Key

3.2 AP Practice Problems (p. 245) – Implicit Differentiation

1. If $\sin(x^2y) = x$, then $\frac{dy}{dx}$ equals

(A) $\cos(x^2y)(2xy + x^2)$ (B) $\frac{1}{\cos(x^2y)(2xy + x^2)}$

$\boxed{(C)}$ $\frac{\sec(x^2y) - 2xy}{x^2}$ (D) $\frac{\sec(x^2y)}{2x}$

*chain Rule out: sin( ) in: x^2y

$\cos(x^2y) \cdot [\overbrace{2x}^{f'} \cdot \overbrace{y}^{g} + \overbrace{x^2}^{f} \cdot \overbrace{(dy/dx)}^{g'}] = 1$

$2xy\cos(x^2y) + x^2\cos(x^2y)(dy/dx) = 1$

$x^2\cos(x^2y)\left(\frac{dy}{dx}\right) = 1 - 2xy\cos(x^2y)$

$\frac{dy}{dx} = \frac{1 - 2xy\cos(x^2y)}{x^2\cos(x^2y)}$

$= \frac{1}{x^2\cos(x^2y)} - \frac{2xy\cos(x^2y)}{x^2\cos(x^2y)}$

$= \frac{\sec(x^2y)}{x^2} - \frac{2xy}{x^2}$

$= \boxed{\frac{\sec(x^2y) - 2xy}{x^2}}$

2. If $\frac{dy}{dx} = \sqrt{1 - 2y^3}$, then $\frac{d^2y}{dx^2}$ equals

(A) $3y^2$ $\boxed{(B)}$ $-3y^2$ (C) $-\frac{3y^2}{\sqrt{x - 2y^3}}$ (D) $-6y^2$

$y' = (1 - 2y^3)^{1/2}$

$\frac{d^2y}{dx^2} = \frac{1}{2}(1 - 2y^3)^{-1/2} \cdot (-6y^2) \cdot \left(\frac{dy}{dx}\right)$

$\frac{d^2y}{dx^2} = \frac{1}{2} \cdot \frac{1}{\cancel{(1-2y^3)^{1/2}}} \cdot -6y^2 \cdot \cancel{(1-2y^3)^{1/2}}$

$\boxed{\frac{d^2y}{dx^2} = -3y^2}$

3. The slope of the tangent line to the graph of $x^2y - xy^3 = 10$ at the point $(-1, 2)$ is

(A) -12 (B) $-\frac{4}{13}$ (C) $-\frac{4}{11}$ $\boxed{(D)}$ $\frac{12}{13}$

$\overbrace{x^2}^{f}\overbrace{y}^{g} - (\overbrace{x}^{f}\overbrace{y^3}^{g}) = 10$

$\overbrace{2x}^{f'} \cdot \overbrace{y}^{g} + \overbrace{x^2}^{f} \cdot \overbrace{(dy/dx)}^{g'} - (\overbrace{(1)}^{f'}\overbrace{(y^3)}^{g} + \overbrace{x}^{f} \cdot \overbrace{3y^2\left(\frac{dy}{dx}\right)}^{g'}) = 0$

$2xy + x^2\left(\frac{dy}{dx}\right) - y^3 - 3xy^2\left(\frac{dy}{dx}\right) = 0$

$x^2\left(\frac{dy}{dx}\right) - 3xy^2\left(\frac{dy}{dx}\right) = y^3 - 2xy$

$dy/dx(x^2 - 3xy^2) = y^3 - 2xy$

$\frac{dy}{dx} = \frac{y^3 - 2xy}{x^2 - 3xy^2}$

$\left.\frac{dy}{dx}\right|_{(-1,2)} = \frac{2^3 - 2(-1)(2)}{(-1)^2 - 3(-1)2^2}$

$= \frac{8+4}{1+12}$

$= \boxed{\frac{12}{13}}$

4. For $f(x) = (x^3 - 4x + 32)^{1/5}$, find $f'(0)$.

(A) $-\frac{1}{4}$ $\boxed{(B)}$ $-\frac{1}{20}$ (C) $\frac{1}{80}$ (D) $\frac{1}{20}$

$f'(x) = \frac{1}{5}(x^3 - 4x + 32)^{-4/5}(3x^2 - 4)$

$f'(0) = \frac{1}{5}(32)^{-4/5}(-4)$

$f'(0) = \frac{1}{5} \cdot \frac{1}{32^{4/5}} \cdot -4$

$f'(0) = \frac{1}{5} \cdot \frac{1}{2^4} \cdot \frac{-4}{1}$

$f'(0) = \frac{1}{5 \cdot -4} = \boxed{\frac{-1}{20}}$

5. Find y' if $x^4 + 4xy + 6y^2 = 12$.

$\boxed{(A)}$ $-\frac{x^3 + y}{x + 3y}$ (B) $-\frac{x^3 + 3y}{x + 12y}$

(C) $\frac{x^3 + y}{x + 3y}$ (D) $\frac{-x^3 + y}{x + 3y}$

$x^4 + \overbrace{4x}^{f}\overbrace{y}^{g} + 6y^2 = 12$

$4x^3 + \overbrace{4}^{f'} \cdot \overbrace{y}^{g} + \overbrace{4x}^{f} \cdot \overbrace{(dy/dx)}^{g'} + 12y(dy/dx) = 0$

$\frac{dy}{dx}(4x + 12y) = -4x^3 - 4y$

$\frac{dy}{dx} = \frac{-4x^3 - 4y}{4x + 12y} \rightarrow \frac{-4(x^3 + y)}{4(x + 3y)}$

$\boxed{\frac{dy}{dx} = \frac{-(x^3 + y)}{x + 3y}}$

6. Find $\frac{dy}{dx}$ at the point (3, 8) on the graph of $\overbrace{5x}^{f}\overbrace{y^{2/3}}^{g} - \overbrace{x^2}^{f}\overbrace{y}^{g} = -12$.

(A) 2 (B) 17 $\boxed{\text{(C)}}$ -7 (D) $-\frac{28}{3}$

$$\overbrace{5}^{f'} \cdot \overbrace{y^{2/3}}^{g} + \overbrace{5x}^{f} \cdot \overbrace{\tfrac{2}{3}y^{-1/3}\left(\tfrac{dy}{dx}\right)}^{g'} - \left(\overbrace{2x}^{f'} \cdot \overbrace{y}^{g} + \overbrace{x^2}^{f}\overbrace{\left(\tfrac{dy}{dx}\right)}^{g'}\right) = 0$$

$$\frac{10x}{3y^{1/3}}\left(\frac{dy}{dx}\right) - x^2\left(\frac{dy}{dx}\right) = -5y^{2/3} + 2xy$$

$$\frac{dy}{dx}\left(\frac{10x}{3y^{1/3}} - x^2\right) = -5y^{2/3} + 2xy$$

$$\frac{dy}{dx} = \frac{-5y^{2/3} + 2xy}{\frac{10x}{3y^{1/3}} - x^2}$$

$$\left.\frac{dy}{dx}\right|_{(3,8)} = \frac{-5(8)^{2/3} + 2(3)(8)}{\frac{10(3)}{3(8)^{1/3}} - 9}$$

$$\left.\frac{dy}{dx}\right|_{(3,8)} = \frac{-5(4)+48}{30/6 - 9} = \frac{28}{-4}$$

$$\boxed{\left.dy/dx\right|_{(3,8)} = -7}$$

7. What is the domain of the derivative of $f(x) = 3x^{2/3}(x+5)^{1/3}$?

(A) The set of all real numbers (B) $\{x \mid x \geq 0\}$

(C) $\{x \mid x \neq 0\}$ $\boxed{\text{(D)}}$ $\{x \mid x \neq 0,\ x \neq -5\}$

$$f'(x) = \overbrace{3 \cdot \tfrac{2}{3}x^{-1/3}}^{f'} \cdot \overbrace{(x+5)^{1/3}}^{g} + \overbrace{3x^{2/3}}^{f} \cdot \overbrace{\tfrac{1}{3}(x+5)^{-2/3}(1)}^{g'}$$

$$f'(x) = \frac{2(x+5)^{1/3}}{x^{1/3}} + \frac{x^{2/3}}{(x+5)^{2/3}} \leftarrow \boxed{\text{Domain: } x \neq 0,\ x \neq -5}$$

8. If $e^y = \tan x$, $0 < x < \frac{\pi}{2}$, what is $\frac{dy}{dx}$ in terms of x?

$\boxed{\text{(A)}}$ $\sec x \csc x$ (B) $\sec^2 x$ (C) $\sec x$ (D) $\sin x \sec x$

$$e^y\left(\frac{dy}{dx}\right) = \sec^2 x \quad \Big| \quad \frac{dy}{dx} = \frac{\sec^2 x}{e^y} \rightarrow \frac{dy}{dx} = \frac{\sec^2 x}{\tan x}$$

$$\frac{dy}{dx} = \frac{\sec^2 x}{\frac{\sin x}{\cos x}} \rightarrow \sec^2 x \cdot \frac{\cos x}{\sin x}$$

$$\frac{dy}{dx} = \frac{\sec x}{\sin x} \rightarrow \boxed{\sec x \csc x}$$

9. $\lim_{h \to 0} \frac{\sqrt[3]{x+h} - \sqrt[3]{x}}{h} =$

(A) $-\frac{3}{x^{2/3}}$, $x \neq 0$ $\boxed{\text{(B)}}$ $\frac{1}{3x^{2/3}}$, $x \neq 0$

(C) $\frac{1}{3x^{2/3}}$, $x \geq 0$ (D) $\frac{x^{2/3}}{3}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x) = \sqrt[3]{x} = x^{1/3}$$

$$f'(x) = \frac{1}{3}x^{-2/3}$$

$$\boxed{f'(x) = \frac{1}{3x^{2/3}}}$$

10. Consider the equation $\overbrace{x^2}^{f}\overbrace{y}^{g} + 3y^3 = 24$.

(a) Find $\frac{dy}{dx}$.

(b) Determine the points, if any, where the tangent line to the graph of the equation is horizontal.

a) $\frac{-2xy}{x^2+9y^2}$ b) $(0, 2)$

$$\overbrace{2x}^{f'} \cdot \overbrace{y}^{g} + \overbrace{x^2}^{f} \cdot \overbrace{\left(dy/dx\right)}^{g'} + 9y^2\left(\frac{dy}{dx}\right) = 0$$

$$\frac{dy}{dx}\left(x^2 + 9y^2\right) = -2xy$$

$$\boxed{\text{a) } \frac{dy}{dx} = \frac{-2xy}{x^2+9y^2}}$$

b) *find horizontal tangent by setting numerator of $\frac{dy}{dx} = 0$

$-2xy = 0$

let $x = 0$

$x^2y + 3y^3 = 24$

$0y + 3y^3 = 24$

$3y^3 = 24$

$y^3 = 8$

$y = 2$

$\boxed{\text{b) Horizontal tangent at point } (0, 2)}$