Section 3.2 Rolle's Theorem and the Mean Value Theorem

1. $f(x) = \left|\frac{1}{x}\right|$

$f(-1) = f(1) = 1$. But, f is not continuous on $[-1, 1]$.

2. Rolle's Theorem does not apply to $f(x) = \cot(x/2)$ over $[\pi, 3\pi]$ because f is not continuous at $x = 2\pi$.

3. Rolle's Theorem does not apply to $f(x) = 1 - |x - 1|$ over $[0, 2]$ because f is not differentiable at $x = 1$.

4. $f(x) = \sqrt{(2 - x^{2/3})^3}$

$f(-1) = f(1) = 1$

$$f'(x) = \frac{-\sqrt{(2 - x^{2/3})}}{x^{1/3}}$$

f is not differentiable at $x = 0$.

5. $f(x) = x^2 - x - 2 = (x - 2)(x + 1)$

x-intercepts: $(-1, 0), (2, 0)$

$f'(x) = 2x - 1 = 0$ at $x = \frac{1}{2}$.

6. $f(x) = x^2 + 6x = x(x + 6)$

x-intercepts: $(0, 0), (-6, 0)$

$f'(x) = 2x + 6 = 0$ at $x = -3$.

7. $f(x) = x\sqrt{x + 4}$

x-intercepts: $(-4, 0), (0, 0)$

$$f'(x) = x\frac{1}{2}(x + 4)^{-1/2} + (x + 4)^{1/2}$$

$$= (x + 4)^{-1/2}\left(\frac{x}{2} + (x + 4)\right)$$

$$f'(x) = \left(\frac{3}{2}x + 4\right)(x + 4)^{-1/2} = 0 \text{ at } x = -\frac{8}{3}$$

8. $f(x) = -3x\sqrt{x + 1}$

x-intercepts: $(-1, 0), (0, 0)$

$$f'(x) = -3x\frac{1}{2}(x + 1)^{-1/2} - 3(x + 1)^{1/2}$$

$$= -3(x + 1)^{-1/2}\left(\frac{x}{2} + (x + 1)\right)$$

$$f'(x) = -3(x + 1)^{-1/2}\left(\frac{3}{2}x + 1\right) = 0 \text{ at } x = -\frac{2}{3}$$

9. $f(x) = -x^2 + 3x, \quad [0, 3]$

$f(0) = -(0)^2 + 3(0)$

$f(3) = -(3)^2 + 3(3) = 0$

f is continuous on $[0, 3]$ and differentiable on $(0, 3)$. Rolle's Theorem applies.

$f'(x) = -2x + 3 = 0$

$-2x = -3 \Rightarrow x = \frac{3}{2}$

c-value: $\frac{3}{2}$

10. $f(x) = x^2 - 8x + 5, [2, 6]$

$f(2) = 4 - 16 + 5 = -7$

$f(6) = 36 - 48 + 5 = -7$

f is continuous on $[2, 6]$ and differentiable on $(2, 6)$. Rolle's Theorem applies.

$f'(x) = 2x - 8 = 0$

$2x = 8 \Rightarrow x = 4$

c-value: 4

11. $f(x) = (x - 1)(x - 2)(x - 3), [1, 3]$

$f(1) = (1 - 1)(1 - 2)(1 - 3) = 0$

$f(3) = (3 - 1)(3 - 2)(3 - 3) = 0$

f is continuous on $[1, 3]$. f is differentiable on $(1, 3)$. Rolle's Theorem applies.

$f(x) = x^3 - 6x^2 + 11x - 6$

$f'(x) = 3x^2 - 12x + 11 = 0$

$$x = \frac{6 \pm \sqrt{3}}{3}$$

c-values: $\frac{6 - \sqrt{3}}{3}, \frac{6 + \sqrt{3}}{3}$

12. $f(x) = (x - 4)(x + 2)^2, [-2, 4]$

$f(-2) = (-2 - 4)(-2 + 2)^2 = 0$

$f(4) = (4 - 4)(4 + 2)^2 = 0$

f is continuous on $[-2, 4]$. f is differentiable on $(-2, 4)$. Rolle's Theorem applies.

$f(x) = (x - 4)(x^2 + 4x + 4) = x^3 - 12x - 16$

$f'(x) = 3x^2 - 12 = 0$

$3x^2 = 12$

$x^2 = 4 \Rightarrow x = \pm 2$

(Note: $x = -2$ is not in the interval.)

c-value: 2

13. $f(x) = x^{2/3} - 1, [-8, 8]$

$f(-8) = (-8)^{2/3} - 1 = 3$

$f(8) = (8)^{2/3} - 1 = 3$

f is continuous on $[-8, 8]$. f is not differentiable on $(-8, 8)$ because $f'(0)$ does not exist. Rolle's Theorem does not apply.

14. $f(x) = 3 - |x - 3|, [0, 6]$

$f(0) = f(6) = 0$

f is continuous on $[0, 6]$. f is not differentiable on $(0, 6)$ because $f'(3)$ does not exist. Rolle's Theorem does not apply.

15. $f(x) = \dfrac{x^2 - 2x - 3}{x + 2}, [-1, 3]$

$f(-1) = \dfrac{(-1)^2 - 2(-1) - 3}{-1 + 2} = 0$

$f(3) = \dfrac{3^2 - 2(3) - 3}{3 + 2} = 0$

f is continuous on $[-1, 3]$.

(**Note:** The discontinuity $x = -2$, is not in the interval.) f is differentiable on $(-1, 3)$. Rolle's Theorem applies.

$$f'(x) = \frac{(x + 2)(2x - 2) - (x^2 - 2x - 3)(1)}{(x + 2)^2} = 0$$

$$\frac{x^2 + 4x - 1}{(x + 2)^2} = 0$$

$$x = \frac{-4 \pm 2\sqrt{5}}{2} = -2 \pm \sqrt{5}$$

(**Note:** $x = -2 - \sqrt{5}$ is not in the interval.)

c-value: $-2 + \sqrt{5}$

16. $f(x) = \dfrac{x^2 - 1}{x}, [-1, 1]$

$f(-1) = \dfrac{(-1)^2 - 1}{-1} = 0$

$f(1) = \dfrac{1^2 - 1}{1} = 0$

f is not continuous on $[-1, 1]$ because $f(0)$ does not exist. Rolle's Theorem does not apply.

17. $f(x) = \sin x, [0, 2\pi]$

$f(0) = \sin 0 = 0$

$f(2\pi) = \sin(2\pi) = 0$

f is continuous on $[0, 2\pi]$. f is differentiable on $(0, 2\pi)$. Rolle's Theorem applies.

$f'(x) = \cos x = 0 \Rightarrow x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

c-values: $\dfrac{\pi}{2}, \dfrac{3\pi}{2}$

18. $f(x) = \cos x, [0, 2\pi]$

$f(0) = \cos 0 = 1$

$f(2\pi) = \cos(2\pi) = 1$

f is continuous on $[0, 2\pi]$. f is differentiable on $(0, 2\pi)$. Rolle's Theorem applies.

$f'(x) = -\sin x = 0 \Rightarrow x = \pi$

c-value: π

19. $f(x) = \sin 3x, \left[0, \frac{\pi}{3}\right]$

$f(0) = \sin(3 \cdot 0) = 0$

$f\left(\frac{\pi}{3}\right) = \sin\left(3 \cdot \frac{\pi}{3}\right) = 0$

f is continuous on $\left[0, \frac{\pi}{3}\right]$. f is differentiable on $\left(0, \frac{\pi}{3}\right)$. Rolle's Theorem applies.

$f'(x) = 3 \cos 3x = 0$

$3x = \frac{\pi}{2} \Rightarrow x = \frac{\pi}{6}$

c-value: $\frac{\pi}{6}$

20. $f(x) = \cos 2x, [-\pi, \pi]$

$f(-\pi) = \cos(-2\pi) = 1$

$f(\pi) = \cos 2\pi = 1$

f is continuous on $[-\pi, \pi]$ and differentiable on $(-\pi, \pi)$. Rolle's Theorem applies.

$f'(x) = -2 \sin 2x$

$-2 \sin 2x = 0$

$\sin 2x = 0$

$x = -\pi, \ -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi$

c-values: $-\frac{\pi}{2}, 0, \frac{\pi}{2}$

21. $f(x) = \tan x, [0, \pi]$

$f(0) = \tan 0 = 0$

$f(\pi) = \tan \pi = 0$

f is not continuous on $[0, \pi]$ because $f(\pi/2)$ does not exist. Rolle's Theorem does not apply.

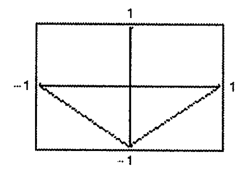
22. $f(x) = \sec x, [\pi, 2\pi]$

f is not continuous on $[\pi, 2\pi]$ because $f(3\pi/2) = \sec(3\pi/2)$ does not exist. Rolle's Theorem does not apply.

23. $f(x) = |x| - 1, [-1, 1]$

$f(-1) = f(1) = 0$

f is continuous on $[-1, 1]$. f is not differentiable on $(-1, 1)$ because $f'(0)$ does not exist. Rolle's Theorem does not apply.



24. $f(x) = x - x^{-1/3}, [0, 1]$

$f(0) = f(1) = 0$

f is continuous on $[0, 1]$. f is differentiable on $(0, 1)$. (**Note:** f is not differentiable at $x = 0$.) Rolle's Theorem applies.

$f'(x) = 1 - \frac{1}{3\sqrt[3]{x^2}} = 0$

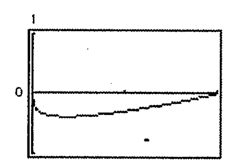
$1 = \frac{1}{3\sqrt[3]{x^2}}$

$\sqrt[3]{x^2} = \frac{1}{3}$

$x^2 = \frac{1}{27}$

$x = \sqrt{\frac{1}{27}} = \frac{\sqrt{3}}{9}$

c-value: $\frac{\sqrt{3}}{9} \approx 0.1925$

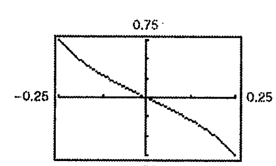


25. $f(x) = x - \tan \pi x, \left[-\frac{1}{4}, \frac{1}{4}\right]$

$f\left(-\frac{1}{4}\right) = -\frac{1}{4} + 1 = \frac{3}{4}$

$f\left(\frac{1}{4}\right) = \frac{1}{4} - 1 = -\frac{3}{4}$

Rolle's Theorem does not apply.



26. $f(x) = \frac{x}{2} - \sin\frac{\pi x}{6}, [-1, 0]$

$f(-1) = f(0) = 0$

f is continuous on $[-1, 0]$. f is differentiable on $(-1, 0)$. Rolle's Theorem applies.

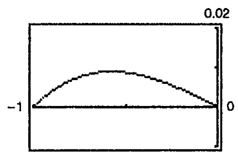
$$f'(x) = \frac{1}{2} - \frac{\pi}{6}\cos\frac{\pi x}{6} = 0$$

$$\cos\frac{\pi x}{6} = \frac{3}{\pi}$$

$$x = -\frac{6}{\pi}\arccos\frac{3}{\pi} \text{ [Value needed in } (-1, 0).]$$

$$\approx -0.5756 \text{ radian}$$

c-value: -0.5756



27. $f(t) = -16t^2 + 48t + 6$

(a) $f(1) = f(2) = 38$

(b) $v = f'(t)$ must be 0 at some time in $(1, 2)$.

$f'(t) = -32t + 48 = 0$

$t = \frac{3}{2}$ sec

28. $C(x) = 10\left(\frac{1}{x} + \frac{x}{x+3}\right)$

(a) $C(3) = C(6) = \frac{25}{3}$

(b) $C'(x) = 10\left(-\frac{1}{x^2} + \frac{3}{(x+3)^2}\right) = 0$

$$\frac{3}{x^2 + 6x + 9} = \frac{1}{x^2}$$

$$2x^2 - 6x - 9 = 0$$

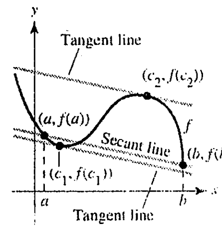
$$x = \frac{6 \pm \sqrt{108}}{4}$$

$$= \frac{6 \pm 6\sqrt{3}}{4} = \frac{3 \pm 3\sqrt{3}}{2}$$

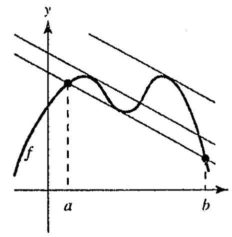
In the interval

$(3, 6)$: $c = \frac{3 + 3\sqrt{3}}{2} \approx 4.098 \approx 410$ components

29.



30.



31. f is not continuous on the interval $[0, 6]$. (f is not continuous at $x = 2$.)

32. f is not differentiable at $x = 2$. The graph of f is not smooth at $x = 2$.

33. $f(x) = \frac{1}{x - 3}, [0, 6]$

f has a discontinuity at $x = 3$.

34. $f(x) = |x - 3|, [0, 6]$

f is not differentiable at $x = 3$.

35. $f(x) = -x^2 + 5$

(a) Slope $= \frac{1 - 4}{2 + 1} = -1$

Secant line: $y - 4 = -(x + 1)$

$$y = -x + 3$$

$$x + y - 3 = 0$$

(b) $f'(x) = -2x = -1 \Rightarrow x = c = \frac{1}{2}$

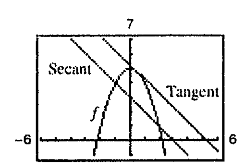
(c) $f(c) = f\left(\frac{1}{2}\right) = -\frac{1}{4} + 5 = \frac{19}{4}$

Tangent line: $y - \frac{19}{4} = -\left(x - \frac{1}{2}\right)$

$$4y - 19 = -4x + 2$$

$$4x + 4y - 21 = 0$$

(d)



36. $f(x) = x^2 - x - 12$

(a) Slope $= \dfrac{-6 - 0}{-2 - 4} = 1$

Secant line: $y - 0 = x - 4$

$x - y - 4 = 0$

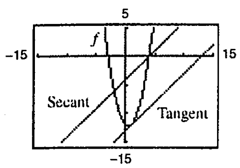
(b) $f'(x) = 2x - 1 = 1 \Rightarrow x = c = 1$

(c) $f(c) = f(1) = -12$

Tangent line: $y + 12 = x - 1$

$x - y - 13 = 0$

(d)



37. $f(x) = x^2$ is continuous on $[-2, 1]$ and differentiable on $(-2, 1)$.

$$\frac{f(1) - f(-2)}{1 - (-2)} = \frac{1 - 4}{3} = -1$$

$$f'(x) = 2x = -1$$

$$x = -\frac{1}{2}$$

$$c = -\frac{1}{2}$$

38. $f(x) = 2x^3$ is continuous on $[0, 6]$ and differentiable on $(0, 6)$.

$$\frac{f(6) - f(0)}{6 - 0} = \frac{432 - 0}{6 - 0} = 72$$

$$f'(x) = 6x^2 = 72$$

$$x^2 = 12$$

$$x = \pm 2\sqrt{3}$$

In the interval $(0, 6)$: $c = 2\sqrt{3}$.

39. $f(x) = x^3 + 2x$ is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$.

$$\frac{f(1) - f(-1)}{1 - (-1)} = \frac{3 - (-3)}{2} = 3$$

$$f'(x) = 3x^2 + 2 = 3$$

$$3x^2 = 1$$

$$x = \pm\frac{1}{\sqrt{3}}$$

$$c = \pm\frac{\sqrt{3}}{3}$$

40. $f(x) = x^4 - 8x$ is continuous on $[0, 2]$ and differentiable on $(0, 2)$.

$$\frac{f(2) - f(0)}{2 - 0} = \frac{0 - 0}{2} = 0$$

$$f'(x) = 4x^3 - 8 = 4(x^3 - 2) = 0$$

$$x^3 = 2$$

$$x = \sqrt[3]{2}$$

$$c = \sqrt[3]{2}$$

41. $f(x) = x^{2/3}$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$.

$$\frac{f(1) - f(0)}{1 - 0} = 1$$

$$f'(x) = \frac{2}{3}x^{-1/3} = 1$$

$$x = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$$

$$c = \frac{8}{27}$$

42. $f(x) = \dfrac{x + 1}{x}$ is not continuous at $x = 0$. The Mean Value Theorem does not apply.

43. $f(x) = |2x + 1|$ is not differentiable at $x = -1/2$. The Mean Value Theorem does not apply.

44. $f(x) = \sqrt{2 - x}$ is continuous on $[-7, 2]$ and differentiable on $(-7, 2)$.

$$\frac{f(2) - f(-7)}{2 - (-7)} = \frac{0 - 3}{9} = -\frac{1}{3}$$

$$f'(x) = \frac{-1}{2\sqrt{2 - x}} = -\frac{1}{3}$$

$$2\sqrt{2 - x} = 3$$

$$\sqrt{2 - x} = \frac{3}{2}$$

$$2 - x = \frac{9}{4}$$

$$x = -\frac{1}{4}$$

$$c = -\frac{1}{4}$$

45. $f(x) = \sin x$ is continuous on $[0, \pi]$ and differentiable on $(0, \pi)$.

$$\frac{f(\pi) - f(0)}{\pi - 0} = \frac{0 - 0}{\pi} = 0$$

$$f'(x) = \cos x = 0$$

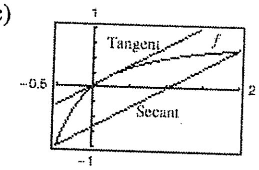
$$x = \pi/2$$

$$c = \frac{\pi}{2}$$

46. $f(x) = \cos x + \tan x$ is not continuous at $x = \pi/2$. The Mean Value Theorem does not apply.

47. $f(x) = \dfrac{x}{x+1}, \left[-\dfrac{1}{2}, 2\right]$

(a)–(c)



(b) Secant line:

$$\text{slope} = \frac{f(2) - f(-1/2)}{2 - (-1/2)} = \frac{2/3 - (-1)}{5/2} = \frac{2}{3}$$

$$y - \frac{2}{3} = \frac{2}{3}(x - 2)$$

$$y = \frac{2}{3}(x - 1)$$

(c) $f'(x) = \dfrac{1}{(x+1)^2} = \dfrac{2}{3}$

$$(x+1)^2 = \frac{3}{2}$$

$$x = -1 \pm \sqrt{\frac{3}{2}} = -1 \pm \frac{\sqrt{6}}{2}$$

In the interval $[-1/2, 2]$: $c = -1 + \left(\sqrt{6}/2\right)$

$$f(c) = \frac{-1 + \left(\sqrt{6}/2\right)}{\left[-1 + \left(\sqrt{6}/2\right)\right] + 1}$$

$$= \frac{-2 + \sqrt{6}}{\sqrt{6}}$$

$$= \frac{-2}{\sqrt{6}} + 1$$

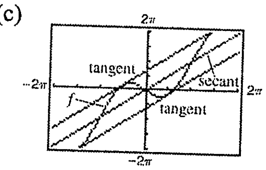
Tangent line: $y - 1 + \dfrac{2}{\sqrt{6}} = \dfrac{2}{3}\left(x - \dfrac{\sqrt{6}}{2} + 1\right)$

$$y - 1 + \frac{\sqrt{6}}{3} = \frac{2}{3}x - \frac{\sqrt{6}}{3} + \frac{2}{3}$$

$$y = \frac{1}{3}\left(2x + 5 - 2\sqrt{6}\right)$$

48. $f(x) = x - 2\sin x, [-\pi, \pi]$

(a)–(c)



(b) Secant line:

$$\text{slope} = \frac{f(\pi) - f(-\pi)}{\pi - (-\pi)} = \frac{\pi - (-\pi)}{2\pi} = 1$$

$$y - \pi = 1(x - \pi)$$

$$y = x$$

(c) $f'(x) = 1 - 2\cos x = 1$

$$\cos x = 0$$

$$x = c = \pm\frac{\pi}{2}$$

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - 2$$

$$f\left(-\frac{\pi}{2}\right) = -\frac{\pi}{2} + 2$$

Tangent lines: $y - \left(\dfrac{\pi}{2} - 2\right) = 1\left(x - \dfrac{\pi}{2}\right)$

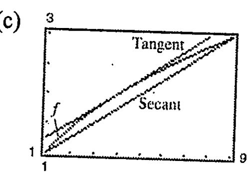
$$y = x - 2$$

$$y - \left(-\frac{\pi}{2} + 2\right) = 1\left(x + \frac{\pi}{2}\right)$$

$$y = x + 2$$

49. $f(x) = \sqrt{x}, [1, 9]$

(a)–(c)



(b) Secant line:

$$\text{slope} = \frac{f(9) - f(1)}{9 - 1} = \frac{3 - 1}{8} = \frac{1}{4}$$

$$y - 1 = \frac{1}{4}(x - 1)$$

$$y = \frac{1}{4}x + \frac{3}{4}$$

(c) $f'(x) = \dfrac{1}{2\sqrt{x}} = \dfrac{1}{4}$

$$x = c = 4$$

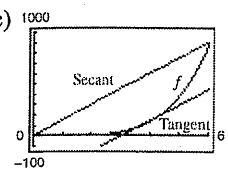
$$f(4) = 2$$

Tangent line: $y - 2 = \dfrac{1}{4}(x - 4)$

$$y = \frac{1}{4}x + 1$$

50. $f(x) = x^4 - 2x^3 + x^2, [0, 6]$

(a)–(c)



(b) Secant line:

$$\text{slope} = \frac{f(6) - f(0)}{6 - 0} = \frac{900 - 0}{6} = 150$$

$$y - 0 = 150(x - 0)$$
$$y = 150x$$

(c) $f'(x) = 4x^3 - 6x^2 + 2x = 150$

Using a graphing utility, there is one solution in $(0, 6)$, $x = c \approx 3.8721$ and $f(c) \approx 123.6721$

Tangent line: $y - 123.6721 = 150(x - 3.8721)$
$$y = 150x - 457.143$$

51. $s(t) = -4.9t^2 + 300$

(a) $v_{avg} = \dfrac{s(3) - s(0)}{3 - 0} = \dfrac{255.9 - 300}{3} = -14.7$ m/sec

(b) $s(t)$ is continuous on $[0, 3]$ and differentiable on $(0, 3)$. Therefore, the Mean Value Theorem applies.

$v(t) = s'(t) = -9.8t = -14.7$ m/sec

$t = \dfrac{-14.7}{-9.8} = 1.5$ sec

52. $S(t) = 200\left(5 - \dfrac{9}{2 + t}\right)$

(a) $\dfrac{S(12) - S(0)}{12 - 0} = \dfrac{200[5 - (9/14)] - 200[5 - (9/2)]}{12} = \dfrac{450}{7}$

(b) $S'(t) = 200\left(\dfrac{9}{(2 + t)^2}\right) = \dfrac{450}{7}$

$$\frac{1}{(2 + t)^2} = \frac{1}{28}$$
$$2 + t = 2\sqrt{7}$$
$$t = 2\sqrt{7} - 2 \approx 3.2915 \text{ months}$$

$S'(t)$ is equal to the average value in April.

53. No. Let $f(x) = x^2$ on $[-1, 2]$.

$f'(x) = 2x$

$f'(0) = 0$ and zero is in the interval $(-1, 2)$ but $f(-1) \neq f(2)$.

54. $f(a) = f(b)$ and $f'(c) = 0$ where c is in the interval (a, b).

(a) $g(x) = f(x) + k$
$g(a) = g(b) = f(a) + k$
$g'(x) = f'(x) \Rightarrow g'(c) = 0$

Interval: $[a, b]$

Critical number of g: c

(b) $g(x) = f(x - k)$
$g(a + k) = g(b + k) = f(a)$
$g'(x) = f'(x - k)$
$g'(c + k) = f'(c) = 0$

Interval: $[a + k, b + k]$

Critical number of g: $c + k$

(c) $g(x) = f(kx)$
$g\left(\dfrac{a}{k}\right) = g\left(\dfrac{b}{k}\right) = f(a)$
$g'(x) = kf'(kx)$
$g'\left(\dfrac{c}{k}\right) = kf'(c) = 0$

Interval: $\left[\dfrac{a}{k}, \dfrac{b}{k}\right]$

Critical number of g: $\dfrac{c}{k}$

55. $f(x) = \begin{cases} 0, & x = 0 \\ 1 - x, & 0 < x \le 1 \end{cases}$

No, this does not contradict Rolle's Theorem. f is not continuous on $[0, 1]$.

56. No. If such a function existed, then the Mean Value Theorem would say that there exists $c \in (-2, 2)$ such that

$$f'(c) = \frac{f(2) - f(-2)}{2 - (-2)} = \frac{6 + 2}{4} = 2.$$

But, $f'(x) < 1$ for all x.

57. Let $S(t)$ be the position function of the plane. If $t = 0$ corresponds to 2 P.M., $S(0) = 0$, $S(5.5) = 2500$ and the Mean Value Theorem says that there exists a time t_0, $0 < t_0 < 5.5$, such that

$$S'(t_0) = v(t_0) = \frac{2500 - 0}{5.5 - 0} \approx 454.54.$$

Applying the Intermediate Value Theorem to the velocity function on the intervals $[0, t_0]$ and $[t_0, 5.5]$, you see that there are at least two times during the flight when the speed was 400 miles per hour. $(0 < 400 < 454.54)$

58. Let $T(t)$ be the temperature of the object. Then $T(0) = 1500°$ and $T(5) = 390°$. The average temperature over the interval $[0, 5]$ is

$$\frac{390 - 1500}{5 - 0} = -222° \text{ F/h}.$$

By the Mean Value Theorem, there exist a time t_0, $0 < t_0 < 5$, such that $T'(t_0) = -222°\text{F/h}$.

59. Let $S(t)$ be the difference in the positions of the 2 bicyclists, $S(t) = S_1(t) - S_2(t)$. Because $S(0) = S(2.25) = 0$, there must exist a time $t_0 \in (0, 2.25)$ such that $S'(t_0) = v(t_0) = 0$. At this time, $v_1(t_0) = v_2(t_0)$.

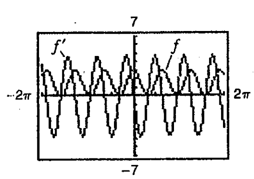
60. Let $t = 0$ correspond to 9:13 A.M. By the Mean Value Theorem, there exists t_0 in $\left(0, \frac{1}{30}\right)$ such that

$$v'(t_0) = a(t_0) = \frac{85 - 35}{1/30} = 1500 \text{ mi/h}^2.$$

61. $f(x) = 3\cos^2\left(\frac{\pi x}{2}\right)$, $f'(x) = 6\cos\left(\frac{\pi x}{2}\right)\left(-\sin\left(\frac{\pi x}{2}\right)\right)\left(\frac{\pi}{2}\right)$

$$= -3\pi\cos\left(\frac{\pi x}{2}\right)\sin\left(\frac{\pi x}{2}\right)$$

(a)



(b) f and f' are both continuous on the entire real line.

(c) Because $f(-1) = f(1) = 0$, Rolle's Theorem applies on $[-1, 1]$. Because $f(1) = 0$ and $f(2) = 3$, Rolle's Theorem does not apply on $[1, 2]$.

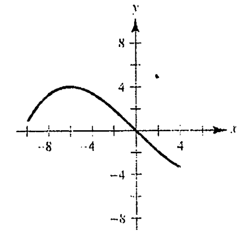
(d) $\lim_{x\to 3^-} f'(x) = 0$

$\lim_{x\to 3^+} f'(x) = 0$

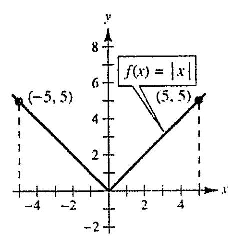
62. (a) f is continuous on $[-10, 4]$ and changes sign, $(f(-8) > 0, f(3) < 0)$. By the Intermediate Value Theorem, there exists at least one value of x in $[-10, 4]$ satisfying $f(x) = 0$.

(b) There exist real numbers a and b such that $-10 < a < b < 4$ and $f(a) = f(b) = 2$. Therefore, by Rolle's Theorem there exists at least one number c in $(-10, 4)$ such that $f'(c) = 0$. This is called a critical number.

(c)

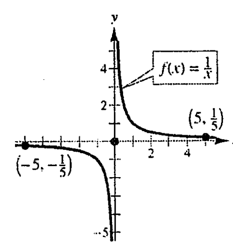


63. f is continuous on $[-5, 5]$ and does not satisfy the conditions of the Mean Value Theorem. $\Rightarrow$ f is not differentiable on $(-5, 5)$. Example: $f(x) = |x|$



64. f is not continuous on $[-5, 5]$.

Example: $f(x) = \begin{cases} 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$



65. $f(x) = x^5 + x^3 + x + 1$

f is differentiable for all x.

$f(-1) = -2$ and $f(0) = 1$, so the Intermediate Value Theorem implies that f has at least one zero c in $[-1, 0]$, $f(c) = 0$.

Suppose f had 2 zeros, $f(c_1) = f(c_2) = 0$. Then Rolle's Theorem would guarantee the existence of a number a such that

$f'(a) = f(c_2) - f(c_1) = 0.$

But, $f'(x) = 5x^4 + 3x^2 + 1 > 0$ for all x. So, f has exactly one real solution.

66. $f(x) = 2x^5 + 7x - 1$

f is differentiable for all x.

$f(0) = -1$ and $f(1) = 8$, so the Intermediate Value Theorem implies that f has at least one zero c in $[0, 1]$, $f(c) = 0$.

Suppose f had 2 zeros, $f(c_1) = f(c_2) = 0$. Then Rolle's Theorem would guarantee the existence of a number a such that

$f'(a) = f(c_2) - f(c_1) = 0.$

But $f'(x) = 10x^4 + 7 > 0$ for all x. So, $f(x) = 0$ has exactly one real solution.

67. $f(x) = 3x + 1 - \sin x$

f is differentiable for all x.

$f(-\pi) = -3\pi + 1 < 0$ and $f(0) = 1 > 0$, so the Intermediate Value Theorem implies that f has at least one zero c in $[-\pi, 0]$, $f(c) = 0$.

Suppose f had 2 zeros, $f(c_1) = f(c_2) = 0$. Then Rolle's Theorem would guarantee the existence of a number a such that

$f'(a) = f(c_2) - f(c_1) = 0.$

But $f'(x) = 3 - \cos x > 0$ for all x. So, $f(x) = 0$ has exactly one real solution.

68. $f(x) = 2x - 2 - \cos x$

$f(0) = -3$, $f(\pi) = 2\pi - 2 + 1 = 2\pi - 1 > 0$. By the Intermediate Value Theorem, f has at least one zero.

Suppose f had 2 zeros, $f(c_1) = f(c_2) = 0$. Then Rolle's Theorem would guarantee the existence of a number a such that

$f'(a) = f(c_2) - f(c_1) = 0.$

But, $f'(x) = 2 + \sin x \geq 1$ for all x. So, f has exactly one real solution.

69. $f'(x) = 0$

$f(x) = c$

$f(2) = 5$

So, $f(x) = 5$.

70. $f'(x) = 4$

$f(x) = 4x + c$

$f(0) = 1 \Rightarrow c = 1$

So, $f(x) = 4x + 1$.

71. $f'(x) = 2x$

$f(x) = x^2 + c$

$f(1) = 0 \Rightarrow 0 = 1 + c \Rightarrow c = -1$

So, $f(x) = x^2 - 1$.

72. $f'(x) = 6x - 1$

$f(x) = 3x^2 - x + c$

$f(2) = 7 \Rightarrow 7 = 3(2^2) - 2 + c$

$= 10 + c \Rightarrow c = -3$

So, $f(x) = 3x^2 - x - 3$.

73. False. $f(x) = 1/x$ has a discontinuity at $x = 0$.

74. False. f must also be continuous *and* differentiable on each interval. Let

$$f(x) = \frac{x^3 - 4x}{x^2 - 1}.$$

75. True. A polynomial is continuous and differentiable everywhere.

76. True

77. Suppose that $p(x) = x^{2n+1} + ax + b$ has two real roots x_1 and x_2. Then by Rolle's Theorem, because $p(x_1) = p(x_2) = 0$, there exists c in (x_1, x_2) such that $p'(c) = 0$. But $p'(x) = (2n + 1)x^{2n} + a \neq 0$, because $n > 0$, $a > 0$. Therefore, $p(x)$ cannot have two real roots.

78. Suppose $f(x)$ is not constant on (a, b). Then there exists x_1 and x_2 in (a, b) such that $f(x_1) \neq f(x_2)$. Then by the Mean Value Theorem, there exists c in (a, b) such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \neq 0.$$

This contradicts the fact that $f'(x) = 0$ for all x in (a, b).

79. If $p(x) = Ax^2 + Bx + C$, then

$$p'(x) = 2Ax + B = \frac{f(b) - f(a)}{b - a}$$
$$= \frac{(Ab^2 + Bb + C) - (Aa^2 + Ba + C)}{b - a}$$
$$= \frac{A(b^2 - a^2) + B(b - a)}{b - a}$$
$$= \frac{(b - a)[A(b + a) + B]}{b - a}$$
$$= A(b + a) + B.$$

So, $2Ax = A(b + a)$ and $x = (b + a)/2$ which is the midpoint of $[a, b]$.

80. (a) $f(x) = x^2, g(x) = -x^3 + x^2 + 3x + 2$

$f(-1) = g(-1) = 1, f(2) = g(2) = 4$

Let $h(x) = f(x) - g(x)$. Then,

$h(-1) = h(2) = 0$. So, by Rolle's Theorem these exists $c \in (-1, 2)$ such that

$h'(c) = f'(c) - g'(c) = 0.$

So, at $x = c$, the tangent line to f is parallel to the tangent line to g.

$h(x) = x^3 - 3x - 2, h'(x)$

$= 3x^2 - 3 = 0 \Rightarrow x = c = 1$

(b) Let $h(x) = f(x) - g(x)$. Then $h(a) = h(b) = 0$ by Rolle's Theorem, there exists c in (a, b) such that

$h'(c) = f'(c) - g'(c) = 0.$

So, at $x = c$, the tangent line to f is parallel to the tangent line to g.

81. Suppose $f(x)$ has two fixed points c_1 and c_2. Then, by the Mean Value Theorem, there exists c such that

$$f'(c) = \frac{f(c_2) - f(c_1)}{c_2 - c_1} = \frac{c_2 - c_1}{c_2 - c_1} = 1.$$

This contradicts the fact that $f'(x) < 1$ for all x.

82. $f(x) = \frac{1}{2}\cos x$ differentiable on $(-\infty, \infty)$.

$f'(x) = -\frac{1}{2}\sin x$

$-\frac{1}{2} \le f'(x) \le \frac{1}{2} \Rightarrow f'(x) < 1$ for all real numbers.

So, from Exercise 62, f has, at most, one fixed point. $(x \approx 0.4502)$

83. Let $f(x) = \cos x$. f is continuous and differentiable for all real numbers. By the Mean Value Theorem, for any interval $[a, b]$, there exists c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$
$$\frac{\cos b - \cos a}{b - a} = -\sin c$$
$$\cos b - \cos a = (-\sin c)(b - a)$$
$$|\cos b - \cos a| = |-\sin c||b - a|$$
$$|\cos b - \cos a| \le |b - a| \text{ since } |-\sin c| \le 1.$$

84. Let $f(x) = \sin x$. f is continuous and differentiable for all real numbers. By the Mean Value Theorem, for any interval $[a, b]$, there exists c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$
$$\sin(b) - \sin(a) = (b - a)\cos(c)$$
$$|\sin(b) - \sin(a)| = |b - a||\cos(c)|$$
$$|\sin a - \sin b| \le |a - b|.$$

85. Let $0 < a < b$. $f(x) = \sqrt{x}$ satisfies the hypotheses of the Mean Value Theorem on $[a, b]$. Hence, there exists c in (a, b) such that

$$f'(c) = \frac{1}{2\sqrt{c}} = \frac{f(b) - f(a)}{b - a} = \frac{\sqrt{b} - \sqrt{a}}{b - a}.$$

So, $\sqrt{b} - \sqrt{a} = (b - a)\frac{1}{2\sqrt{c}} < \frac{b - a}{2\sqrt{a}}$.

Section 3.3 Increasing and Decreasing Functions and the First Derivative Test

1. (a) Increasing: $(0, 6)$ and $(8, 9)$. Largest: $(0, 6)$

(b) Decreasing: $(6, 8)$ and $(9, 10)$. Largest: $(6, 8)$

2. (a) Increasing: $(4, 5), (6, 7)$. Largest: $(4, 5), (6, 7)$

(b) Decreasing: $(-3, 1), (1, 4), (5, 6)$. Largest: $(-3, 1)$