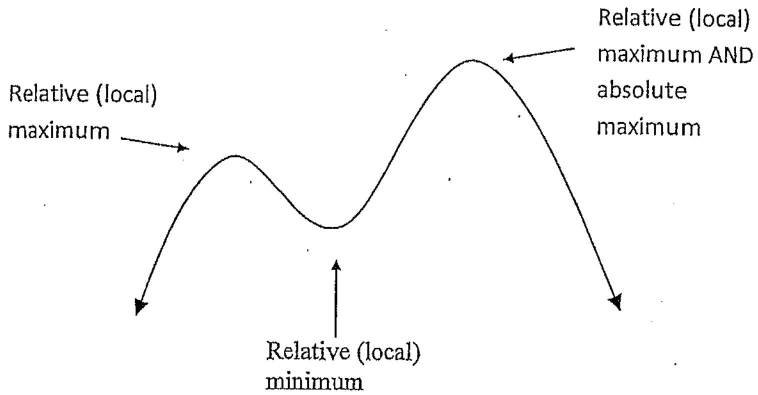
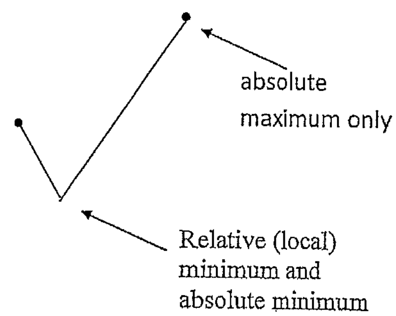
Calculus Ch. 3.1 Notes: Extrema and Critical Points

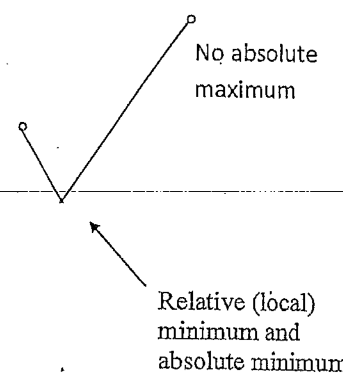
Extrema : maximums and minimums



Closed interval



Open Interval



Relative (local) extrema: any "hills and valleys" of graph

Absolute (global) extrema: highest or lowest points on the entire graph

*holes and $\pm\infty$ can not be considered as absolute extrema.

(EVT)

Extreme Value Theorem: If a function is continuous on a closed interval, then it has **both** an (absolute) minimum and an (absolute) maximum on that interval.

Fermat's Theorem: If a function is continuous on a closed interval, then the absolute extreme will **either** be at the **a)** critical numbers or **b)** at an endpoint.

Critical numbers (values) : x-values in the domain of a function where the derivative of a function is either 0 or undefined.

*Relative extrema ONLY occur at critical numbers, but not all critical numbers are where relative extrema occur.

*Maximum and minimum values refer to the **y-values** of the point.

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Steps: ✱ Confirm continuous function on closed interval

1. Find critical points
   a. Set $f'(x) = 0$
   b. Find where $f'(x)$ is undefined (Set denominator of $f'(x) = 0$)

2. Plug all critical points and endpoints into f(x)

3. Compare y-values to determine absolute maximum(s) and absolute minimum(s)

Find all critical numbers for each. What are the values of the absolute extrema?

Example 1: $f(x) = 3x^4 - 4x^3$ on [0, 2]

Example 2: $f(x) = (x-1)^{\frac{2}{3}}$ on [-1, 0]

Example 3: $f(x) = \frac{4}{3}x\sqrt{3-x}$ on [0, 3]

Chapter 3 Curve Sketching 3.1 EVT Classwork Problems

Finding Extrema on a Closed Interval In Exercises 17–36, find the absolute extrema of the function on the closed interval.

19. $g(x) = 2x^2 - 8x,\ [0, 6]$

21. $f(x) = x^3 - \frac{3}{2}x^2,\ [-1, 2]$

23. $y = 3x^{2/3} - 2x,\ [-1, 1]$

24. $g(x) = \sqrt[3]{x},\ [-8, 8]$

26. $f(x) = \frac{2x}{x^2 + 1},\ [-2, 2]$

28. $h(t) = \frac{t}{t + 3},\ [-1, 6]$

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If the Mean Value Theorem cannot be applied, explain why not.

39. $f(x) = x^3 + 2x, \quad [-1, 1]$

40. $f(x) = x^4 - 8x, \quad [0, 2]$

41. $f(x) = x^{2/3}, \quad [0, 1]$

42. $f(x) = \dfrac{x+1}{x}, \quad [-1, 2]$

35. **Mean Value Theorem** Consider the graph of the function $f(x) = -x^2 + 5$ (see figure on next page).

(a) Find the equation of the secant line joining the points $(-1, 4)$ and $(2, 1)$.

(b) Use the Mean Value Theorem to determine a point c in the interval $(-1, 2)$ such that the tangent line at c is parallel to the secant line.

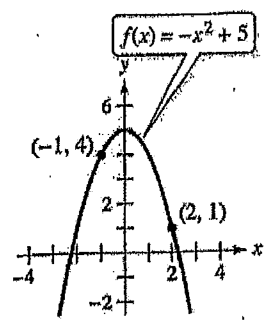

Figure for 35

Notes: Mean Value Theorem (MVT) and Rolle's Theorem

Mean Value Theorem (MVT): If a function, f(x), is **continuous** on [a, b] and **differentiable** on (a, b), then there must be at least one point, c in (a, b) where the slope of the tangent (derivative) is equal to the slope of the secant. $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

*In other words, set the derivative equal to the slope between endpoints ($m_{avg.}$) *

MVT Steps:

1. Check Continuity (no breaks between endpoints)
   a. Does f(x) have variables in the denominator? (V.A. or holes)
   b. If so, then look to see if the x-value lies in the **closed** interval [a, b]
   c. If the x lies between the interval, then function is not continuous on the interval, MVT fails
2. Check Differentiability (smooth curve between endpoints)
   a. Does f'(x) have variables in the denominator? (sharp points, slope undefined)
   b. If yes, then look to see if the x-value lies in the **open** interval (a, b)
   c. If the x lies between the interval, then function is not differentiable on the interval, MVT fails

Note, all polynomials are continuous and differentiable everywhere

3. Find $m_{avg.}$ (This is the slope between your endpoints, slope of secant line)
4. Set $f'(x) = m_{avg}$ and solve for x

Example 1: Determine if the mean value theorem can be applied to $f(x) = 2x^3 + x + 4$ on the interval [-2, 1]. If so, find the value of c based on the theorem.

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Rolle's Theorem: If a function, f(x) is continuous on [a, b], differentiable on (a, b), and f(a) = f(b), then there must be at least one point on the function where the slope of the tangent (derivative) is 0.

*In other words, if the endpoints have the same y-values, then we can guarantee a relative maximum or relative minimum somewhere between the endpoints

*Rolle's Theorem is just a specific case of the Mean Value Theorem

Rolle's Theorem Steps:

1. Check Continuity (no breaks between endpoints)
2. Check Differentiability (smooth curve between endpoints)

Note, all polynomials are continuous and differentiable everywhere

3. Test endpoints. Does f(a) = f(b)? If not, then Rolle's fails / does not apply
4. If yes, then set f'(x) = 0 and solve for x

Example 2: Determine if Rolle's theorem can be applied to $f(x) = x^2 - 3x + 2$ on the interval [1, 2]. If so, find the value of c such that f '(c) = 0.

Example 3: Determine if Rolle's theorem can be applied for $f(x) = 3 - |x - 3|$ on [0, 6]

Chapter 3 Curve Sketching 3.2 Rolle's Classwork Problems

Using Rolle's Theorem In Exercises 9–22, determine whether Rolle's Theorem can be applied to f on the closed interval $[a, b]$. If Rolle's Theorem can be applied, find all values of c in the open interval (a, b) such that $f'(c) = 0$. If Rolle's Theorem cannot be applied, explain why not.

9. $f(x) = -x^2 + 3x, \quad [0, 3]$

10. $f(x) = x^2 - 8x + 5, \quad [2, 6]$

13. $f(x) = x^{2/3} - 1, \quad [-8, 8]$

15. $f(x) = \dfrac{x^2 - 2x - 3}{x + 2}, \quad [-1, 3]$

16. $f(x) = \dfrac{x^2 - 1}{x}, \quad [-1, 1]$

Calculus Ch. 3.3 **Notes: First Derivative Test**

What does the derivative represent? ______________________

When the function is **increasing**, what is common about the derivatives at those points? __________

When the function is **decreasing**, what is common about the derivatives at those points? __________

When $f'(x) > 0$, ______________________

When $f'(x) < 0$, ______________________

When $f'(x) = 0$, ______________________

First Derivative Test Steps (Finds inc/dec and relative max/min)

1. Find f '(x), set equal to zero
   a. Find critical points from BOTH numerator and denominator
   b. Remember, critical points also exist where function is not differentiable (sharp point)
2. Put all critical points on sign line
3. Test intervals
   a. Plug values into f '(x) to determine slope
      i. Positive (+) means increasing slope
      ii. Negative(–) means decreasing slope
4. Write Because Statements
   a. f(x) increasing in interval (a,b) b/c $f'(x) > 0$
   b. f(x) decreasing in interval (a,b) b/c $f'(x) < 0$
   c. Relative max at (**a, f(a)**) b/c f '(x) changes from + to –
   d. Relative min at (**a, f(a)**) b/c f '(x) changes from – to +

Example 1: Determine the intervals at which the function $f(x) = \frac{1}{3}x^3 + \frac{5}{2}x^2 - 6x - 3$ is increasing and decreasing.. Locate the relative extrema.

First Derivative Test Steps (Finds inc/dec and relative max/min).

1. Find f '(x), set equal to zero
   a. Find critical points from BOTH numerator and denominator
   b. Remember, critical points also exist where function is not differentiable (sharp point)
2. Put all critical points on sign line
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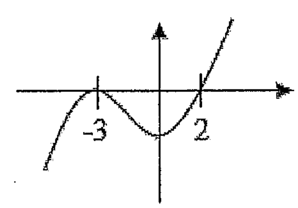
   a. f(x) increasing in interval (a,b) b/c f '(x) > 0
   b. f(x) decreasing in interval (a,b) b/c f '(x) < 0
   c. Relative max at (**a, f(a)**) b/c f '(x) changes from + to –
   d. Relative min at (**a, f(a)**) b/c f '(x) changes from – to +

Example 2: Determine the intervals at which the function $f(x)=\dfrac{5x+2}{x-3}$ is increasing and decreasing. Locate the relative extrema.

Example 3: Make a first derivative sign line for the following graph of f '(x):



AB Calculus Ch. 3.3 Select HW Problems

Applying the First Derivative Test In Exercises 17–40, (a) find the critical numbers of f (if any), (b) find the open interval(s) on which the function is increasing or decreasing, (c) apply the First Derivative Test to identify all relative extrema, and (d) use a graphing utility to confirm your results.

19. $f(x) = -2x^2 + 4x + 3$

21. $f(x) = 2x^3 + 3x^2 - 12x$

25. $f(x) = \dfrac{x^5 - 5x}{5}$

27. $f(x) = x^{1/3} + 1$

29. $f(x) = (x + 2)^{2/3}$

33. $f(x) = 2x + \dfrac{1}{x}$

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AB Calculus Ch. 3.4 Select HW Problems

Finding Points of Inflection **In Exercises 15–30, find the points of inflection and discuss the concavity of the graph of the function.**

15. $f(x) = x^3 - 6x^2 + 12x$

17. $f(x) = \frac{1}{2}x^4 + 2x^3$

19. $f(x) = x(x - 4)^3$

21. $f(x) = x\sqrt{x + 3}$

23. $f(x) = \dfrac{4}{x^2 + 1}$

Calculus Ch. 3.4 **Notes: "Concavity Test" and the 2nd Derivative Test**

Are both of these functions increasing? ________ What do we know about their derivatives? ________

1) If f "(x) is > 0, then f '(x) is increasing and f(x) is concave up.
2) If f "(x) is < 0, then f '(x) is decreasing and f(x) is concave down.
3) A Point of Inflection (POI) occurs whenever f "(x) <u>changes sign</u>. (f(x) changes concavity)

<u>"Concavity Test" Steps</u> (Finding interval Concave Up/Down and POI)

1. Find $f''(x)$, set equal to zero
   a. Find critical points from BOTH numerator <u>and</u> denominator
2. Put all critical points on sign line
3. Test intervals
   a. Plug values into $f''(x)$ to determine concavity
      i. Positive (+) means concave up
      ii. Negative(–) means concave down
4. Write Because Statements
   a. Concave up in interval (a,b) b/c f''(x) > 0
   b. Concave down in interval (a,b) b/c f''(x) < 0
5. Point of Inflection at (**a, f(a)**) b/c $f''(x)$ changes signs

*Note: POI may exist on graph where $f''(x)$ does not exist (sharp point). POI exists as long as graph is continuous and $f''(x)$ changes concavity (change of signs)

Example 1: Find the points of inflection if $f(x) = -2x^5 + \frac{5}{3}x^3$

The 2^{nd} Derivative Test

The 2^{nd} derivative test is a test for relative extrema (max/min) and NOT for Point of Inflection

*The 2^{nd} derivative test achieves the same as the 1^{st} derivative test.

1) If you plug a critical number from f'(x) into $f''(x)$ and if $f''(x) > 0$, then that is the x-value of the relative **minimum**

2) If you plug a critical number from f'(x) into $f''(x)$ and if $f''(x) < 0$, then that is the x-value of the relative **maximum**

3) If you plug a critical number from f'(x) into $f''(x)$ and if $f''(x) = 0$, then the test is inconclusive. We need the first derivative test to determine if critical number is a relative extrema.

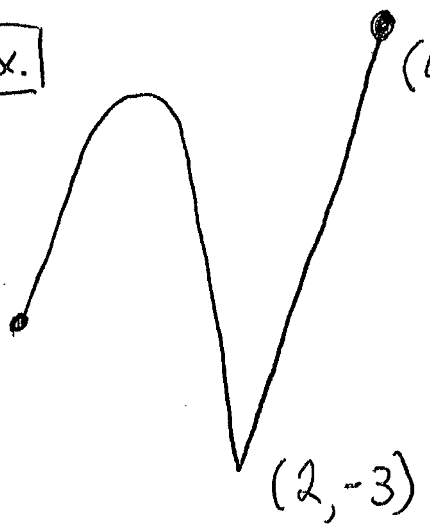
2^{nd} Derivative Test Steps (Test for Relative Extrema, **NOT** Point of Inflection)

1. Find f'(x) , set equal to zero
   a. Find critical points. Set numerator and denominator of f '(x) = 0. (These are candidates for relative max/min)
2. Find $f''(x)$
3. Plug the critical points (from step #1) into $f''(x)$.
   a. If result is positive value, then f''(x) > 0, concave up, and therefore relative minimum exists at x-value
   b. If result is negative value, then f''(x) < 0, concave down, and therefore relative maximum exists at x-value
   c. f result is zero, then since f''(x) = 0, then this test is inconclusive. We cannot determine whether relative extrema exists. (Use First Derivative Test)

Example 2: Find the relative extrema of $f(x) = x^3 - 4x^2 - 3x$

Extreme Value Theorem (EVT) Ch. 3.1

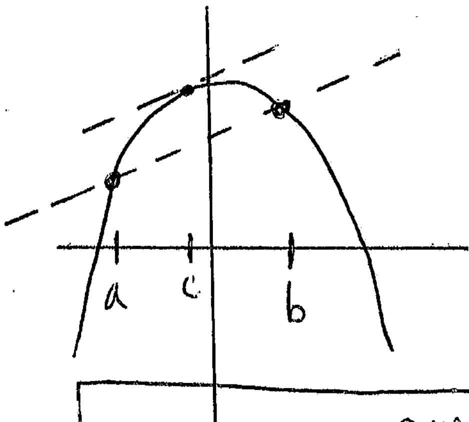
Ex.



Purpose: Find Abs max/min on closed interval

* $f(x)$ continuous $[a, b]$
* find critical points
  - a) set $f'(x) = 0$
  - b) set denominator of $f'(x) = 0$
* test critical points and endpoints into $f(x)$ to find absolute max/min
* Abs max is 7 at $x = 4$
  Abs min is -3 at $x = 2$

3.2a Mean Value Theorem (MVT)



purpose: find the location on the curve where the guaranteed slope occurs.

Conditions:

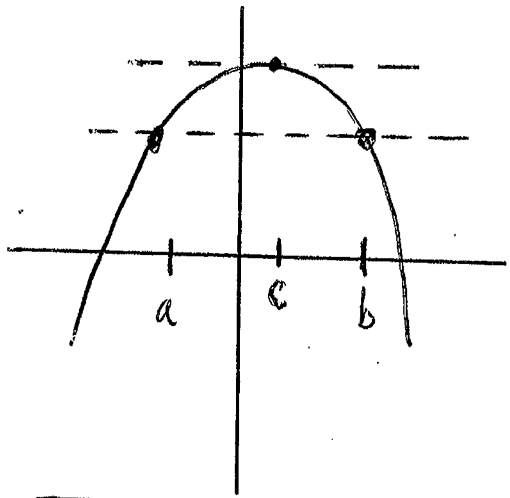
* $f(x)$ continuous on $[a,b]$ (no VA, no holes on interval)
* $f(x)$ differentiable on (a,b) (no sharp turns, no slope undefined on (a,b))

MVT: $f'(c) = \frac{f(b)-f(a)}{b-a}$

Steps:

1) find slope between endpoints $\left[\frac{f(b)-f(a)}{b-a}\right]$
2) find $f'(x)$
3) set $f'(x)$ = slope value, solve for x (c-value)
4) Keep the c-values in interval (a,b)

3.2b Rolle's Theorem



Rolle's Theorem: $f'(c) = 0$

Steps:

1) confirm endpts have same y-values
2) find $f'(x)$
3) set numerator of $f'(x) = 0$, solve for x (c-value)
4) Keep the c-values in interval (a, b)

Purpose: Find the location on the curve where the guaranteed slope of 0 occurs.

Conditions:

* $f(x)$ continuous $[a, b]$
  (no breaks, no vertical asymptote, no holes)
* $f(x)$ differentiable (a, b)
  (no sharp turns, no location with undefined slope)
* $f(a) = f(b)$
  (endpoints with same y-value)

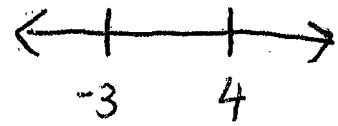
Ch. 3.3 1st Derivative Test

Purpose: Use $f'(x)$ to determine slope behavior of graph and find relative max, relative mins of graph

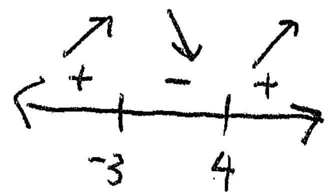
1) Find critical points
   a) find $f'(x)$
   b) set numerator of $f'(x) = 0$
   c) set denominator of $f'(x) = 0$

Ex:

2) Place critical values on $f'(x)$ sign line



3) Test intervals, plug in x-values into $f'(x)$



   a) Rel. max at $(-3, \_)$ b/c $f'(x)$ changes from + to -

   b) Rel. min at $(4, \_)$ b/c $f'(x)$ changes from - to +

   c) $f(x)$ increasing $(-\infty, 3), (4, \infty)$ b/c $f'(x) > 0$

   d) $f(x)$ decreasing $(-3, 4)$ b/c $f'(x) < 0$

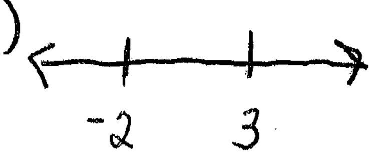
Ch. 3.4 Concavity Test:

Purpose: Use $f''(x)$ to determine concavity behavior of graph and find Points of Inflection (POI)

1) Find critical points
   a) find $f''(x)$
   b) set numerator, denominator of $f''(x) = 0$

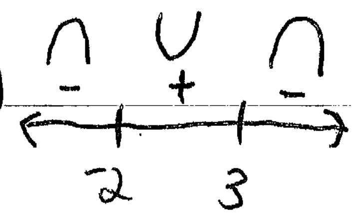
Example

2) Place critical points on $f''(x)$ sign line. $f''(x)$



3) Test intervals, plug x-values into $f''(x)$

$f''(x)$



   a) POI at $(-2, \_)$ and $(3, \_)$ b/c $f''(x)$ change signs.
   b) $f(x)$ concave up $(-2, 3)$ b/c $f''(x) > 0$
   c) $f(x)$ concave down $(-\infty, -2), (3, \infty)$ b/c $f''(x) < 0$

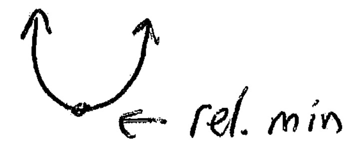
Ch. 3.4 2^{nd} derivative test

Purpose: Use $f''(x)$ to determine relative max/mins of graph

Steps:

example

1) find $f'(x)$ and critical points (set $f'(x) = 0$) $x = a, b$
2) find $f''(x)$
3) plug in critical points from $f'(x)$ <u>into</u> $f''(x)$
4) If $f''(a) > 0$, concave up, Rel. min at $x = a$



5) If $f''(b) < 0$, concave down, Rel. max at $x = b$