Theorems (IVT, EVT, and MVT)

Students should be able to apply and have a geometric understanding of the following:

- Intermediate Value Theorem
- Mean Value Theorem for derivatives
- Extreme Value Theorem

Multiple Choice

1. (calculator not allowed)
   If f is continuous for $a \le x \le b$ and differentiable for $a < x < b$, which of the following could be false?

   (A) $f'(c) = \frac{f(b) - f(a)}{b - a}$ for some c such that $a < c < b$. MVT applies ✓

   $\boxed{\text{(B)}}$ $f'(c) = 0$ for some c such that $a < c < b$. Rolle's may not apply since $f(a) = f(b)$ has not been established

   (C) f has a minimum value on $a \le x \le b$. EVT applies

   (D) f has a maximum value on $a \le x \le b$. EVT applies

   (E) $\int_a^b f(x)\,dx$ exists.

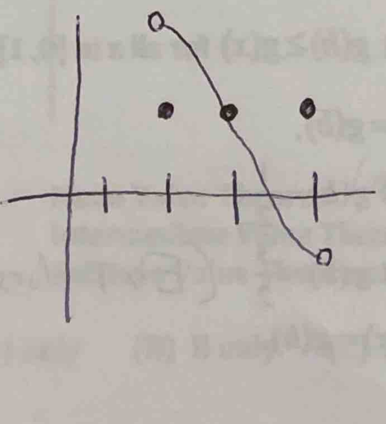
2. (calculator not allowed)
   The function f is defined on the closed interval $[2,4]$ and $f(2) = f(3) = f(4)$. On the open interval $(2,4)$, f is continuous and strictly decreasing. Which of the following statements is true?

   $\boxed{\text{(A)}}$ f attains neither a minimum value nor a maximum value on the closed interval $[2,4]$.

   ~~(B)~~ f attains a minimum value but does not attain a maximum value on the closed interval $[2,4]$.

   ~~(C)~~ f attains a maximum value but does not attain a minimum value on the closed interval $[2,4]$.

   ~~(D)~~ f attains both a minimum value and a maximum value on the closed interval $[2,4]$.



3. (calculator not allowed)

Let f be a function with first derivative defined by $f'(x)=\dfrac{2x^2-5}{x^2}$ for $x>0$. It is known that $f(1)=7$ and $f(5)=11$. What value of x in the open interval $(1,5)$ satisfies the conclusion of the Mean Value Theorem for f on the closed interval $[1,5]$?

$\dfrac{11-7}{5-1}=\dfrac{4}{4}=1$

(A) 1 (B) $\sqrt{\dfrac{5}{2}}$ (C) $\sqrt[3]{10}$ $\boxed{\text{(D) } \sqrt{5}}$

MVT:

$f'(c)=\dfrac{f(b)-f(a)}{b-a}$

$\dfrac{2x^2-5}{x^2}=1$

$2x^2-5=x^2$

$x^2=5$

$x=\pm\sqrt{5}$

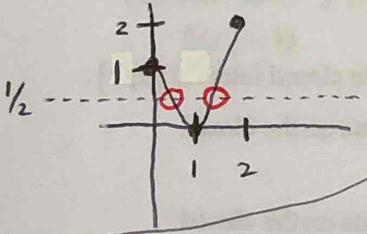
$\boxed{c=\sqrt{5}}$ $\cancel{c=-\sqrt{5}}$

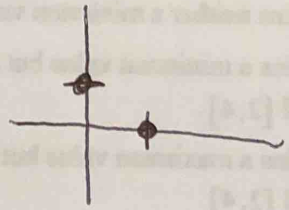
4. (calculator not allowed)

x	0	1	2
$f(x)$	1	k	2

The function f is continuous on the closed interval [0, 2] and has values that are given in the table above. The equation $f(x)=\dfrac{1}{2}$ must have at least two solutions in the interval [0, 2] if $k=$

$\boxed{\text{(A) } 0}$ (B) $\dfrac{1}{2}$ (C) 1 (D) 2 (E) 3





5. (calculator not allowed)
Let g be a continuous function on the closed interval [0, 1]. Let $g(0)=1$ and $g(1)=0$. Which of the following is NOT necessarily true?

(A) There exists a number h in [0, 1] such that $g(h)\geq g(x)$ for all x in [0, 1].

(B) For all a and b in [0, 1], if $a=b$, then $g(a)=g(b)$.

(C) There exists a number h in [0, 1] such that $g(h)=\dfrac{1}{2}$.

$\boxed{\text{(D)}}$ There exists a number h in [0, 1] such that $g(h)=\dfrac{3}{2}$. (EVT does not apply)

(E) For all h in the open interval (0, 1), $\lim\limits_{x\to h} g(x)=g(h)$.

$$MVT: f'(c) = \frac{f(b)-f(a)}{b-a}$$

6. (calculator not allowed)

If $f(x)=\sin\left(\frac{x}{2}\right)$, then there exists a number c in the interval $\frac{\pi}{2}<x<\frac{3\pi}{2}$ that satisfies the conclusion of the Mean Value Theorem. Which of the following could be c?

(A) $\frac{2\pi}{3}$

(B) $\frac{3\pi}{4}$

(C) $\frac{5\pi}{6}$

(D) π

(E) $\frac{3\pi}{2}$

$f(\pi/2) = \sin\left(\frac{\pi/2}{2}\right) = \sin(\pi/4) = \frac{\sqrt{2}}{2}$

$f(3\pi/2) = \sin\left(\frac{3\pi/2}{2}\right) = \sin(3\pi/4) = +\frac{\sqrt{2}}{2}$

$\rightarrow$ slope: $\frac{\sqrt{2}/2 - \sqrt{2}/2}{3\pi/2 - \pi/2} = 0$

$f'(x) = \cos(x/2)\cdot\frac{1}{2}$

$\frac{1}{2}\cos(x/2) = 0$

$\cos(x/2) = 0$

$x/2 = \cos^{-1}(0)$

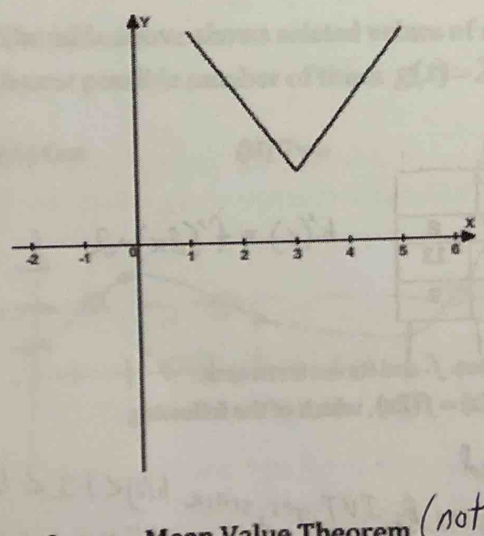
$x/2 = \pi/2,\ 3\pi/2$

$x = \pi,\ 3\pi$

$\boxed{c=\pi}$

7. (calculator not allowed)

Which of the following theorems may be applied to the graph below, $y=|x-3|+b,\ b>0$, over the interval $[2,\ 4]$?



I. Mean Value Theorem (not differentiable)

✓ II. Intermediate Value Theorem

✓ III. Extreme Value Theorem

(A) I only (B) II only (C) III only $\boxed{\text{(D) II and III only}}$ (E) I, II, and III

8. (calculator not allowed) $\quad$ MVT: $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

The function f is defined by $f(x) = 4x^2 - 5x + 1$. The application of the Mean Value Theorem to f on the interval $0 < x < 2$ guarantees the existence of a value c, where $0 < c < 2$ such that $f'(c) =$

$\boxed{\text{(A) 1}}$ (B) 3 (C) 7 (D) 8

$f(0) = 4(0)^2 - 5(0) + 1 = 1$

$f(2) = 4(2)^2 - 5(2) + 1 = 7$

slope: $\dfrac{7-1}{2-0} = \dfrac{6}{2} = 3$

$f'(x) = 8x - 5$

$8x - 5 = 3$

$8x = 8$

$x = 1$

$\boxed{c = 1}$

9. (calculator not allowed)

A function of f is continuous on the closed interval $[2,5]$ with $f(2)=17$ and $f(5)=17$. Which of the following additional conditions guarantees that there is a number c in the open interval $(2,5)$ such that $f'(c)=0$?

(A) No additional conditions are necessary

(B) f has a relative extremum on the open interval $(2,5)$.

$\boxed{\text{(C) } f \text{ is differentiable on the open interval } (2,5).}$ condition needed for Rolle's Theorem

(D) $\int_2^5 f(x)dx$ exists

10. (calculator not allowed)

$h(x)$	3	9	13	
x	0	2	4	8
$f(x)$	3	4	9	13
$f'(x)$	0	1	1	2
$h'(x)$	0	2	4	

$h'(x) = f'(2x) \cdot 2$

The table above gives values of a differentiable function f and its derivatives at selected values of x. If h is the function given by $h(x) = f(2x)$, which of the following statements must be true?

~~(I)~~ h is increasing on $2 < x < 4$. not guaranteed

(II) There exists c, where $0 < c < 4$, such that $h(c) = 12$. By IVT yes, since $h(2) < 12 < h(4)$

(III) There exists c, where $0 < c < 2$, such that $h'(c) = 3$. By IVT yes, since $h'(2) < 3 < h'(4)$

(A) II only

(B) I and III only

$\boxed{\text{(C)}}$ II and III only

(D) I, II, and III

11. (calculator allowed)

The function f is continuous for $-2 \le x \le 1$ and differentiable for $-2 < x < 1$. If $f(-2) = -5$ and $f(1) = 4$, which of the following statements could be false?

(A) There exists c, where $-2<c<1$, such that $f(c)=0$. IVT

(B) There exists c, where $-2<c<1$, such that $f'(c)=0$. Rolle's theorem does not apply $f(a) \neq f(b)$

(C) There exists c, where $-2<c<1$, such that $f(c)=3$. ✓ (IVT)

(D) There exists c, where $-2<c<1$, such that $f'(c)=3$. ✓ (MVT)

(E) There exists c, where $-2 \le c \le 1$ such that $f(c) \ge f(x)$ ✓ IVT
for all x on the closed interval $-2 \le x \le 1$.

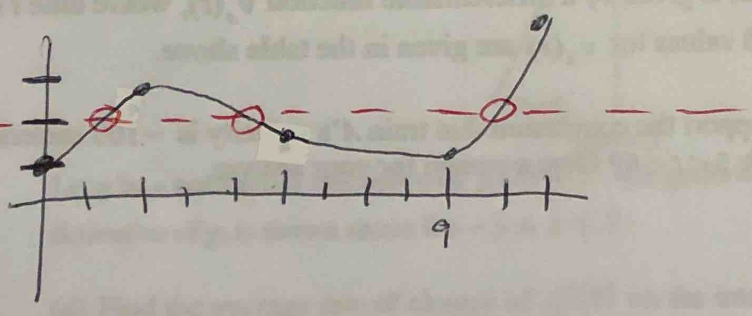
MVT: slope: $\frac{4 - -5}{1 - -2} = \frac{9}{3} = 3$

12. (calculator not allowed)

x	0	2	5	9	11
$g(x)$	1	2.8	1.7	1	3.4

The table above shows selcted values of a continuous function g. For $0 \le x \le 11$, what is the fewest possible number of times $g(x)=2$?

(A) One (B) Two (C) Three (D) Four



Free Response

13. (calculator not allowed)

t (minutes)	0	1	2	3	4	5	6
$C(t)$ (ounces)	0	5.3	8.8	11.2	12.8	13.8	14.5

Hot water is dripping through a coffeemaker, filling a large cup with coffee. The amount of coffee in the cup at time t, $0 \le t \le 6$, is given by a differentiable function C, where t is measured in minutes. Selected values of $C(t)$, measured in ounces, are given in the table above.

Is there a time t, $2 \le t \le 4$, at which $C'(t) = 2$? Justify your answer.

yes, by MVT, since $\frac{f(4)-f(2)}{4-2} = \frac{12.8-8.8}{4-2} = \frac{4}{2} = 2$

then $C'(t) = 2$ at some t-value

14. (calculator not allowed)

t (minutes)	0	2	5	8	12
$v_A(t)$ (meters/minute)	0	100	40	-120	-150

Train A runs back and forth on an east-west section of railroad track. Train A's velocity, measured in meters per minute, is given by a differentiable function $v_A(t)$, where time t is measured in minutes. Selected values for $v_A(t)$ are given in the table above.

(b) Do the data in the table support the conclusion that train A's velocity is -100 meters per minute at some time t with $5 < t < 8$? Give a reason for your answer.

By IVT,