_0^{1} \frac{1}{x^2}dx$ diverges

When we recognize an infinite discontinuity at an endpoint, we have to set up a one-sided limit. When the infinite discontinuity is on the interior, we have to set up two integrals, approaching the VA from each side in each integral.

Example 7:

(a) $\int_0^1 \frac{1}{x^{1/3}}\,dx =$

$\lim_{b\to 0^+} \int_b^1 \frac{1}{x^{1/3}}\,dx = \int x^{-1/3}$

$\lim_{b\to 0^+} \frac{x^{2/3}}{2/3}\Big]_b^1 = \lim_{b\to 0^+} \frac{3}{2}(1)^{2/3} - \frac{3}{2}(b)^{2/3}$

$= \boxed{\frac{3}{2}}$

(b) $\int_0^1 \frac{1}{x^3}\,dx =$

$\lim_{b\to 0^+} \int_b^1 x^{-3}\,dx = \frac{x^{-2}}{-2}\Big]_b^1$

$\lim_{b\to 0^+} \frac{-1}{2x^2}\Big]_b^1$

$\lim_{b\to 0^+} \frac{-1}{2(1)^2} - \left(\frac{-1}{2(b)^2}\right)$

$-\frac{1}{2} + \infty$ $\boxed{\text{diverges}}$

Example 8:

(a) $\int_0^{27} \frac{dx}{\sqrt[3]{27-x}} =$

$\lim_{b\to 27^-} \int_0^b \frac{dx}{\sqrt[3]{27-x}}$

$u = 27 - x$

$\frac{du}{dx} = -1$

$dx = -du$

$\int \frac{-du}{u^{1/3}} = \int -u^{-1/3}\,du$

$-\frac{3}{2}u^{2/3} = -\frac{3}{2}\left[27-x\right]^{2/3}\Big]_0^b$

$\lim_{b\to 27^-} -\frac{3}{2}\left[27-b\right]^{2/3} - -\frac{3}{2}\left[27-0\right]^{2/3}$

$= -\frac{3}{2}(0)^{2/3} + \frac{3}{2}\left[27\right]^{2/3}$

$0 + \frac{3}{2}\left[3\right]^2$

$= \boxed{\frac{27}{2}}$

(b) $\int_0^3 \frac{dx}{(x-1)^{2/3}} =$

$u = x - 1$

$\frac{du}{dx} = 1$

$\int \frac{du}{u^{2/3}} = \int u^{-2/3}\,du$

$= 3u^{1/3}$

$\lim_{b\to 1^-} \int_0^b \frac{dx}{(x-1)^{2/3}} + \lim_{b\to 1^+} \int_b^3 \frac{dx}{(x-1)^{2/3}}$

$\lim_{b\to 1^-} 3(x-1)^{1/3}\Big]_0^b \quad \lim_{b\to 1^+} 3(x-1)^{1/3}\Big]_b^3$

$3(b-1)^{1/3} - 3(0-1)^{1/3} + 3(3-1)^{1/3} - 3(1-1)^{1/3}$

$0 + 3 + 3\sqrt[3]{2} - 0$

$\boxed{3 + 3\sqrt[3]{2}}$