3.2 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Writing **In Exercises 1–4, explain why Rolle's Theorem does not apply to the function even though there exist a and b such that $f(a) = f(b)$.**

1. $f(x) = \left|\frac{1}{x}\right|$, $[-1, 1]$

2. $f(x) = \cot\frac{x}{2}$, $[\pi, 3\pi]$

3. $f(x) = 1 - |x - 1|$, $[0, 2]$

4. $f(x) = \sqrt{(2 - x^{2/3})^3}$, $[-1, 1]$

Intercepts and Derivatives **In Exercises 5–8, find the two x-intercepts of the function f and show that $f'(x) = 0$ at some point between the two x-intercepts.**

5. $f(x) = x^2 - x - 2$

6. $f(x) = x^2 + 6x$

7. $f(x) = x\sqrt{x + 4}$

8. $f(x) = -3x\sqrt{x + 1}$

Using Rolle's Theorem **In Exercises 9–22, determine whether Rolle's Theorem can be applied to f on the closed interval $[a, b]$. If Rolle's Theorem can be applied, find all values of c in the open interval (a, b) such that $f'(c) = 0$. If Rolle's Theorem cannot be applied, explain why not.**

9. $f(x) = -x^2 + 3x$, $[0, 3]$

10. $f(x) = x^2 - 8x + 5$, $[2, 6]$

11. $f(x) = (x - 1)(x - 2)(x - 3)$, $[1, 3]$

12. $f(x) = (x - 4)(x + 2)^2$, $[-2, 4]$

13. $f(x) = x^{2/3} - 1$, $[-8, 8]$

14. $f(x) = 3 - |x - 3|$, $[0, 6]$

15. $f(x) = \dfrac{x^2 - 2x - 3}{x + 2}$, $[-1, 3]$

16. $f(x) = \dfrac{x^2 - 1}{x}$, $[-1, 1]$

17. $f(x) = \sin x$, $[0, 2\pi]$

18. $f(x) = \cos x$, $[0, 2\pi]$

19. $f(x) = \sin 3x$, $\left[0, \frac{\pi}{3}\right]$

20. $f(x) = \cos 2x$, $[-\pi, \pi]$

21. $f(x) = \tan x$, $[0, \pi]$

22. $f(x) = \sec x$, $[\pi, 2\pi]$

Using Rolle's Theorem **In Exercises 23–26, use a graphing utility to graph the function on the closed interval $[a, b]$. Determine whether Rolle's Theorem can be applied to f on the interval and, if so, find all values of c in the open interval (a, b) such that $f'(c) = 0$.**

23. $f(x) = |x| - 1$, $[-1, 1]$

24. $f(x) = x - x^{1/3}$, $[0, 1]$

25. $f(x) = x - \tan \pi x$, $\left[-\frac{1}{4}, \frac{1}{4}\right]$

26. $f(x) = \dfrac{x}{2} - \sin\dfrac{\pi x}{6}$, $[-1, 0]$

27. Vertical Motion The height of a ball t seconds after it is thrown upward from a height of 6 feet and with an initial velocity of 48 feet per second is $f(t) = -16t^2 + 48t + 6$.

(a) Verify that $f(1) = f(2)$.

(b) According to Rolle's Theorem, what must the velocity be at some time in the interval $(1, 2)$? Find that time.

28. Reorder Costs The ordering and transportation cost C for components used in a manufacturing process is approximated by

$$C(x) = 10\left(\frac{1}{x} + \frac{x}{x + 3}\right)$$

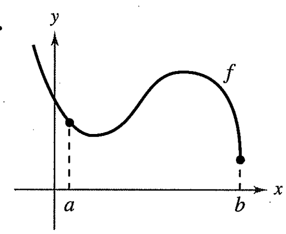
where C is measured in thousands of dollars and x is the order size in hundreds.

(a) Verify that $C(3) = C(6)$.

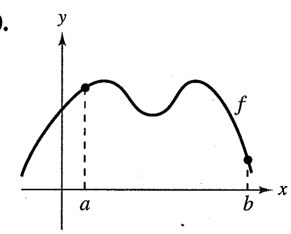
(b) According to Rolle's Theorem, the rate of change of the cost must be 0 for some order size in the interval $(3, 6)$. Find that order size.

Mean Value Theorem **In Exercises 29 and 30, copy the graph and sketch the secant line to the graph through the points $(a, f(a))$ and $(b, f(b))$. Then sketch any tangent lines to the graph for each value of c guaranteed by the Mean Value Theorem. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

29.

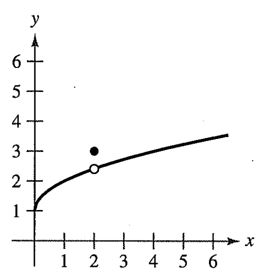


30.

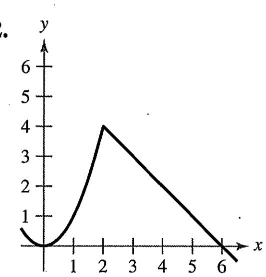


Writing **In Exercises 31–34, explain why the Mean Value Theorem does not apply to the function f on the interval $[0, 6]$.**

31.



32.



33. $f(x) = \dfrac{1}{x - 3}$

34. $f(x) = |x - 3|$

35. Mean Value Theorem Consider the graph of the function $f(x) = -x^2 + 5$ (see figure on next page).

(a) Find the equation of the secant line joining the points $(-1, 4)$ and $(2, 1)$.

(b) Use the Mean Value Theorem to determine a point c in the interval $(-1, 2)$ such that the tangent line at c is parallel to the secant line.

(c) Find the equation of the tangent line through c.

(d) Then use a graphing utility to graph f, the secant line, and the tangent line.

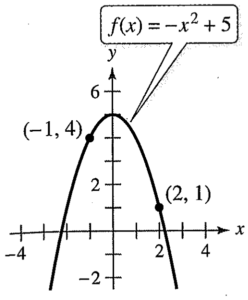


Figure for 35

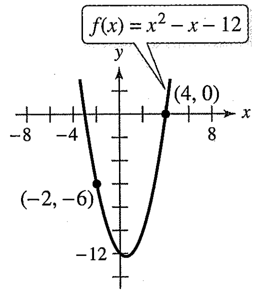


Figure for 36

36. Mean Value Theorem Consider the graph of the function $f(x) = x^2 - x - 12$ (see figure).

(a) Find the equation of the secant line joining the points $(-2, -6)$ and $(4, 0)$.

(b) Use the Mean Value Theorem to determine a point c in the interval $(-2, 4)$ such that the tangent line at c is parallel to the secant line.

(c) Find the equation of the tangent line through c.

(d) Then use a graphing utility to graph f, the secant line, and the tangent line.

Using the Mean Value Theorem **In Exercises 37–46, determine whether the Mean Value Theorem can be applied to f on the closed interval $[a, b]$. If the Mean Value Theorem can be applied, find all values of c in the open interval (a, b) such that**

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

If the Mean Value Theorem cannot be applied, explain why not.

37. $f(x) = x^2, \quad [-2, 1]$

38. $f(x) = 2x^3, \quad [0, 6]$

39. $f(x) = x^3 + 2x, \quad [-1, 1]$

40. $f(x) = x^4 - 8x, \quad [0, 2]$

41. $f(x) = x^{2/3}, \quad [0, 1]$

42. $f(x) = \dfrac{x + 1}{x}, \quad [-1, 2]$

43. $f(x) = |2x + 1|, \quad [-1, 3]$

44. $f(x) = \sqrt{2 - x}, \quad [-7, 2]$

45. $f(x) = \sin x, \quad [0, \pi]$

46. $f(x) = \cos x + \tan x, \quad [0, \pi]$

Using the Mean Value Theorem **In Exercises 47–50, use a graphing utility to (a) graph the function f on the given interval, (b) find and graph the secant line through points on the graph of f at the endpoints of the given interval, and (c) find and graph any tangent lines to the graph of f that are parallel to the secant line.**

47. $f(x) = \dfrac{x}{x + 1}, \quad \left[-\dfrac{1}{2}, 2\right]$

48. $f(x) = x - 2\sin x, \quad [-\pi, \pi]$

49. $f(x) = \sqrt{x}, \quad [1, 9]$

50. $f(x) = x^4 - 2x^3 + x^2, \quad [0, 6]$

51. Vertical Motion The height of an object t seconds after it is dropped from a height of 300 meters is

$$s(t) = -4.9t^2 + 300.$$

(a) Find the average velocity of the object during the first 3 seconds.

(b) Use the Mean Value Theorem to verify that at some time during the first 3 seconds of fall, the instantaneous velocity equals the average velocity. Find that time.

52. Sales A company introduces a new product for which the number of units sold S is

$$S(t) = 200\left(5 - \frac{9}{2 + t}\right)$$

where t is the time in months.

(a) Find the average rate of change of $S(t)$ during the first year.

(b) During what month of the first year does $S'(t)$ equal the average rate of change?

WRITING ABOUT CONCEPTS

53. Converse of Rolle's Theorem Let f be continuous on $[a, b]$ and differentiable on (a, b). If there exists c in (a, b) such that $f'(c) = 0$, does it follow that $f(a) = f(b)$? Explain.

54. Rolle's Theorem Let f be continuous on $[a, b]$ and differentiable on (a, b). Also, suppose that $f(a) = f(b)$ and that c is a real number in the interval such that $f'(c) = 0$. Find an interval for the function g over which Rolle's Theorem can be applied, and find the corresponding critical number of g (k is a constant).

(a) $g(x) = f(x) + k$ (b) $g(x) = f(x - k)$

(c) $g(x) = f(kx)$

55. Rolle's Theorem The function

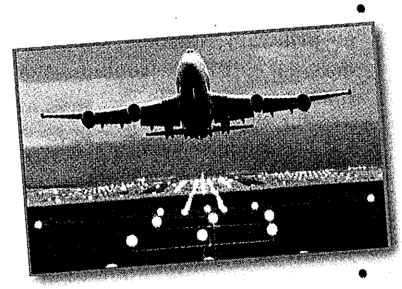
$$f(x) = \begin{cases} 0, & x = 0 \\ 1 - x, & 0 < x \le 1 \end{cases}$$

is differentiable on $(0, 1)$ and satisfies $f(0) = f(1)$. However, its derivative is never zero on $(0, 1)$. Does this contradict Rolle's Theorem? Explain.

56. Mean Value Theorem Can you find a function f such that $f(-2) = -2$, $f(2) = 6$, and $f'(x) < 1$ for all x? Why or why not?

57. Speed

A plane begins its take-off at 2:00 P.M. on a 2500-mile flight. After 5.5 hours, the plane arrives at its destination. Explain why there are at least two times during the flight when the speed of the plane is 400 miles per hour.



58. Temperature When an object is removed from a furnace and placed in an environment with a constant temperature of 90°F, its core temperature is 1500°F. Five hours later, the core temperature is 390°F. Explain why there must exist a time in the interval when the temperature is decreasing at a rate of 222°F per hour.

59. Velocity Two bicyclists begin a race at 8:00 A.M. They both finish the race 2 hours and 15 minutes later. Prove that at some time during the race, the bicyclists are traveling at the same velocity.

60. Acceleration At 9:13 A.M., a sports car is traveling 35 miles per hour. Two minutes later, the car is traveling 85 miles per hour. Prove that at some time during this two-minute interval, the car's acceleration is exactly 1500 miles per hour squared.

61. Using a Function Consider the function

$$f(x) = 3\cos^2\left(\frac{\pi x}{2}\right).$$

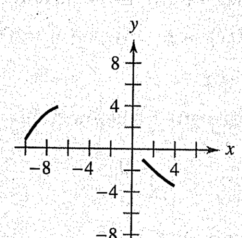
(a) Use a graphing utility to graph f and f'.

(b) Is f a continuous function? Is f' a continuous function?

(c) Does Rolle's Theorem apply on the interval $[-1, 1]$? Does it apply on the interval $[1, 2]$? Explain.

(d) Evaluate, if possible, $\lim_{x\to 3^-} f'(x)$ and $\lim_{x\to 3^+} f'(x)$.

62. HOW DO YOU SEE IT? The figure shows two parts of the graph of a continuous differentiable function f on $[-10, 4]$. The derivative f' is also continuous. To print an enlarged copy of the graph, go to *MathGraphs.com*.



(a) Explain why f must have at least one zero in $[-10, 4]$.

(b) Explain why f' must also have at least one zero in the interval $[-10, 4]$. What are these zeros called?

(c) Make a possible sketch of the function with one zero of f' on the interval $[-10, 4]$.

Think About It **In Exercises 63 and 64, sketch the graph of an arbitrary function f that satisfies the given condition but does not satisfy the conditions of the Mean Value Theorem on the interval $[-5, 5]$.**

63. f is continuous on $[-5, 5]$.

64. f is not continuous on $[-5, 5]$.

Finding a Solution **In Exercises 65–68, use the Intermediate Value Theorem and Rolle's Theorem to prove that the equation has exactly one real solution.**

65. $x^5 + x^3 + x + 1 = 0$ **66.** $2x^5 + 7x - 1 = 0$

67. $3x + 1 - \sin x = 0$ **68.** $2x - 2 - \cos x = 0$

Differential Equation **In Exercises 69–72, find a function f that has the derivative $f'(x)$ and whose graph passes through the given point. Explain your reasoning.**

69. $f'(x) = 0, \quad (2, 5)$ **70.** $f'(x) = 4, \quad (0, 1)$

71. $f'(x) = 2x, \quad (1, 0)$ **72.** $f'(x) = 6x - 1, \quad (2, 7)$

True or False? **In Exercises 73–76, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

73. The Mean Value Theorem can be applied to

$$f(x) = \frac{1}{x}$$

on the interval $[-1, 1]$.

74. If the graph of a function has three x-intercepts, then it must have at least two points at which its tangent line is horizontal.

75. If the graph of a polynomial function has three x-intercepts, then it must have at least two points at which its tangent line is horizontal.

76. If $f'(x) = 0$ for all x in the domain of f, then f is a constant function.

77. Proof Prove that if $a > 0$ and n is any positive integer, then the polynomial function $p(x) = x^{2n+1} + ax + b$ cannot have two real roots.

78. Proof Prove that if $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

79. Proof Let $p(x) = Ax^2 + Bx + C$. Prove that for any interval $[a, b]$, the value c guaranteed by the Mean Value Theorem is the midpoint of the interval.

80. Using Rolle's Theorem

(a) Let $f(x) = x^2$ and $g(x) = -x^3 + x^2 + 3x + 2$. Then $f(-1) = g(-1)$ and $f(2) = g(2)$. Show that there is at least one value c in the interval $(-1, 2)$ where the tangent line to f at $(c, f(c))$ is parallel to the tangent line to g at $(c, g(c))$. Identify c.

(b) Let f and g be differentiable functions on $[a, b]$ where $f(a) = g(a)$ and $f(b) = g(b)$. Show that there is at least one value c in the interval (a, b) where the tangent line to f at $(c, f(c))$ is parallel to the tangent line to g at $(c, g(c))$.

81. Proof Prove that if f is differentiable on $(-\infty, \infty)$ and $f'(x) < 1$ for all real numbers, then f has at most one fixed point. A fixed point of a function f is a real number c such that $f(c) = c$.

82. Fixed Point Use the result of Exercise 81 to show that $f(x) = \frac{1}{2}\cos x$ has at most one fixed point.

83. Proof Prove that $|\cos a - \cos b| \le |a - b|$ for all a and b.

84. Proof Prove that $|\sin a - \sin b| \le |a - b|$ for all a and b.

85. Using the Mean Value Theorem Let $0 < a < b$. Use the Mean Value Theorem to show that

$$\sqrt{b} - \sqrt{a} < \frac{b - a}{2\sqrt{a}}.$$