Key

AP Calculus – 3.2 Notes – Implicit Differentiation

Recall:

Explicit equation	Implicit equation
$y = x + 16$	$x^2 + y^2 = 16$

Chain Rule and Implicit Differentiation

In terms of x	In terms of y
$\frac{d}{dx}x = 1\left(\frac{dx}{dx}\right) = 1$	$\frac{d}{dx}y = 1\left(\frac{dy}{dx}\right)$
$\frac{d}{dx}x^2 = 2x\left(\frac{dx}{dx}\right) = 2x$	$\frac{d}{dx}y^2 = 2y\left(\frac{dy}{dx}\right)$
$\frac{d}{dx}e^{5x} = e^{5x}\cdot 5\left(\frac{dx}{dx}\right) = 5e^{5x}$	$\frac{d}{dx}e^{5y} = 5e^{5y}\left(\frac{dy}{dx}\right)$

Implicit Differentiation Example: Find $\frac{dy}{dx}$ for $y^2 - 5x^3 = 3y$

2 instances of y in the starting equation means that 2 instances of $\frac{dy}{dx}$ will show in the derivative equation

Step 1: Take the derivative. Each time the derivative of "y" is involved, include a $\frac{dy}{dx}$.

$$2y\left(\frac{dy}{dx}\right) - 15x^2 = 3\left(\frac{dy}{dx}\right)$$

Step 2: Gather all terms with $\frac{dy}{dx}$ on the left side, everything else on the right.

$$2y\left(\frac{dy}{dx}\right) - 3\left(\frac{dy}{dx}\right) = 15x^2$$

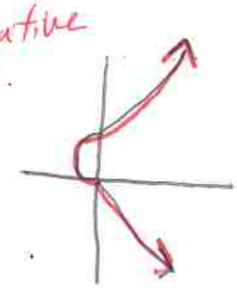
Step 3: Factor out the $\frac{dy}{dx}$ if necessary, to create only one $\frac{dy}{dx}$ term.

$$\frac{dy}{dx}(2y - 3) = 15x^2$$

Step 4. Solve for $\frac{dy}{dx}$.

$$\boxed{\frac{dy}{dx} = \frac{15x^2}{2y-3}}$$

this is simply the slope formula or derivative equation for the original equation curve.



1. $y^3 - 2x = x^4 + 2y$

$$3y^2\left(\frac{dy}{dx}\right) - 2 = 4x^3 + 2\left(\frac{dy}{dx}\right)$$

$$3y^2\left(\frac{dy}{dx}\right) - 2\left(\frac{dy}{dx}\right) = 4x^3 + 2$$

$$\frac{dy}{dx}\left(3y^2 - 2\right) = 4x^3 + 2$$

$$\boxed{\frac{dy}{dx} = \frac{4x^3+2}{3y^2-2}}$$

2. $\sin(xy) = 10x$

*chain Rule $\frac{d}{dx}\sin u = \cos u \cdot u'$

*product Rule

$$\cos(xy)\cdot\left[\underbrace{(1)}_{f'}\underbrace{(y)}_{g} + \underbrace{(x)}_{f}\cdot\underbrace{\left(\frac{dy}{dx}\right)}_{g'}\right] = 10$$

$$y\cos(xy) + x\cos(xy)\cdot\frac{dy}{dx} = 10$$

$$\frac{dy}{dx}\cdot x\cos(xy) = 10 - y\cos(xy)$$

$$\frac{dy}{dx} = \boxed{\frac{10 - y\cos(xy)}{x\cos(xy)}} \rightarrow \frac{10}{x\cos(xy)} - \frac{y\cancel{\cos(xy)}}{x\cancel{\cos(xy)}}$$

$$\rightarrow \boxed{\frac{10\sec(xy) - y}{x}}$$

Derivative at a point – implicit differentiation

3. Find the equation of all tangent lines for $x^2 + y^2 = 4$ when $x = 1$

$2x + 2y\left(\frac{dy}{dx}\right) = 0$

$2y\left(\frac{dy}{dx}\right) = -2x$

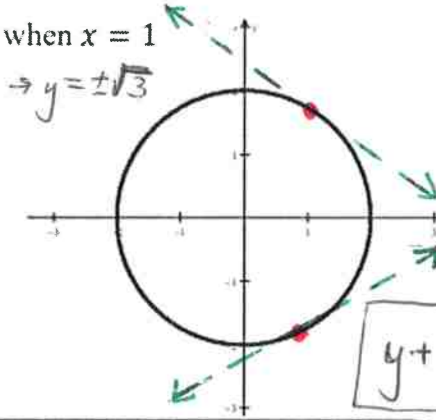
$\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$

point: $(1)^2 + y^2 = 4 \quad y^2 = 3 \rightarrow y = \pm\sqrt{3}$

point: $(1, \sqrt{3})$

slope: $\left.\frac{dy}{dx}\right|_{(1,\sqrt{3})} = \frac{-1}{\sqrt{3}}$

$\boxed{y - \sqrt{3} = \frac{-1}{\sqrt{3}}(x-1)}$



point: $(1, -\sqrt{3})$

$\left.\frac{dy}{dx}\right|_{(1,-\sqrt{3})} = \frac{-x}{y} \rightarrow \frac{1}{\sqrt{3}}$

$\boxed{y + \sqrt{3} = \frac{1}{\sqrt{3}}(x-1)}$

Horizontal and Vertical Tangent Lines

Horizontal tangent lines exist when the slope, $\frac{dy}{dx} = 0$ *set numerator of $\frac{dy}{dx} = 0$

Vertical tangent lines exist when the slope, $\frac{dy}{dx}$ is undefined *set denominator of $dy/dx = 0$

4. Find all ***horizontal*** tangent lines of the graph $3x^2 + 2y^2 = 16$.

$6x + 4y\left(\frac{dy}{dx}\right) = 0$

$4y\left(\frac{dy}{dx}\right) = -6x$

$\frac{dy}{dx} = \frac{-6x}{4y} = \frac{-3x}{2y}$

$\boxed{\frac{dy}{dx} = \frac{\boxed{-3x}}{2y}}$

$-3x = 0$

$x = 0$

$3(0)^2 + 2y^2 = 16$

$2y^2 = 16$

$y^2 = 8$

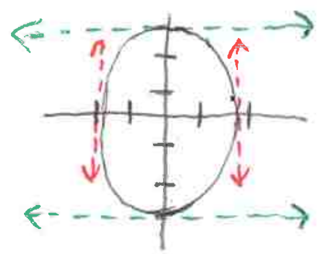
$\boxed{y = \pm\sqrt{8}}$

5. Find all ***vertical*** tangent lines of the graph $3x^2 + 2y^2 = 16$.

$\frac{dy}{dx} = \frac{-3x}{\boxed{2y}}$

$2y = 0$

$y = 0$



$3x^2 + 2(0)^2 = 16$

$3x^2 = 16$

$x^2 = \frac{16}{3}$

$x = \pm\sqrt{\frac{16}{3}}$

$\boxed{x = \pm\frac{4}{\sqrt{3}}}$

Practice Problems:

chain Rule $\frac{d}{dx}\sin u = \cos u \cdot u'$

Find $\frac{dy}{dx}$.

1. $5x^2 + 2y^3 = 4$

$10x + 6y^2\left(\frac{dy}{dx}\right) = 0$

$6y^2\left(\frac{dy}{dx}\right) = -10x$

$\frac{dy}{dx} = \frac{-10x}{6y^2} \rightarrow \boxed{\frac{-5x}{3y^2}}$

2. $5y^2 + 3 = x^2$

$10y\left(\frac{dy}{dx}\right) + 0 = 2x$

$\frac{dy}{dx} = \frac{2x}{10y} = \boxed{\frac{x}{5y}}$

3. $\sin(x + y) = 2x$

$\cos(x+y) \cdot \left[1 + 1\left(\frac{dy}{dx}\right)\right] = 2$

$\cos(x+y) + \frac{dy}{dx}\left[\cos(x+y)\right] = 2$

$\frac{dy}{dx}\left[\cos(x+y)\right] = 2 - \cos(x+y)$

$\boxed{\frac{dy}{dx} = \frac{2-\cos(x+y)}{\cos(x+y)} \text{ or } 2\sec(x+y) - 1}$

4. $4x + 1 = \cos y^2$

$$4 + 0 = -\sin(y^2)\cdot 2y\left(\frac{dy}{dx}\right)$$

$$4 = -2y\sin(y^2)\left(\frac{dy}{dx}\right)$$

$$\frac{4}{-2y\sin(y^2)} = \frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{-2\csc(y^2)}{y}}$$

5. $5x^2 - e^{4y^2} = -6$

$$10x - e^{4y^2}\cdot 8y\left(\frac{dy}{dx}\right) = 0$$

$$-8ye^{4y^2}\left(\frac{dy}{dx}\right) = -10x$$

$$\boxed{\frac{dy}{dx} = \frac{5x}{4ye^{4y^2}}}$$

6. $\ln(y^3) = 5x + 3$

$$3\ln y = 5x + 3$$

$$3\left(\frac{1}{y}\right)\left(\frac{dy}{dx}\right) = 5$$

$$\frac{3}{y}\left(\frac{dy}{dx}\right) = 5$$

$$\frac{dy}{dx} = 5\cdot\frac{y}{3} \rightarrow \boxed{\frac{dy}{dx} = \frac{5y}{3}}$$

7. $x^2 = 4y^3 + 5y^2$

$$2x = 12y^2\left(\frac{dy}{dx}\right) + 10y\left(\frac{dy}{dx}\right)$$

$$2x = \frac{dy}{dx}(12y^2 + 10y)$$

$$\frac{2x}{12y^2+10y} = \frac{dy}{dx} \rightarrow \boxed{\frac{dy}{dx} = \frac{x}{6y^2+5y}}$$

8. $5x^3 - 2y = 5y^3$

$$15x^2 - 2\left(\frac{dy}{dx}\right) = 15y^2\left(\frac{dy}{dx}\right)$$

$$15x^2 = 2\frac{dy}{dx} + 15y^2\left(\frac{dy}{dx}\right)$$

$$15x^2 = \frac{dy}{dx}(2 + 15y^2)$$

$$\boxed{\frac{15x^2}{2+15y^2} = \frac{dy}{dx}}$$

9. $\ln y^2 + \cos^2 x = 1 - y$

$$2\ln y + [\cos x]^2 = 1 - y$$

$$2\left(\frac{1}{y}\right)\left(\frac{dy}{dx}\right) + 2[\cos x]\cdot -\sin x = 0 - 1\left(\frac{dy}{dx}\right)$$

$$\frac{2}{y}\left(\frac{dy}{dx}\right) + 1\left(\frac{dy}{dx}\right) = 2\sin x\cos x$$

$$\frac{dy}{dx}\left(\frac{2}{y} + 1\right) = 2\sin x\cos x \quad \boxed{\frac{dy}{dx} = \frac{2\sin x\cos x}{\frac{2}{y}+1}}$$

10. $\sin\left(\frac{y}{2}\right) + e^y = 4x$

$$\cos\left(\frac{y}{2}\right)\cdot\frac{1}{2}\left(\frac{dy}{dx}\right) + e^y\left(\frac{dy}{dx}\right) = 4$$

$$\frac{dy}{dx}\left[\frac{1}{2}\cos\left(\frac{y}{2}\right) + e^y\right] = 4$$

$$\boxed{\frac{dy}{dx} = \frac{4}{\frac{1}{2}\cos\left(\frac{y}{2}\right) + e^y}}$$

11. $x^3 + y^3 = \underbrace{6}_{f}\underbrace{xy}_{g}$ ← product Rule

$$3x^2 + 3y^2\left(\frac{dy}{dx}\right) = \underbrace{6}_{f'}\cdot\underbrace{y}_{g} + \underbrace{6x}_{f}\cdot\underbrace{\left(\frac{dy}{dx}\right)}_{g'}$$

$$3y^2\left(\frac{dy}{dx}\right) - 6x\left(\frac{dy}{dx}\right) = 6y - 3x^2$$

$$\frac{dy}{dx}(3y^2 - 6x) = 6y - 3x^2$$

$$\frac{dy}{dx} = \frac{6y-3x^2}{3y^2-6x} \rightarrow \frac{\cancel{3}(2y-x^2)}{\cancel{3}(y^2-2x)} \rightarrow \boxed{\frac{dy}{dx} = \frac{2y-x^2}{y^2-2x}}$$

12. $\frac{x}{\sin y} = 5$

$$5\sin y = x$$

$$5\cos y\left(\frac{dy}{dx}\right) = 1$$

$$\boxed{\frac{dy}{dx} = \frac{1}{5\cos y}}$$

$$\boxed{\frac{dy}{dx} = \frac{\sec y}{5}}$$

13. $\underbrace{\ln x}_{f}\,\underbrace{e^{3y}}_{g} = 2y^2$

$$\underbrace{\frac{1}{x}}_{f'}\cdot\underbrace{e^{3y}}_{g} + \underbrace{\ln x}_{f}\cdot\underbrace{e^{3y}\cdot 3\left(\frac{dy}{dx}\right)}_{g'} = 4y\left(\frac{dy}{dx}\right)$$

$$3e^{3y}\ln x\left(\frac{dy}{dx}\right) - 4y\left(\frac{dy}{dx}\right) = \frac{-e^{3y}}{x}$$

$$\frac{dy}{dx}(3e^{3y}\ln x - 4y) = \frac{-e^{3y}}{x}$$

$$\boxed{\frac{dy}{dx} = \frac{-e^{3y}}{x(3e^{3y}\ln x - 4y)}}$$

Find the slope of the tangent line at the given point. Show work.

14. $2 = 3x^4 + xy^4$ at $(-1, 1)$

$$0 = 12x^3 + \overbrace{(1)}^{f'} \cdot \overbrace{y^4}^{g} + \overbrace{x}^{f} \cdot \overbrace{4y^3\left(\frac{dy}{dx}\right)}^{g'}$$

$$-4xy^3\left(\frac{dy}{dx}\right) = 12x^3 + y^4$$

$$\frac{dy}{dx} = \frac{12x^3 + y^4}{-4xy^3}$$

$$\left.\frac{dy}{dx}\right|_{(-1,1)} = \frac{12(-1)^3 + 1^4}{-4(-1)(1)^3} = \frac{-12+1}{4} \rightarrow \boxed{\frac{-11}{4}}$$

15. $\underbrace{x}_{f} \underbrace{\ln y}_{g} = 4 - 2x$ at $(2, 1)$

$$\overbrace{1}^{f'} \cdot \overbrace{\ln y}^{g} + \overbrace{x}^{f} \cdot \overbrace{\frac{1}{y}\left(\frac{dy}{dx}\right)}^{g'} = 0 - 2$$

$$\frac{x}{y}\left(\frac{dy}{dx}\right) = -2 - \ln y$$

$$\frac{dy}{dx} = \frac{y(-2 - \ln y)}{x}$$

$$\left.\frac{dy}{dx}\right|_{(2,1)} = \frac{1(-2 - \ln 1)}{2}$$

$$= \frac{-2}{2} \rightarrow \boxed{-1}$$

Find the equation of the tangent line at the given point.

16. $x^2 + y^2 + 19 = 2x + 12y$ at $(4, 3)$

$$2x + 2y\left(\frac{dy}{dx}\right) + 0 = 2 + 12\left(\frac{dy}{dx}\right)$$

$$2y\left(\frac{dy}{dx}\right) - 12\left(\frac{dy}{dx}\right) = 2 - 2x$$

$$\frac{dy}{dx}(2y - 12) = 2 - 2x$$

$$\frac{dy}{dx} = \frac{2 - 2x}{2y - 12}$$

$$\left.\frac{dy}{dx}\right|_{(4,3)} = \frac{2 - 2(4)}{2(3) - 12} \rightarrow \frac{-6}{-6} = 1$$

point: $(4, 3)$

slope: $m = 1$

$$\boxed{y - 3 = 1(x - 4)}$$

17. $\underbrace{x}_{f} \underbrace{\sin 2y}_{g} = \underbrace{y}_{f} \underbrace{\cos 2x}_{g}$ at $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

$$\overbrace{1}^{f'} \cdot \overbrace{\sin 2y}^{g} + \overbrace{x}^{f} \cdot \overbrace{\cos(2y)(2)\frac{dy}{dx}}^{g'} = \overbrace{1\left(\frac{dy}{dx}\right)}^{f'} \overbrace{\cos(2x)}^{g} + \overbrace{y}^{f} \cdot \overbrace{-\sin 2x \cdot 2}^{g'}$$

$$\frac{dy}{dx}(2x\cos 2y - \cos 2x) = -\sin 2y - 2y\sin 2x$$

$$\left.\frac{dy}{dx}\right|_{(\pi/4, \pi/2)} = \frac{-\sin 2(\pi/2) - 2(\pi/2)\sin(2 \cdot \pi/4)}{2(\pi/4)\cos(2 \cdot \pi/2) - \cos(2 \cdot \pi/4)} \rightarrow \frac{0 - \pi}{-\frac{\pi}{2} - 0} \rightarrow \boxed{2}$$

point: $(\pi/4, \pi/2)$

slope: $m = 2$

$$\rightarrow \boxed{y - \frac{\pi}{2} = 2(x - \pi/4)}$$

Find the equations of all horizontal and vertical tangent lines. Calculator allowed. Round to three decimals.

18. $x^2 + x + 2y^2 = 8$

$$2x + 1 + 4y\left(\frac{dy}{dx}\right) = 0$$

$$4y\left(\frac{dy}{dx}\right) = -2x - 1$$

$$\boxed{\frac{dy}{dx} = \frac{-2x - 1}{4y}}$$

set numerator $= 0$ to find horizontal tangent

set denominator $= 0$ for vertical tangents

$-2x - 1 = 0 \quad -2x = 1$

$x = -1/2$ when $x = -1/2$

$$\left(\frac{-1}{2}\right)^2 + \left(\frac{-1}{2}\right) + 2y^2 = 8$$

$$2y^2 = \frac{+1}{4} + 8$$

$$y = \pm\sqrt{4.125}$$

$4y = 0 \rightarrow y = 0$

$x^2 + x + 0 = 8$

Horizontal: $y = \pm\sqrt{4.125}$

Vertical: $x = \frac{-1 \pm \sqrt{33}}{2}$ or $x = -3.372, 2.372$

$$x^2 + x - 8 = 0 \quad \frac{-1 \pm \sqrt{1^2 - 4(1)(-8)}}{2(1)} \rightarrow \frac{-1 \pm \sqrt{33}}{2}$$

19. $x + y = y^2$

$$1 + 1\left(\frac{dy}{dx}\right) = 2y\left(\frac{dy}{dx}\right)$$

$$\frac{dy}{dx}(1 - 2y) = -1$$

$$\frac{dy}{dx} = \frac{-1}{1 - 2y}$$

set $= 0$

set $= 0$

$1 - 2y = 0 \quad 1 = 2y$

$y = 1/2$

$$x + \frac{1}{2} = \left(\frac{1}{2}\right)^2$$

$$x = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$$

Horizontal: none

Vertical: $x = -1/4$

slope perpendicular to to tangent line

20. Find the slope of the normal line to $y = x + \cos(xy)$ at $(0,1)$.

$1\left(\frac{dy}{dx}\right) = 1 + -\sin(xy)\cdot\left[1y + x\left(\frac{dy}{dx}\right)\right]$

$1\left(\frac{dy}{dx}\right) = 1 - y\sin(xy) - x\sin(xy)\left(\frac{dy}{dx}\right)$

$\frac{dy}{dx}\left(1 + x\sin(xy)\right) = 1 - y\sin(xy)$

$\frac{dy}{dx} = \frac{1 - y\sin(xy)}{1 + x\sin(xy)}$

$\left.\frac{dy}{dx}\right|_{(0,1)} = \frac{1 - 1\sin(0)}{1 + 0\sin(0)} \to 1$

$m_1 = 1$ so $m_2 = \frac{-1}{1} = -1$

(A) 1 $\qquad$ $\boxed{\text{(B) } -1}$ $\qquad$ (C) 0 $\qquad$ (D) 2 $\qquad$ (E) Undefined

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21. The graph of $f(x)$, shown below, consists of a semicircle and two-line segments. $f'(1) =$

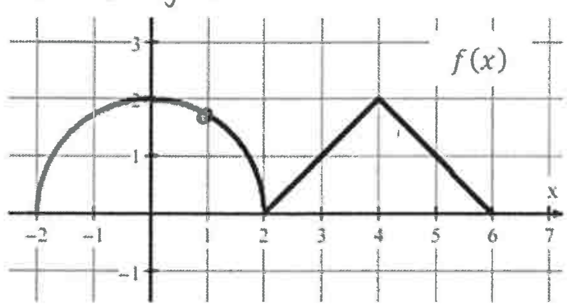
Circle equation: $x^2 + y^2 = r^2 \to (x-h)^2 + (y-k)^2 = r^2$

$(x-0)^2 + (y-0)^2 = 2^2$

$x^2 + y^2 = 4$

$2x + 2y\left(\frac{dy}{dx}\right) = 0$

$2y\left(\frac{dy}{dx}\right) = -2x$



(A) -1 $\qquad$ $\boxed{\text{(B) } -\frac{1}{\sqrt{3}}}$ $\qquad$ (C) $\frac{1}{\sqrt{3}}$ $\qquad$ (D) 1 $\qquad$ (E) $\sqrt{3}$

$\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$

*point of graph at $x = 1 \to (1)^2 + y^2 = 4 \to y^2 = 4 - 1 = 3$, $y = \sqrt{3}$

$\left.\frac{dy}{dx}\right|_{(1,\sqrt{3})} = \boxed{\frac{-1}{\sqrt{3}}}$

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point is $x^2(1) + (1)^2 = 5$

$x^2 = 4 \to x = \pm\sqrt{4} \to x = 2, -2$

22. Find the value(s) of $\frac{dy}{dx}$ of $x^2y + y^2 = 5$ at $y = 1$.

$\underbrace{2x}_{f'}\cdot\underbrace{y}_{g} + \underbrace{x^2}_{f}\cdot\underbrace{\left(\frac{dy}{dx}\right)}_{g'} + 2y\left(\frac{dy}{dx}\right) = 0$

$\frac{dy}{dx}(x^2 + 2y) = -2xy$

$\boxed{\frac{dy}{dx} = \frac{-2xy}{x^2 + 2y}}$

$\left.\frac{dy}{dx}\right|_{(2,1)} = \frac{-2(2)(1)}{2^2 + 2(1)} \to \frac{-4}{6} = \boxed{\frac{-2}{3}}$

$\left.\frac{dy}{dx}\right|_{(-2,1)} = \frac{-2(-2)(1)}{(-2)^2 + 2(1)} \to \frac{4}{6} = \boxed{\frac{2}{3}}$

(A) $-\frac{3}{2}$ only $\qquad$ (B) $-\frac{2}{3}$ only $\qquad$ (C) $\frac{2}{3}$ only $\qquad$ $\boxed{\text{(D) } \pm\frac{2}{3}}$ $\qquad$ (E) $\pm\frac{3}{2}$