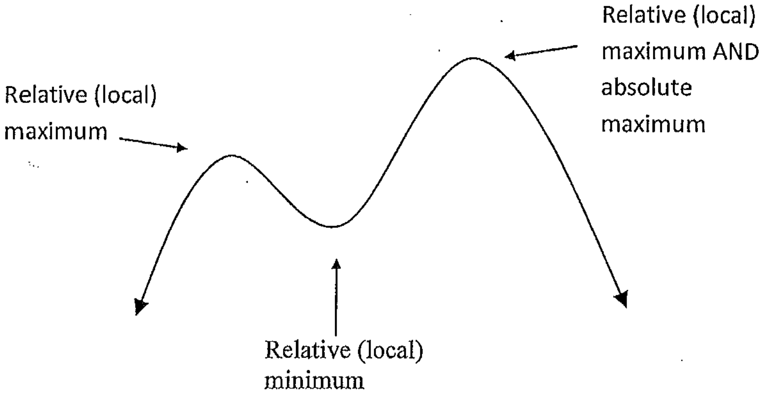
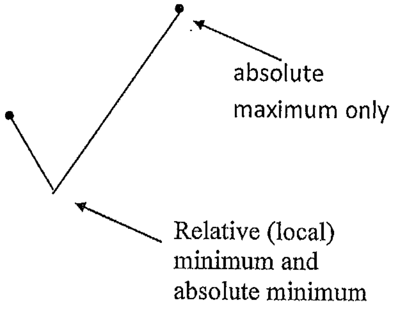
Calculus Ch. 3.1 Notes: Extrema and Critical Points

Key

Extrema : maximums and minimums



Closed interval

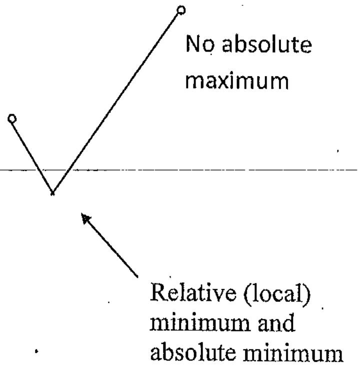


Relative (local) extrema: any "hills and valleys" of graph

Absolute (global) extrema: highest or lowest points on the entire graph

*holes and $\pm\infty$ can **not** be considered as absolute extrema.

Open Interval



Extreme Value Theorem: If a function is continuous on a closed interval, then it has **both** an (absolute) minimum and an (absolute) maximum on that interval.

Fermat's Theorem: If a function is continuous on a closed interval, then the absolute extreme will **either** be at the **a)** critical numbers or **b)** at an endpoint.

Critical numbers (values) : x-values in the domain of a function where the derivative of a function is either 0 or undefined.

*Relative extrema ONLY occur at critical numbers, but not all critical numbers are where relative extrema occur.

*Maximum and minimum values refer to the **y-values** of the point.

Steps:

1. Find critical points
   a. Set f'(x) = 0
   b. Find where f'(x) is undefined (Set denominator of f'(x) = 0)

2. Plug all critical points and endpoints into f(x)

3. Compare y-values to determine absolute maximum(s) and absolute minimum(s)

Find all critical numbers for each. What are the values of the absolute extrema?

Example 1: $f(x) = 3x^4 - 4x^3$ on [0, 2]

$f'(x) = 12x^3 - 12x^2$

$0 = 12x(x-1)$

$x = 0, 1$

$f(0) = 0$

$f(1) = -1$ (min)

$f(2) = 16$ (max)

Example 2: $f(x) = (x-1)^{\frac{2}{3}}$ on [-1, 0] $f(x)$ continuous on $[-1, 0]$

$f'(x) = \frac{2}{3}(x-1)^{-1/3}(1)$

$f'(x) = \frac{2}{3(x-1)^{1/3}}$

$x = 1$ (critical pt.)

$f(-1) = \sqrt[3]{4}$ (Abs. max is $\sqrt[3]{4}$ at $x = -1$)

$f(0) = 1$ (Abs. min is 1 at $x = 0$) $f(x)$ continuous on $[0, 3]$

Example 3: $f(x) = \frac{4}{3}x\sqrt{3-x}$ on [0, 3] $f(x) = \frac{4}{3}x(3-x)^{1/2}$

$f'(x) = \frac{4}{3}(3-x)^{1/2} + \frac{4}{3}x \cdot \frac{1}{2}(3-x)^{-1/2}(-1)$

$0 = \frac{4\sqrt{3-x}}{3} - \frac{2x}{3\sqrt{3-x}}$

$= \frac{4(3-x) - 2x}{3\sqrt{3-x}} = \frac{12 - 4x - 2x}{3\sqrt{3-x}}$

$f'(x) = \frac{12 - 6x}{3\sqrt{3-x}}$

$12 - 6x = 0$ $6x = 12$ $\underline{x = 2}$

$3\sqrt{3-x} = 0$ $\underline{x = 3}$

$f(0) = 0$] (Abs. min)

$f(3) = 0$

$f(2) = \frac{4}{3}(2)\sqrt{1} = \frac{8}{3}$ (Abs. max)

Key

Chapter 3 Curve Sketching 3.1 EVT Classwork Problems

Finding Extrema on a Closed Interval In Exercises 17–36, find the absolute extrema of the function on the closed interval.

19. $g(x) = 2x^2 - 8x,\ [0, 6]$

$g(x)$ continuous $[0, 6]$

$g'(x) = 4x - 8$

$0 = 4x - 8$

$8 = 4x$

$x = 2$

$g(0) = 0$

$g(2) = -8$

$g(6) = 24$

Abs max is 24 at $x = 6$

Abs min is -8 at $x = 2$

21. $f(x) = x^3 - \frac{3}{2}x^2,\ [-1, 2]$

$f(x)$ continuous $[-1, 2]$

$f'(x) = 3x^2 - 3x$

$0 = 3x(x - 1)$

$x = 0, 1$

$f(-1) = -5/2$

$f(0) = 0$

$f(1) = -1/2$

$f(2) = 2$

Abs max is 2 at $x = 2$

Abs min is $-5/2$ at $x = -1$

23. $y = 3x^{2/3} - 2x,\ [-1, 1]$

$f(x)$ continuous on $[-1, 1]$

*set denom = 0

$x = 0$

$y'(x) = 3 \cdot \frac{2}{3}x^{-1/3} - 2$

$y'(x) = \frac{2}{x^{1/3}} - 2$

$2 = \frac{2}{x^{1/3}}$

$2x^{1/3} = 2$

$x^{1/3} = 1$

$x = 1$

$f(0) = 0$

$f(-1) = 5$

$f(1) = 1$

Abs max is 5 at $x = -1$

Abs min is 0 at $x = 0$

24. $g(x) = \sqrt[3]{x},\ [-8, 8]$

$g(x)$ continuous $[-8, 8]$

$g(x) = x^{1/3}$

$g'(x) = \frac{1}{3}x^{-2/3}$

$g'(x) = \frac{1}{3x^{2/3}}$

$x = 0$

$g(-8) = \sqrt[3]{-8} = -2$

$g(0) = 0$

$g(8) = \sqrt[3]{8} = 2$

Abs max is 2 at $x = 8$

Abs min is -2 at $x = -8$

26. $f(x) = \frac{2x}{x^2 + 1},\ [-2, 2]$

$f(x)$ continuous $[-2, 2]$

$$f'(x) = \frac{\overbrace{2}^{f'}\overbrace{(x^2+1)}^{g} - \overbrace{2x}^{f}\overbrace{(2x)}^{g'}}{\underbrace{(x^2+1)^2}_{g^2}} = \frac{2x^2 + 2 - 4x^2}{(x^2+1)^2}$$

$f'(x) = \frac{2 - 2x^2}{(x^2+1)^2}$

$2 - 2x^2 = 0$

$2 = 2x^2$

$x^2 = 1$

$x = \pm 1$

$f(-2) = -4/5$ $f(1) = 1$

$f(-1) = -1$ $f(2) = 4/5$

Abs max is 1 at $x = 1$

Abs min is -1 at $x = -1$

28. $h(t) = \frac{t}{t + 3},\ [-1, 6]$

VA: $t = -3$

$h(t)$ continuous $[-1, 6]$

$h'(t) = \frac{(1)(t+3) - (t)(1)}{(t+3)^2}$

$h'(t) = \frac{t + 3 - t}{(t+3)^2} = \frac{3}{(t+3)^2}$

No critical point

$0 \neq \frac{+3}{(t+3)^2}$

$h(-1) = -1/2$

$h(6) = 2/3$

Abs max is $2/3$ at $t = 6$

Abs min is $-1/2$ at $t = -1$

④

Key

Chapter 3 Curve Sketching 3.2 MVT and Rolle's Classwork Problems

If the Mean Value Theorem cannot be applied, explain why not. $f'(c)=\frac{f(b)-f(a)}{b-a}$

39. $f(x) = x^3 + 2x, \quad [-1, 1]$

$f(x)$ continuous $[-1,1]$, differentiable $(-1,1)$

$f(-1)=-3$

$f(1)=3$

slope: $\frac{3--3}{1--1}=\frac{6}{2}$

slope $=3$

$f'(x)=3x^2+2$

$3=3x^2+2$

$1=3x^2$

$\frac{1}{3}=x^2$

$x=\pm\sqrt{1/3}$

$\boxed{c=\sqrt{\frac{1}{3}}, -\sqrt{\frac{1}{3}}}$

40. $f(x) = x^4 - 8x, \quad [0, 2]$

$f(x)$ continuous $[0,2]$, differentiable $(0,2)$

$f(0)=0$

$f(2)=0$

slope: $\frac{0-0}{2-0}=0$

$f'(x)=4x^3-8$

$0=4x^3-8$

$8=4x^3$

$2=x^3$

$x=\sqrt[3]{2}$

$\boxed{c=\sqrt[3]{2}}$

41. $f(x) = x^{2/3}, \quad [0, 1]$

$f(x)$ continuous $[0,1]$, differentiable $(0,1)$

$f(0)=0$

$f(1)=1$

slope:

$m=\frac{1-0}{1-0}=1$

$f'(x)=\frac{2}{3}x^{-1/3}$

$1=\frac{2}{3x^{1/3}}$

$3x^{1/3}=2$

$x^{1/3}=\frac{2}{3}$

$\left(x^{1/3}\right)^3=\left(\frac{2}{3}\right)^3$

$x=\frac{8}{27}$

$\boxed{c=\frac{8}{27}}$

42. $f(x) = \frac{x+1}{x}, \quad [-1, 2]$

V.A. at $x=0$

$f(x)$ not continuous $[-1,2]$

MVT does not apply.

35. Mean Value Theorem Consider the graph of the function $f(x) = -x^2 + 5$ (see figure on next page).

(a) Find the equation of the secant line joining the points $(-1, 4)$ and $(2, 1)$.

(b) Use the Mean Value Theorem to determine a point c in the interval $(-1, 2)$ such that the tangent line at c is parallel to the secant line. $f(x)$ continuous $[-1,2]$, differentiable $(-1,2)$

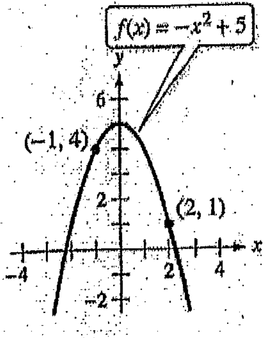


Figure for 35

$f(-1)=4$

$f(2)=1$

slope: $m=\frac{1-4}{2-(-1)}$

$m=\frac{-3}{3}=-1$

$f(x)=-x^2+5$

$f'(x)=-2x$

$-1=-2x$

$\frac{1}{2}=x$

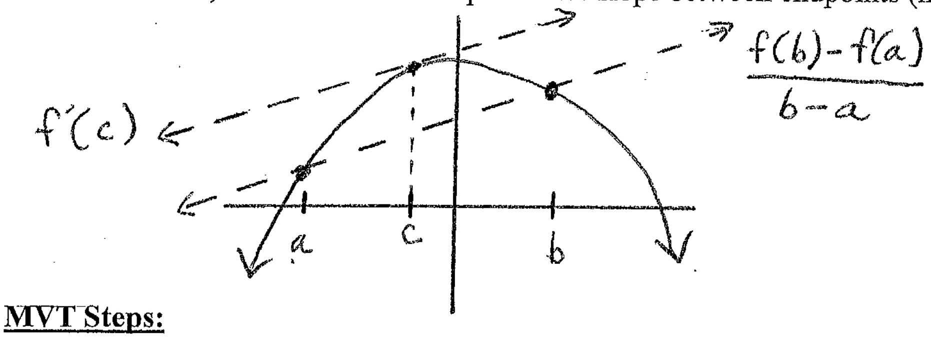
$\boxed{c=1/2}$

Calculus Ch. 3.2

Notes: Mean Value Theorem (MVT) and Rolle's Theorem

Mean Value Theorem (MVT): If a function, f(x), is **continuous** on [a, b] and **differentiable** on (a, b), then there must be at least one point, c in (a, b) where the slope of the tangent (derivative) is equal to the slope of the secant. $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

*In other words, set the derivative equal to the slope between endpoints ($m_{avg.}$) *



MVT Steps:

1. Check Continuity (no breaks between endpoints)
    a. Does f(x) have variables in the denominator? (V.A. or holes)
    b. If so, then look to see if the x-value lies in the **closed** interval [a, b]
    c. If the x lies between the interval, then function is not continuous on the interval, MVT fails
2. Check Differentiability (smooth curve between endpoints)
    a. Does f'(x) have variables in the denominator? (sharp points, slope undefined)
    b. If yes, then look to see if the x-value lies in the **open** interval (a, b)
    c. If the x lies between the interval, then function is not differentiable on the interval, MVT fails

Note, all polynomials are continuous and differentiable everywhere

3. Find $m_{avg.}$ (This is the slope between your endpoints, slope of secant line)
4. Set $f'(x) = m_{avg}$ and solve for x

Example 1: Determine if the mean value theorem can be applied to $f(x) = 2x^3 + x + 4$ on the interval [-2, 1]. If so, find the value of c based on the theorem.

f(x) is continuous, differentiable on [-2, 1]

$f(-2) = -14$

$f(1) = 7$

$m_{Avg} = \dfrac{-14-7}{-2-1}$

$m_{Avg} = \dfrac{-21}{-3} = 7$

$f'(x) = 6x^2 + 1$

set $f'(x) = m_{Avg}$

$6x^2 + 1 = 7$

$6x^2 = +6$

$x = \pm 1$

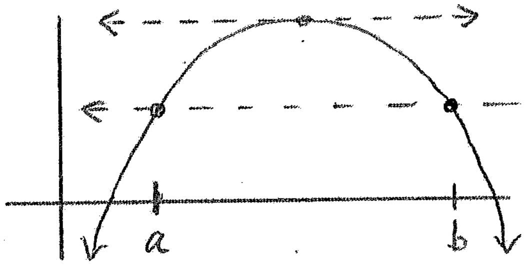
* we are only looking for c-value in the open interval (-2, 1)

~~c = 1~~, $\boxed{c = -1}$

Rolle's Theorem: If a function, f(x) is continuous on [a, b], differentiable on (a, b), and f(a) = f(b), then there must be at least one point on the function where the slope of the tangent (derivative) is 0.

*In other words, if the endpoints have the same y-values, then we can guarantee a relative maximum or relative minimum somewhere between the endpoints

*Rolle's Theorem is just a specific case of the Mean Value Theorem



$$\frac{f(b)-f(a)}{b-a} = 0$$

Rolle's Theorem Steps:

1. Check Continuity (no breaks between endpoints)
2. Check Differentiability (smooth curve between endpoints)

Note, all polynomials are continuous and differentiable everywhere

3. Test endpoints. Does f(a) = f(b)? If not ,then Rolle's fails / does not apply
4. If yes, then set f'(x) = 0 and solve for x

Example 2: Determine if Rolle's theorem can be applied to $f(x) = x^2 - 3x + 2$ on the interval [1, 2]. If so, find the value of c such that f '(c) = 0.

f(x) is continuous and differentiable on [1, 2]

$f(1) = 0$

$f(2) = 0$

$m_{Avg} = \frac{0-0}{2-1} = 0$

set $f'(c) = m_{Avg}$

$f'(x) = 2x - 3$

$2x - 3 = 0$

$x = 3/2$

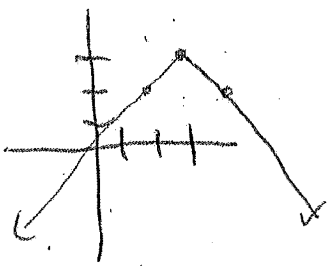
$\boxed{c = 3/2}$

Example 3: Determine if Rolle's theorem can be applied for f(x) = 3 – |x – 3| on [0, 6]

V(3, 3)

f(x) not differentiable at x=3, not differentiable on interval [0, 6]

Rolle's theorem does not apply.



Key

Chapter 3 Curve Sketching 3.2 Rolle's Classwork Problems

Using Rolle's Theorem In Exercises 9–22, determine whether Rolle's Theorem can be applied to f on the closed interval $[a, b]$. If Rolle's Theorem can be applied, find all values of c in the open interval (a, b) such that $f'(c) = 0$. If Rolle's Theorem cannot be applied, explain why not.

9. $f(x) = -x^2 + 3x, \quad [0, 3]$

f(x) continuous $[0,3]$, differentiable $(0,3)$

$f(0)=0$

$f(3)=0$

$f'(x)=-2x+3$

$0=-2x+3$

$2x=3$

$x=3/2$

$\boxed{c=\frac{3}{2}}$

10. $f(x) = x^2 - 8x + 5, \quad [2, 6]$

f(x) continuous $[2,6]$, differentiable $(2,6)$

$f(2)=2^2-16+5=-7$

$f(6)=6^2-48=-7$

$m=\frac{-7-(-7)}{6-2}=\frac{0}{4}=\underline{0}$

$\underline{f'(x)}=2x-8$

$0=2x-8$

$8=2x$

$\frac{8}{2}=x$

$\boxed{c=4}$

13. $f(x) = x^{2/3} - 1, \quad [-8, 8]$

f(x) continuous $[-8,8]$ f(x) $\boxed{\text{not}}$ differentiable $(-8,8)$

$f(-8)=3$

$f(8)=3$

$f'(x)=\frac{2}{3}x^{-1/3}$

$0=\frac{2}{3x^{1/3}}$

$2\neq0$

Rolle's theorem does not apply, f(x) not differentiable at $x=0$

15. $f(x) = \frac{x^2 - 2x - 3}{x + 2}, \quad [-1, 3]$

f(x) continuous $[-1,3]$, differentiable $(-1,3)$

$f(-1)=0$

$f(3)=0$

$$f'(x)=\frac{\overbrace{(2x-2)}^{f'}\overbrace{(x+2)}^{g}-\overbrace{(x^2-2x-3)}^{f}\overbrace{(1)}^{g'}}{\underbrace{(x+2)^2}_{g^2}}$$

$$f'(x)=\frac{2x^2+4x-2x-4-x^2+2x+3}{(x+2)^2}$$

$$f'(x)=\frac{x^2+4x-1}{(x+2)^2}$$

$x^2+4x-1=0$

$\frac{-4\pm\sqrt{4^2-4(1)(-1)}}{2(1)}$

$x=-2\pm\sqrt{5}$

$\boxed{c=-2+\sqrt{5}}$

16. $f(x) = \frac{x^2 - 1}{x}, \quad [-1, 1]$

V.A. at $x=0$

f(x) not continuous $[-1,1]$

Rolle's Theorem does not apply.

Calculus Ch. 3.3 **Notes: First Derivative Test**

What does the derivative represent? slope of tangent line to function (curve)

When the function is **increasing**, what is common about the derivatives at those points? $f'(x) > 0$

When the function is **decreasing**, what is common about the derivatives at those points? $f'(x) < 0$

When $f'(x) > 0$, f(x) is increasing

When $f'(x) < 0$, f(x) is decreasing

When $f'(x) = 0$, f(x) is constant (horizontal or changing direction)

First Derivative Test Steps (Finds inc/dec and relative max/min)

1. Find f '(x), set equal to zero
   a. Find critical points from BOTH numerator and denominator
   b. Remember, critical points also exist where function is not differentiable (sharp point)
2. Put all critical points on sign line
3. Test intervals
   a. Plug values into f '(x) to determine slope
      i. Positive (+) means increasing slope
      ii. Negative(–) means decreasing slope

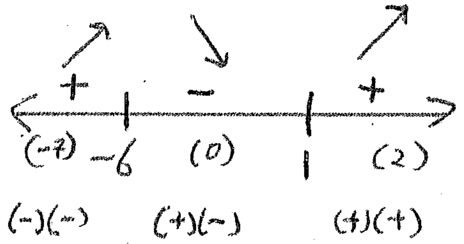
*4. Write Because Statements *
   a. f(x) increasing in interval (a,b) b/c $f'(x) > 0$
   b. f(x) decreasing in interval (a,b) b/c $f'(x) < 0$
   c. Relative max at (**a, f(a)**) b/c f '(x) changes from + to –
   d. Relative min at (**a, f(a)**) b/c f '(x) changes from – to +

Example 1: Determine the intervals at which the function $f(x) = \frac{1}{3}x^3 + \frac{5}{2}x^2 - 6x - 3$ is increasing and decreasing. Locate the relative extrema.

$f'(x) = x^2 + 5x - 6$

$0 = (x+6)(x-1)$

$x = 1,\ x = -6$



f(x) is increasing on $(-\infty, -6) \cup (1, \infty)$ b/c $f'(x) > 0$

f(x) is decreasing on $(-6, 1)$ b/c $f'(x) < 0$

Relative max at $(-6, 51)$ b/c $f'(x)$ changes from + to –

Relative min at $(1, -6.167)$ b/c $f'(x)$ changes from – to +

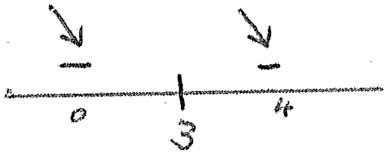
First Derivative Test Steps (Finds inc/dec and relative max/min)

1. Find f '(x), set equal to zero
   - a. Find critical points from BOTH numerator and denominator
   - b. Remember, critical points also exist where function is not differentiable (sharp point)
2. Put all critical points on sign line
3. Test intervals
   - a. Plug values into f '(x) to determine slope
     - i. Positive (+) means increasing slope
     - ii. Negative(–) means decreasing slope
4. Write Because Statements
   - a. f(x) increasing in interval (a,b) b/c f '(x) > 0
   - b. f(x) decreasing in interval (a,b) b/c f '(x) < 0
   - c. Relative max at (**a, f(a)**) b/c f '(x) changes from + to –
   - d. Relative min at (**a, f(a)**) b/c f '(x) changes from – to +

Example 2: Determine the intervals at which the function $f(x) = \frac{5x+2}{x-3}$ is increasing and decreasing. Locate the relative extrema.

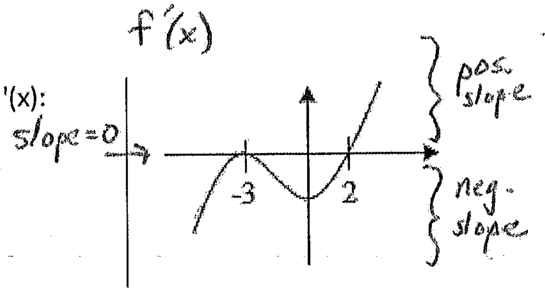
$$f'(x) = \frac{5(x-3)-(5x+2)(1)}{(x-3)^2} = \frac{5x-15-5x-2}{(x-3)^2} = \frac{-17}{(x-3)^2}$$

critical pt:
$x = 3$

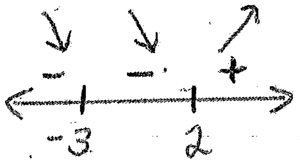


f(x) is decreasing on $(-\infty, 3) \cup (3, \infty)$ b/c $f'(x) < 0$
No relative extrema for f(x)

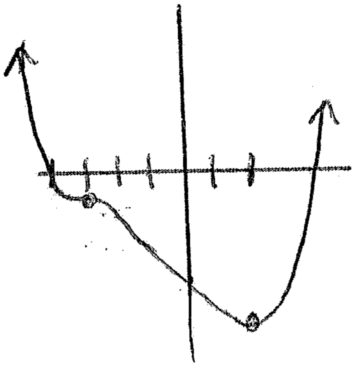
Example 3: Make a first derivative sign line for the following graph of f '(x):



1) Make sign line:



2) sketch graph using sign line



AB Calculus Ch. 3.3 Select HW Problems Key

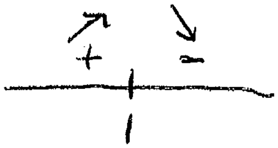
Applying the First Derivative Test In Exercises 17–40, (a) find the critical numbers of f (if any), (b) find the open interval(s) on which the function is increasing or decreasing, (c) apply the First Derivative Test to identify all relative extrema, and (d) use a graphing utility to confirm your results.

19. $f(x) = -2x^2 + 4x + 3$

$f'(x) = -4x+4$

$0 = -4(x-1)$

$x = 1$



Increasing on $(-\infty, 1)$ b/c $f'(x) > 0$

Decreasing on $(1, \infty)$ b/c $f'(x) < 0$

Relative max at $(1, \underline{5})$ b/c $f'(x)$ changes from + to −

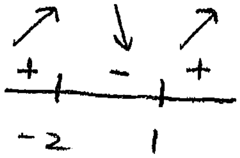
21. $f(x) = 2x^3 + 3x^2 - 12x$

$f'(x) = 6x^2+6x-12$

$0 = 6(x^2+x-2)$

$0 = 6(x+2)(x-1)$

$x = 1, -2$



Increasing on $(-\infty, -2) \cup (1, \infty)$ b/c $f'(x) > 0$

Decreasing on $(-2, 1)$ b/c $f'(x) < 0$

Rel. max $(-2, \underline{20})$ b/c $f'(x)$ changes from + to −

Rel. min $(1, \underline{-7})$ b/c $f'(x)$ changes from − to +

25. $f(x) = \dfrac{x^5 - 5x}{5} = \frac{1}{5}(x^5 - 5x)$

$f(x) = \frac{1}{5}x^5 - \frac{5}{5}x = \frac{1}{5}x^5 - x$

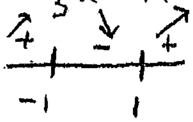
$f'(x) = 5 \cdot \frac{1}{5}x^4 - 1$

$f'(x) = x^4 - 1$

$0 = (x^2+1)(x^2-1)$

$0 = (x^2+1)(x-1)(x+1)$

$x = 1, -1$



Inc: $(-\infty, -1) \cup (1, \infty)$

Dec: $(-1, 1)$

Rel. max: $(-1, 4/5)$

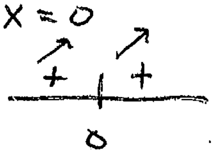
Rel. min: $(1, -4/5)$

27. $f(x) = x^{1/3} + 1$

$f'(x) = \frac{1}{3}x^{-2/3} + 0$

$0 = \frac{1}{3x^{2/3}}$

$x = 0$



Increasing: $(-\infty, 0] \cup (0, \infty)$ b/c $f'(x) > 0$

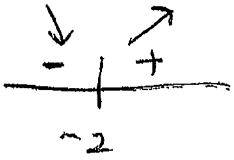
No relative extrema

29. $f(x) = (x + 2)^{2/3}$ * Apply chain rule

$f'(x) = \frac{2}{3}(x+2)^{-1/3}(1)$

$0 = \dfrac{2}{3(x+2)^{1/3}}$

$x = -2$



Dec: $(-\infty, -2)$ b/c $f'(x) < 0$

Inc: $(-2, \infty)$ b/c $f'(x) > 0$

Rel. min $(-2, \underline{0})$ b/c $f'(x)$ changes from − to +

33. $f(x) = 2x + \dfrac{1}{x}$

$f(x) = 2x + x^{-1}$

$f'(x) = 2 - 1x^{-2}$

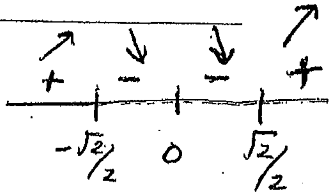
$f'(x) = 2 - \frac{1}{x^2}$

$f'(x) = \dfrac{2x^2-1}{x^2}$

$2x^2 - 1 = 0$ | $x^2 = 0$

$x^2 = 1/2$

$x = \pm\sqrt{1/2} = \frac{\sqrt{2}}{2}$



Inc: $(-\infty, -\sqrt{2}/2) \cup (\sqrt{2}/2, \infty)$ b/c $f'(x) > 0$

Dec: $(-\sqrt{2}/2, 0) \cup (0, \sqrt{2}/2)$ b/c $f'(x) < 0$

Rel. max: $(-\sqrt{2}/2, -2\sqrt{2})$

Rel. min: $(\sqrt{2}/2, 2\sqrt{2})$

⑫

2017

AB Calculus Ch. 3.4 Select HW Problems Key

Steps:
1) find $f''(x)$ (second derivative)
2) set $f''(x)=0$, find critical pts.
3) create sign line, evaluate concavity in intervals

Finding Points of Inflection In Exercises 15–30, find the points of inflection and discuss the concavity of the graph of the function.

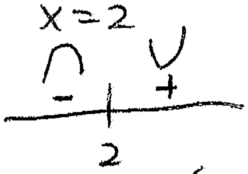
15. $f(x) = x^3 - 6x^2 + 12x$

$f'(x) = 3x^2 - 12x + 12$

$f''(x) = 6x - 12$

$0 = 6(x-2)$

$x = 2$



concave up: $(2,\infty)$ b/c $f''(x) > 0$
concave down: $(-\infty,2)$ b/c $f''(x) < 0$
POI: $(2, \underline{8})$ b/c $f''(x)$ change signs

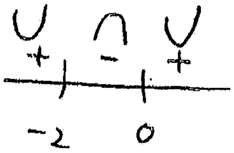
17. $f(x) = \frac{1}{2}x^4 + 2x^3$

$f'(x) = 4\cdot\frac{1}{2}x^3 + 6x^2 = 2x^3 + 6x^2$

$f''(x) = 6x^2 + 12x$

$0 = 6x(x+2)$

$x = 0, -2$



Concave up $(-\infty,-2) \cup (0,\infty)$ b/c $f''(x) > 0$

Concave down $(-2,0)$ b/c $f''(x) < 0$

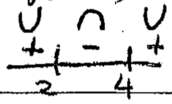
POI: $(-2, \underline{-8})$ and $(0, \underline{0})$ b/c $f''(x)$ change signs

19. $f(x) = x(x-4)^3$ *apply product, chain rule

$f'(x) = 1\cdot(x-4)^3 + x\cdot 3(x-4)^2(1)$
$= (x-4)^2[x-4+3x] = (x-4)^2(4x-4)$

$f''(x) = 2(x-4)\cdot(4x-4) + (x-4)^2(4)$
$= 2(x-4)\cdot 4(x-1) + (x-4)^2\cdot 4$
$= 4(x-4)[\overset{2x-2+x-4}{2(x-1)+x-4}] = 4(x-4)(3x-6)$

$f''(x) = 12(x-4)(x-2)$
$0 = 12(x-4)(x-2)$
$x = 2, 4$



concave up $(-\infty,2) \cup (4,\infty)$
concave down $(2,4)$
POI: $(2, \underline{-16})$
$(4, \underline{0})$

Domain: $[-3,\infty)$

21. $f(x) = x\sqrt{x+3}$ *Apply product, chain rule

$f(x) = x(x+3)^{1/2}$

$f'(x) = 1(x+3)^{1/2} + x\cdot\frac{1}{2}(x+3)^{-1/2}(1)$

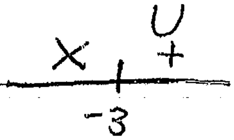
$f'(x) = \sqrt{x+3} + \frac{x}{2\sqrt{x+3}} = \frac{2(x+3)+x}{2\sqrt{x+3}} = \frac{3x+6}{2\sqrt{x+3}}$

$f''(x) = \frac{3\cdot 2\sqrt{x+3} - (3x+6)\cdot 2\cdot\frac{1}{2}(x+3)^{-1/2}}{[2\sqrt{x+3}]^2}$

$= \frac{6\sqrt{x+3} - \frac{3x+6}{\sqrt{x+3}}}{4(x+3)} \cdot \frac{\sqrt{x+3}}{\sqrt{x+3}} = \frac{\overset{6x+18-3x-6}{6(x+3)-(3x+6)}}{4(x+3)^{3/2}}$

$f''(x) = \frac{3x+12}{4(x+3)^{3/2}}$

$x = -4, -3$
$\nearrow$
not in domain



Concave up: $(-3,\infty)$
No POI.

23. $f(x) = \frac{4}{x^2+1} = 4(x^2+1)^{-1}$

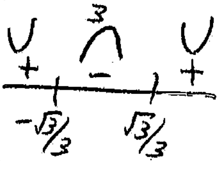
$f'(x) = -4(x^2+1)^{-2}(2x) = \frac{-8x}{(x^2+1)^2}$

$f''(x) = \frac{-8(x^2+1)^2 - 8x\cdot 2(x^2+1)(2x)}{(x^2+1)^4}$

$= \frac{-8(x^2+1)[x^2+1-4x^2]}{(x^2+1)^4}$

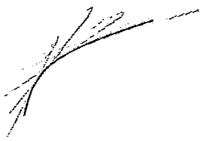
$f''(x) = \frac{-8(1-3x^2)}{(x^2+1)^3}$

$x = \pm\frac{\sqrt{3}}{3}$



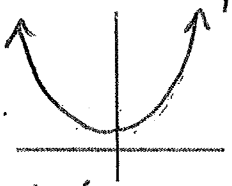
Concave up: $(-\infty, -\sqrt{3}/3) \cup (\sqrt{3}/3, \infty)$
Concave down: $(-\sqrt{3}/3, \sqrt{3}/3)$
POI: $(\frac{-\sqrt{3}}{3}, \underline{3})$ and $(\frac{\sqrt{3}}{3}, \underline{3})$

Calculus Ch. 3.4 **Notes: "Concavity Test" and the 2^{nd} Derivative Test**



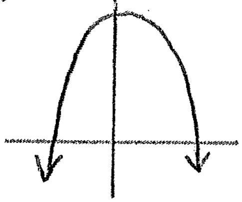
Are both of these functions increasing? yes What do we know about their derivatives? rate of slope is changing

concave up



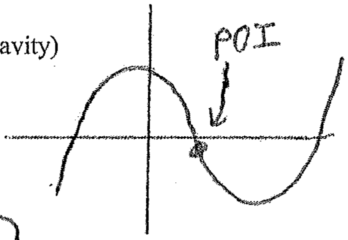
slope is becoming more positive

concave down



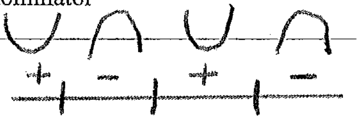
slope is becoming more negative

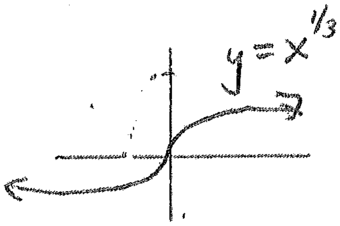
1) If f "(x) is > 0, then f '(x) is increasing and f(x) is concave up.
2) If f "(x) is < 0, then f '(x) is decreasing and f(x) is concave down.
3) A Point of Inflection (POI) occurs whenever f "(x) changes sign. (f(x) changes concavity)



"Concavity Test" Steps (Finding interval Concave Up/Down and POI)

1. Find $f''(x)$, set equal to zero
   a. Find critical points from BOTH numerator and denominator
2. Put all critical points on sign line
3. Test intervals
   a. Plug values into $f''(x)$ to determine concavity
      i. Positive (+) means concave up
      ii. Negative(–) means concave down
4. Write Because Statements
   a. Concave up in interval (a,b) b/c f ''(x) > 0
   b. Concave down in interval (a,b) b/c f ''(x) < 0
5. Point of Inflection at (**a, f(a)**) b/c $f''(x)$ changes signs





*Note: POI may exist on graph where $f''(x)$ does not exist (sharp point). POI exists as long as graph is continuous and $f''(x)$ changes concavity (change of signs) or vertical tangent line

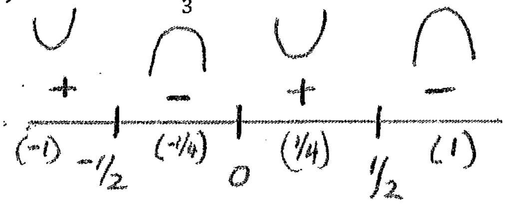
Example 1: Find the points of inflection if $f(x) = -2x^5 + \frac{5}{3}x^3$ * and find intervals concave up/down

$f'(x) = -10x^4 + 5x^2$

$f''(x) = -40x^3 + 10x$

$0 = -10x(4x^2 - 1)$

$x = 0, \quad x = \pm 1/2$



$f''(x)$ concave up on $(-\infty, -1/2) \cup (0, 1/2)$ b/c $f''(x) > 0$

$f''(x)$ concave down on $(-1/2, 0) \cup (1/2, \infty)$ b/c $f''(x) < 0$

POI at $(-1/2, -0.146), (0,0), (1/2, 0.146)$ b/c $f''(x)$ changes signs.

The 2^{nd} Derivative Test

The 2^{nd} derivative test is a test for relative extrema (max/min) and NOT for Point of Inflection

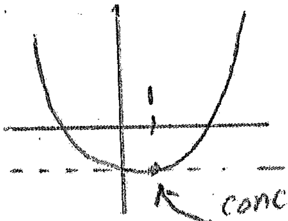
*The 2^{nd} derivative test achieves the same as the 1^{st} derivative test.

1) If you plug a critical number from f'(x) into $f''(x)$ and if $f''(x) > 0$, then that is the x-value of the relative **minimum**

2) If you plug a critical number from f'(x) into $f''(x)$ and if $f''(x) < 0$, then that is the x-value of the relative **maximum**

3) If you plug a critical number from f'(x) into $f''(x)$ and if $f''(x) = 0$, then the test is inconclusive. We need the first derivative test to determine if critical number is a relative extrema.

*critical numbers from f'(x) indicate candidates for relative max/min.

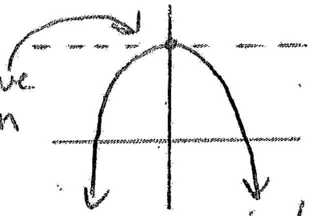


$f'(1)=0$

$f''(1)>0$

so the graph has relative min at $x=1$

concave up



concave down

$f'(0)=0$

$f''(0)<0$

* graph has relative max at $x=0$

2^{nd} Derivative Test Steps (Test for Relative Extrema, **NOT** Point of Inflection)

1. Find f'(x), set equal to zero
   a. Find critical points. Set numerator and denominator of f'(x) = 0. (These are candidates for relative max/min)
2. Find $f''(x)$
3. Plug the critical points (from step #1) into $f''(x)$.
   a. If result is positive value, then f''(x) > 0, concave up, and therefore relative minimum exists at x-value
   b. If result is negative value, then f''(x) < 0, concave down, and therefore relative maximum exists at x-value
   c. f result is zero, then since f''(x) = 0, then this test is inconclusive. We cannot determine whether relative extrema exists. (Use First Derivative Test)

Example 2: Find the relative extrema of $f(x) = x^3 - 4x^2 - 3x$ *using 2nd derivative test.

$f'(x) = 3x^2 - 8x - 3$

$0 = (3x+1)(x-3)$

$x = -1/3, 3$

$f''(x) = 6x - 8$

*Test critical points using f''(x).

$f''(x) = 6x - 8$

$f''(-\frac{1}{3}) = 6(-\frac{1}{3}) - 8 < 0$

Relative max at $x = -\frac{1}{3}$
b/c $f'(-1/3) = 0$ and $f''(-\frac{1}{3}) < 0$

$f''(3) = 6(3) - 8 = 10 > 0$

Relative min at $x = 3$
b/c $f'(3) = 0$ and $f''(3) > 0$